PRINCIPIA MATHEMATICA

TO ∗56

BY

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AND

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PREFACE

THE mathematical treatment of the principles of mathematics, which is the subject of the present work, has arisen from the conjunction of two different studies, both in the main very modern. On the one hand we have the work of analysts and geometers, in the way of formulating and systematising their axioms, and the work of Cantor and others on such matters as the theory of aggregates. On the other hand we have symbolic logic, which, after a necessary period of growth, has now, thanks to Peano and his followers, acquired the technical adaptability and the logical comprehensiveness that are essential to a mathematical instrument for dealing with what have hitherto been the beginnings of mathematics. From the combination of these two studies two results emerge, namely (1) that what were formerly taken, tacitly or explicitly, as axioms, are either unnecessary or demonstrable; (2) that the same methods by which supposed axioms are demonstrated will give valuable results in regions, such as infinite number, which had formerly been regarded as inaccessible to human knowledge. Hence the scope of mathematics is enlarged both by the addition of new subjects and by a backward extension into provinces hitherto abandoned to philosophy.

The present work was originally intended by us to be comprised in a second volume of *The Principles of Mathematics*. With that object in view, the writing of it was begun in 1900. But as we advanced, it became increasingly evident that the subject is a very much larger one than we had supposed; moreover on many fundamental questions which had been left obscure and doubtful in the former work, we have now arrived at what we believe to be satisfactory solutions. It therefore became necessary to make our book independent of *The Principles of Mathematics*. We have, however, avoided both controversy and general philosophy, and made our statements dogmatic in form. The justification for this is that the chief reason in favour of any theory on the principles of mathematics must always be inductive, *i.e.* it must lie in the fact that the theory in question enables us to deduce ordinary mathematics. In mathematics, the greatest degree of self-evidence is usually not to be found quite at the beginning, but at some later point; hence the early deductions, until they reach this point, give reasons rather for believing the premisses because true consequences follow from them, than for believing the consequences because they follow from the premisses.

In constructing a deductive system such as that contained in the present work, there are two opposite tasks which have to be concurrently performed. On the one hand, we have to analyse existing mathematics, with a view to discovering what premisses are employed, whether these premisses are mutually consistent, and whether they are capable of reduction to more fundamental premisses. On the other hand, when we have decided upon our premisses, we have to build up again as much as may seem necessary of the data previously analysed, and as many other consequences of our premisses as are of sufficient general interest to deserve statement. The preliminary labour of analysis does not appear in the final presentation, which merely sets forth the outcome of the analysis in certain undefined ideas and identical with a or not identical with a. It follows (as will be proved in ∗20·81) that, if "ϕa" and "ψa" are both significant, the class of values of x for which "ϕx" is significant is the same as the class of those for which "ψx" is significant, *i.e.* two types which have a common member are identical.

In the following proof, the chief point to observe is the use of ∗10·221. There are two variables, a and x, to be identified. In the first use, we depend upon the fact that ϕa and $x = a$ both occur in both (4) and (5): the occurrence of ϕa in both justifies the identification of the two a's, and when these have been identified, the occurrence of $x = a$ in both justifies the identification of the two x's. (Unless the a's had been already identified, this would not be legitimate, because "$x = a$" is typically ambiguous if neither x nor a is of given type.) The second use of ∗10·221 is justified by the fact that both ϕa and ϕx occur in both (2) and (6).

∗13·3. $\vdash :: \phi a \vee \sim\phi a \,.\supset :.\, \phi x \vee \sim \phi x \,.\equiv :\, x = a \,.\vee .\, x \neq a$

Dem.

$\vdash . *2{\cdot}11 . \quad \supset \vdash . \phi x \vee \sim \phi x \quad (1)$

$\vdash . (1) . \text{Simp} . \quad \supset \vdash : \phi a \vee \sim\phi a \,.\supset .\, \phi x \vee \sim\phi x \quad (2)$

$\vdash . *2{\cdot}11 . \quad \supset \vdash : x = a \,.\vee .\, x \neq a \quad (3)$

$\vdash . (3) . \text{Simp} . \quad \supset \vdash :. \phi a \vee \sim\phi a \,.\supset :\, x = a \,.\vee .\, x \neq a \quad (4)$

$\vdash . *13{\cdot}101 . \text{Comm} . \supset \vdash :. \phi a \vee \sim\phi a \,.\supset :\, x = a \,.\supset .\, \phi x \vee \sim\phi x \quad (5)$

$\vdash . (4) . (5) . *10{\cdot}13{\cdot}221 . \supset$

$\vdash :: \phi a \vee \sim\phi a \,.\supset :\, x = a \,.\vee .\, x \neq a :. \phi a \vee \sim\phi a \,.\supset :\, x = a \,.\supset .\, \phi x \vee \sim\phi x \quad (6)$

$\vdash . (2) . (6) . *10{\cdot}13{\cdot}221 . \supset$

$\vdash :: \phi a \vee \sim\phi a \,.\supset .\, \phi x \vee \sim\phi x :. \phi a \vee \sim\phi a \,.\supset :\, x = a \,.\vee .\, x \neq a :.$
$\phi a \vee \sim\phi a \,.\supset :\, x = a \,.\supset .\, \phi x \vee \sim\phi x \quad (7)$

$\vdash . (7) . \text{Simp} . \supset$

$\vdash :: \phi a \vee \sim\phi a \,.\supset .\, \phi x \vee \sim\phi x :. \phi a \vee \sim\phi a \,.\supset :\, x = a \,.\vee .\, x \neq a \quad (8)$

$\vdash . (8) . *5{\cdot}35 . \quad \supset \vdash :: \phi a \vee \sim\phi a \,.\supset :.\, \phi x \vee \sim\phi x \,.\equiv :\, x = a \,.\vee .\, x \neq a ::$
$\supset \vdash . \text{Prop}$

∗14. DESCRIPTIONS

Summary of ∗14.

A *description* is a phrase of the form "the term which etc.," or, more explicitly, "the term x which satisfies $\phi\hat{x}$," where $\phi\hat{x}$ is some function satisfied by one and only one argument. For reasons explained in the Introduction (Chapter III), we do not define "the x which satisfies $\phi\hat{x}$," but we define any proposition in which this phrase occurs. Thus when we say: "The term x which satisfies $\phi\hat{x}$ satisfies $\psi\hat{x}$," we shall mean: "There is a term b such that ϕx is true when, and only when, x is b, and ψb is true." That is, writing "$(\iota x)(\phi x)$" for "the term x which satisfies ϕx," $\psi(\iota x)(\phi x)$ is to mean

$$(\exists b) : \phi x \,.\equiv_x .\, x = b : \psi b.$$

This, however, is not yet quite adequate as a definition, for when $(\iota x)(\phi x)$ occurs in a proposition which is part of a larger proposition, there is doubt whether the smaller or the larger proposition is to be taken as the "$\psi(\iota x)(\phi x)$." Take, for example, $\psi(\iota x)(\phi x) \,.\supset .\, p$. This may be either

$$(\exists b) : \phi x \,.\equiv_x .\, x = b : \psi b : \supset . p$$

or

$$(\exists b) :. \phi x \,.\equiv_x .\, x = b : \psi b \,.\supset .\, p.$$

If "$(\exists b) : \phi x \,.\equiv_x .\, x = b$" is false, the first of these must be true, while the second must be false. Thus it is very necessary to distinguish them.

The proposition which is to be treated as the "$\psi(\iota x)(\phi x)$" will be called the *scope* of $(\iota x)(\phi x)$. Thus in the first of the above two propositions, the scope of $(\iota x)(\phi x)$ is $\psi(\iota x)(\phi x)$, while in the second it is $\psi(\iota x)(\phi x) \,.\supset .\, p$. In order to avoid ambiguities as to scope, we shall indicate the scope by writing "$[(\iota x)(\phi x)]$" at the beginning of the scope, followed by enough dots to extend to the end of the scope. Thus of the above two propositions the first is

$$[(\iota x)(\phi x)] \,.\, \psi(\iota x)(\phi x) \,.\supset .\, p,$$

while the second is

$$[(\iota x)(\phi x)] : \psi(\iota x)(\phi x) \,.\supset .\, p.$$

Thus we arrive at the following definition:

∗14·01. $[(\iota x)(\phi x)] \,.\, \psi(\iota x)(\phi x) \,.=:\, (\exists b) : \phi x \,.\equiv_x .\, x = b : \psi b \quad \text{Df}$

It will be found in practice that the scope usually required is the smallest proposition enclosed in dots or brackets in which "$(\iota x)(\phi x)$" occurs. Hence when this scope is to be given to $(\iota x)(\phi x)$, we shall usually omit explicit mention of the scope. Thus *e.g.* we shall have

$$a \neq (\iota x)(\phi x) \,.=:\, (\exists b) : \phi x \,.\equiv_x .\, x = b : a \neq b,$$
$$\sim\{a = (\iota x)(\phi x)\} \,.=.\, \sim\{(\exists b) : \phi x \,.\equiv_x .\, x = b : a = b\}.$$

Of these the first necessarily implies $(\exists b) : \phi x . \equiv_x . x = b$, while the second does not. We put

$*14{\cdot}02.$ $E\,!\,(\imath x)(\phi x) . = : (\exists b) : \phi x . \equiv_x . x = b$ Df

This defines: "The x satisfying $\phi\hat{x}$ exists," which holds when, and only when, $\phi\hat{x}$ is satisfied by one value of x and by no other value.

When two or more descriptions occur in the same proposition, there is need of avoiding ambiguity as to which has the larger scope. For this purpose, we put

$*14{\cdot}03.$ $[(\imath x)(\phi x), (\imath x)(\psi x)] . f\{(\imath x)(\phi x), (\imath x)(\psi x)\} . = :$

$$[(\imath x)(\phi x)] : [(\imath x)(\psi x)] . f\{(\imath x)(\phi x), (\imath x)(\psi x)\} \quad \text{Df}$$

It will be shown ($*14{\cdot}113$) that the truth-value of a proposition containing two descriptions is unaffected by the question which has the larger scope. Hence we shall in general adopt the convention that the description occurring first typographically is to have the larger scope, unless the contrary is expressly indicated. Thus *e.g.*

$$(\imath x)(\phi x) = (\imath x)(\psi x)$$

will mean $(\exists b) : \phi x . \equiv_x . x = b : b = (\imath x)(\psi x)$,

i.e. $(\exists b) :. \phi x . \equiv_x . x = b :. (\exists c) : \psi x . \equiv_x . x = c : b = c.$

By this convention we are able almost always to avoid explicit indication of the order of elimination of two or more descriptions. If, however, we require a larger scope for the later description, we put

$*14{\cdot}04.$ $[(\imath x)(\psi x)] . f\{(\imath x)(\phi x), (\imath x)(\psi x)\} . = .$

$$[(\imath x)(\psi x), (\imath x)(\phi x)] . f\{(\imath x)(\phi x), (\imath x)(\psi x)\} \quad \text{Df}$$

Whenever we have $E\,!\,(\imath x)(\phi x)$, $(\imath x)(\phi x)$ behaves, formally, like an ordinary argument to any function in which it may occur. This fact is embodied in the following proposition:

$*14{\cdot}18.$ $\vdash :. E\,!\,(\imath x)(\phi x) . \supset : (x) . \psi x . \supset . \psi(\imath x)(\phi x)$

That is to say, when $(\imath x)(\phi x)$ exists, it has any property which belongs to everything. This does not hold when $(\imath x)(\phi x)$ does not exist; for example, the present King of France does not have the property of being either bald or not bald.

If $(\imath x)(\phi x)$ has any property whatever, it must exist. This fact is stated in the proposition:

$*14{\cdot}21.$ $\vdash : \psi(\imath x)(\phi x) . \supset . E\,!\,(\imath x)(\phi x)$

This proposition is obvious, since "$E\,!\,(\imath x)(\phi x)$" is, by the definitions, part of "$\psi(\imath x)(\phi x)$." When, in ordinary language or in philosophy, something is said to "exist," it is always something *described*, *i.e.* it is not something immediately presented, like a taste or a patch of colour, but something like "matter" or "mind" or "Homer" (meaning "the author of the Homeric

poems"), which is known by description as "the so-and-so," and is thus of the form $(\imath x)(\phi x)$. Thus in all such cases, the existence of the (grammatical) subject $(\imath x)(\phi x)$ can be analytically inferred from any true proposition having this grammatical subject. It would seem that the word "existence" cannot be significantly applied to subjects immediately given; *i.e.* not only does our definition give no meaning to "$E\,!\,x$," but there is no reason, in philosophy, to suppose that a meaning of existence could be found which would be applicable to immediately given subjects.

Besides the above, the following are among the more useful propositions of the present number.

$*14{\cdot}202.$ $\vdash :. \phi x . \equiv_x . x = b : \equiv : (\imath x)(\phi x) = b : \equiv : \phi x . \equiv_x . b = x : \equiv : b = (\imath x)(\phi x)$

From the first equivalence in the above, it follows that

$*14{\cdot}204.$ $\vdash : E\,!\,(\imath x)(\phi x) . \equiv . (\exists b) . (\imath x)(\phi x) = b$

I.e. $(\imath x)(\phi x)$ exists when there is something which $(\imath x)(\phi x)$ is.

We have

$*14{\cdot}205.$ $\vdash : \psi(\imath x)(\phi x) . \equiv . (\exists b) . b = (\imath x)(\phi x) . \psi b$

I.e. $(\imath x)(\phi x)$ has the property ψ when there is something which is $(\imath x)(\phi x)$ and which has the property ψ.

We have to prove that such symbols as "$(\imath x)(\phi x)$" obey the same rules with regard to identity as symbols which directly represent objects. To this, however, there is one partial exception, for instead of having

$$(\imath x)(\phi x) = (\imath x)(\phi x),$$

we only have

$*14{\cdot}28.$ $\vdash : E\,!\,(\imath x)(\phi x) . \equiv . (\imath x)(\phi x) = (\imath x)(\phi x)$

I.e. "$(\imath x)(\phi x)$" only satisfies the reflexive property of identity if $(\imath x)(\phi x)$ exists.

The symmetrical property of identity holds for such symbols as $(\imath x)(\phi x)$, without the need of assuming existence, *i.e.* we have

$*14{\cdot}13.$ $\vdash : a = (\imath x)(\phi x) . \equiv . (\imath x)(\phi x) = a$

$*14{\cdot}131.$ $\vdash : (\imath x)(\phi x) = (\imath x)(\psi x) . \equiv . (\imath x)(\psi x) = (\imath x)(\phi x)$

Similarly the transitive property of identity holds without the need of assuming existence. This is proved in $*14{\cdot}14{\cdot}142{\cdot}144$.

$*14{\cdot}01.$ $[(\imath x)(\phi x)] . \psi(\imath x)(\phi x) . = : (\exists b) : \phi x . \equiv_x . x = b : \psi b$ Df

$*14{\cdot}02.$ $E\,!\,(\imath x)(\phi x) . = : (\exists b) : \phi x . \equiv_x . x = b$ Df

$*14{\cdot}03.$ $[(\imath x)(\phi x), (\imath x)(\psi x)] . f\{(\imath x)(\phi x), (\imath x)(\psi x)\} . = :$

$$[(\imath x)(\phi x)] : [(\imath x)(\psi x)] . f\{(\imath x)(\phi x), (\imath x)(\psi x)\} \quad \text{Df}$$

$*14{\cdot}04.$ $[(\imath x)(\psi x)] . f\{(\imath x)(\phi x), (\imath x)(\psi x)\} . = .$

$$[(\imath x)(\psi x), (\imath x)(\phi x)] . f\{(\imath x)(\phi x), (\imath x)(\psi x)\} \quad \text{Df}$$

***14·1.** $\vdash :. [(\iota x)(\phi x)] . \psi(\iota x)(\phi x) . \equiv : (\exists b) : \phi x . \equiv_x . x = b : \psi b$ [*4·2 . (*14·01)]

In virtue of our conventions as to the scope intended when no scope is explicitly indicated, the above proposition is the same as the following:

***14·101.** $\vdash :. \psi(\iota x)(\phi x) . \equiv : (\exists b) : \phi x . \equiv_x . x = b : \psi b$ [*14·1]

***14·11.** $\vdash :. E!(\iota x)(\phi x) . \equiv : (\exists b) : \phi x . \equiv_x . x = b$ [*4·2 . (*14·02)]

***14·111.** $\vdash :. [(\iota x)(\psi x)] . f\{(\iota x)(\phi x), (\iota x)(\psi x)\} . \equiv :$

$(\exists b, c) : \phi x . \equiv_x . x = b : \psi x . \equiv_x . x = c : f(b, c)$

Dem.

$\vdash . *4{\cdot}2 . (*14{\cdot}04{\cdot}03) . \supset$

$\vdash :: [(\iota x)(\psi x)] . f\{(\iota x)(\phi x), (\iota x)(\psi x)\} . \equiv :.$

$[(\iota x)(\psi x)] : [(\iota x)(\phi x)] . f\{(\iota x)(\phi x), (\iota x)(\psi x)\} :.$

[*14·1] $\equiv :. [(\iota x)(\psi x)] :. (\exists b) : \phi x . \equiv_x . x = b : f\{b, (\iota x)(\psi x)\} :.$

[*14·1] $\equiv :. (\exists c) :. \psi x . \equiv_x . x = c :. (\exists b) : \phi x . \equiv_x . x = b : f(b, c) :.$

[*11·55] $\equiv :. (\exists b, c) : \phi x . \equiv_x . x = c : \psi x . \equiv_x . x = b : f(b, c) :: \supset \vdash .$ Prop

***14·112.** $\vdash :. f\{(\iota x)(\phi x), (\iota x)(\psi x)\} . \equiv :$

$(\exists b, c) : \phi x . \equiv_x . x = b : \psi x . \equiv_x . x = c : f(b, c)$

[Proof as in *14·111]

In the above proposition, we assume the convention explained on p. 174, after the statement of *14·03.

***14·113.** $\vdash : [(\iota x)(\psi x)] . f\{(\iota x)(\phi x), (\iota x)(\psi x)\} . \equiv . f\{(\iota x)(\phi x), (\iota x)(\psi x)\}$

[*14·111·112]

This proposition shows that when two descriptions occur in the same proposition, the truth-value of the proposition is unaffected by the question which has the larger scope.

***14·12.** $\vdash :. E!(\iota x)(\phi x) . \supset : \phi x . \phi y . \supset_{x,y} . x = y$

Dem.

$\vdash . *14{\cdot}11$ $\supset \vdash :. \text{Hp} . \supset : (\exists b) : \phi x . \equiv_x . x = b$ (1)

$\vdash . *4{\cdot}38 . *10{\cdot}1 . *11{\cdot}11{\cdot}3 . \supset$

$\vdash :. \phi x . \equiv_x . x = b : \supset : \phi x . \phi y . \equiv_{x,y} . x = b . y = b .$

[*13·172] $\supset_{x,y} . x = y$ (2)

$\vdash . (2) . *10{\cdot}11{\cdot}23 .$ $\supset \vdash :. (\exists b) : \phi x . \equiv_x . x = b : \supset : \phi x . \phi y . \supset_{x,y} . x = y$ (3)

$\vdash . (1) . (3) .$ $\supset \vdash .$ Prop

***14·121.** $\vdash :. \phi x . \equiv_x . x = b : \phi x . \equiv_x . x = c : \supset . b = c$

Dem.

$\vdash . *10{\cdot}1 . \supset \vdash :. \text{Hp} . \supset : \phi b . \equiv . b = b : \phi b . \equiv . b = c :$

[*13·15] $\supset : \phi b : \phi b . \equiv . b = c :$

[Ass] $\supset : b = c :. \supset \vdash .$ Prop

***14·122.** $\vdash :. \phi x . \equiv_x . x = b : \equiv : \phi x . \supset_x . x = b : \phi b :$

$\equiv : \phi x . \supset_x . x = b : (\exists x) . \phi x$

Dem.

$\vdash . *10{\cdot}22 .$ $\supset \vdash :. \phi x . \equiv_x . x = b : \equiv : \phi x . \supset_x . x = b : x = b . \supset_x . \phi x :$

[*13·191] $\equiv : \phi x . \supset_x . x = b : \phi b$ (1)

$\vdash . *4{\cdot}71 .$ $\supset \vdash :. \phi x . \supset . x = b : \supset : \phi x . \equiv . \phi x . x = b :.$

[*10·11·27] $\supset \vdash :. \phi x . \supset_x . x = b : \supset : \phi x . \equiv_x . \phi x . x = b :$

[*10·281] $\supset : (\exists x) . \phi x . \equiv . (\exists x) . \phi x . x = b .$

[*13·195] $\equiv . \phi b$ (2)

$\vdash . (2) . *5{\cdot}32 . \supset \vdash :. \phi x . \supset_x . x = b : (\exists x) . \phi x : \equiv : \phi x . \supset_x . x = b : \phi b$ (3)

$\vdash . (1) . (3) .$ $\supset \vdash .$ Prop

The two following propositions (*14·123·124) are placed here because of the analogy with *14·122, but they are not used until we come to the theory of couples (*55 and *56).

***14·123.** $\vdash :. \phi(z, w) . \equiv_{z,w} . z = x . w = y :$

$\equiv : \phi(z, w) . \supset_{z,w} . z = x . w = y : \phi(x, y) :$

$\equiv : \phi(z, w) . \supset_{z,w} . z = x . w = y : (\exists z, w) . \phi(z, w)$

Dem.

$\vdash . *11{\cdot}31 .$ $\supset \vdash :. \phi(z, w) . \equiv_{z,w} . z = x . w = y :$

$\equiv : \phi(z, w) . \supset_{z,w} . z = x . w = y : z = x . w = y . \supset_{z,w} . \phi(z, w) :$

[*13·21] $\equiv : \phi(z, w) . \supset_{z,w} . z = x . w = y : \phi(x, y)$ (1)

$\vdash . *4{\cdot}71 .$ $\supset \vdash :. \phi(z, w) . \supset . z = x . w = y :$

$\supset : \phi(z, w) . \equiv . \phi(z, w) . z = x . w = y :.$

[*11·11·32] $\supset \vdash :. \phi(z, w) . \supset_{z,w} . z = x . w = y :$

$\supset : \phi(z, w) . \equiv_{z,w} . \phi(z, w) . z = x . w = y :$

[*11·341] $\supset : (\exists z, w) . \phi(z, w) . \equiv . (\exists z, w) . \phi(z, w) . z = x . w = y .$

[*13·22] $\equiv . \phi(x, y)$ (2)

$\vdash . (2) . *5{\cdot}32 . \supset \vdash :. \phi(z, w) . \supset_{z,w} . z = x . w = y : (\exists z, w) . \phi(z, w) :$

$\equiv : \phi(z, w) . \supset_{z,w} . z = x . w = y : \phi(x, y)$ (3)

$\vdash . (1) . (3) .$ $\supset \vdash .$ Prop

***14·124.** $\vdash :. (\exists x, y) : \phi(z, w) . \equiv_{z,w} . z = x . w = y :$

$\equiv : (\exists x, y) . \phi(x, y) : \phi(z, w) . \phi(u, v) . \supset_{z,w,u,v} . z = u . w = v$

Dem.

$\vdash . *14{\cdot}123 . *3{\cdot}27 . \supset$

$\vdash :. (\exists x, y) : \phi(z, w) . \equiv_{z,w} . z = x . w = y : \supset . (\exists x, y) . \phi(x, y)$ (1)

$\vdash . *11{\cdot}1 . *3{\cdot}47 . \supset \vdash :. \phi(z, w) . \equiv_{z,w} . z = x . w = y :$

$\supset : \phi(z, w) . \phi(u, v) . \supset . z = x . w = y . u = x . v = y .$

[*13·172] $\supset . z = u . w = v$ (2)

$\vdash . (2) . *11{\cdot}11{\cdot}35 . \supset$

$\vdash :. (\exists x, y) : \phi(z, w) . \equiv_{z,w} . z = x . w = y :$

$\supset : \phi(z, w) . \phi(u, v) . \supset . z = u . w = v$ (3)

$\vdash . (3) . $ *11·11·3 $. \supset$

$\vdash :. (\exists x, y) : \phi(z, w) . \equiv_{z,w} . z = x . w = y :$

$\supset : \phi(z, w) . \phi(u, v) . \supset_{z,w,u,v} . z = u . w = v$ (4)

$\vdash . $ *11·1 $. \supset \vdash :. \phi(x, y) : \phi(z, w) . \phi(u, v) . \supset_{z,w,u,v} . z = u . w = v :$

$\supset : \phi(x, y) : \phi(z, w) . \phi(x, y) . \supset_{z,w} . z = x . w = y :$

[*5·33] $\supset : \phi(x, y) : \phi(z, w) . \supset_{z,w} . z = x . w = y :$

[*14·123] $\supset : \phi(z, w) . \equiv_{z,w} . z = x . w = y$ (5)

$\vdash . (5) . $ *11·11·34·45 $. \supset$

$\vdash :. (\exists x, y) . \phi(x, y) : \phi(z, w) . \phi(u, v) . \supset_{z,w,u,v} . z = u . w = v :$

$\supset : (\exists x, y) : \phi(z, w) . \equiv_{z,w} . z = x . w = y$ (6)

$\vdash . (1) . (4) . (6) . \supset \vdash . \mathrm{Prop}$

***14·13.** $\vdash : a = (\iota x)(\phi x) . \equiv . (\iota x)(\phi x) = a$

Dem.

$\vdash . $ *14·1 . $\supset \vdash :. a = (\iota x)(\phi x) . \equiv : (\exists b) : \phi x . \equiv_x . x = b : a = b$ (1)

$\vdash . $ *13·16 . *4·36 . $\supset \vdash :. \phi x . \equiv_x . x = b : a = b : \equiv : \phi x . \equiv_x . x = b : b = a :$

[*10·11·281] $\supset \vdash :. (\exists b) : \phi x . \equiv_x . x = b : a = b :$

$\equiv : (\exists b) : \phi x . \equiv_x . x = b : b = a :$

[*14·1] $\equiv : (\iota x)(\phi x) = a$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

This proposition is not an *immediate* consequence of *13·16, because "$a = (\iota x)(\phi x)$" is not a value of the function "$x = y$." Similar remarks apply to the following propositions.

***14·131.** $\vdash : (\iota x)(\phi x) = (\iota x)(\psi x) . \equiv . (\iota x)(\psi x) = (\iota x)(\phi x)$

Dem.

$\vdash . $ *14·1 . $\supset \vdash :: (\iota x)(\phi x) = (\iota x)(\psi x) . \equiv :. (\exists b) : \phi x . \equiv_x . x = b : b = (\iota x)(\psi x) :.$

[*14·1] $\equiv :. (\exists b) :. \phi x . \equiv_x . x = b :. (\exists c) : \psi x . \equiv_x . x = c : b = c :.$

[*11·6] $\equiv :. (\exists c) :. \psi x . \equiv_x . x = c :. (\exists b) : \phi x . \equiv_x . x = b : b = c :.$

[*14·1] $\equiv :. (\exists c) :. \psi x . \equiv_x . x = c : (\iota x)(\phi x) = c :.$

[*14·13] $\equiv :. (\exists c) :. \psi x . \equiv_x . x = c : c = (\iota x)(\phi x) :.$

[*14·1] $\equiv :. (\iota x)(\psi x) = (\iota x)(\phi x) :: \supset \vdash . \mathrm{Prop}$

In the above proposition, in accordance with our convention, the descriptive expression $(\iota x)(\phi x)$ is eliminated before $(\iota x)(\psi x)$, because it occurs first in "$(\iota x)(\phi x) = (\iota x)(\psi x)$"; but in "$(\iota x)(\psi x) = (\iota x)(\phi x)$," $(\iota x)(\psi x)$ is to be first eliminated. The order of elimination makes no difference to the truth-value, as was proved in *14·113.

The above proposition may also be proved as follows:

$\vdash . $ *14·111 . $\supset \vdash :. (\iota x)(\phi x) = (\iota x)(\psi x) .$

$\equiv : (\exists b, c) : \phi x . \equiv_x . x = b : \psi x . \equiv_x . x = c : b = c :$

[*4·3.*13·16.*11·11·341] $\equiv : (\exists b, c) : \psi x . \equiv_x . x = c : \phi x . \equiv_x . x = b : c = b :$

[*11·2.*14·111] $\equiv : (\iota x)(\psi x) = (\iota x)(\phi x) :. \supset \vdash . \mathrm{Prop}$

***14·14.** $\vdash : a = b . b = (\iota x)(\phi x) . \supset . a = (\iota x)(\phi x)$ [*13·13]

***14·142.** $\vdash : a = (\iota x)(\phi x) . (\iota x)(\phi x) = (\iota x)(\psi x) . \supset . a = (\iota x)(\psi x)$

Dem.

$\vdash . $ *14·1 . $\supset \vdash :: \mathrm{Hp} . \supset :. (\exists b) : \phi x . \equiv_x . x = b : a = b :.$

$(\exists c) : \psi x . \equiv_x . x = c : c = (\iota x)(\psi x) :.$

[*13·195] $\supset :. \phi x . \equiv_x . x = a :. (\exists c) : \psi x . \equiv_x . x = c : c = (\iota x)(\psi x) :.$

[*10·35] $\supset :. (\exists c) :. \phi x . \equiv_x . x = a : \psi x . \equiv_x . x = c : c = (\iota x)(\psi x) :.$

[*14·121] $\supset :. (\exists c) :. \phi x . \equiv_x . x = a : a = c : c = (\iota x)(\psi x) :.$

[*3·27.*13·195] $\supset :. a = (\iota x)(\psi x) :: \supset \vdash . \mathrm{Prop}$

***14·144.** $\vdash : (\iota x)(\phi x) = (\iota x)(\psi x) . (\iota x)(\psi x) = (\iota x)(\chi x) . \supset . (\iota x)(\phi x) = (\iota x)(\chi x)$

Dem.

$\vdash . $ *14·111 . $\supset \vdash :: \mathrm{Hp} . \supset :. (\exists a, b) : \phi x . \equiv_x . x = a : \psi x . \equiv_x . x = b : a = b :.$

$(\exists c, d) : \psi x . \equiv_x . x = c : \chi x . \equiv_x . x = d : c = d :.$

[*13·195] $\supset :. (\exists a) : \phi x . \equiv_x . x = a : \psi x . \equiv_x . x = a :.$

$(\exists c) : \psi x . \equiv_x . x = c : \chi x . \equiv_x . x = c :.$

[*11·54] $\supset :. (\exists a, c) : \phi x . \equiv_x . x = a : \psi x . \equiv_x . x = a :$

$\psi x . \equiv_x . x = c : \chi x . \equiv_x . x = c :.$

[*14·121.*11·42] $\supset :. (\exists a, c) : \phi x . \equiv_x . x = a : \chi x . \equiv_x . x = c : a = c :.$

[*14·111] $\supset :. (\iota x)(\phi x) = (\iota x)(\chi x) :: \supset \vdash . \mathrm{Prop}$

***14·145.** $\vdash : a = (\iota x)(\phi x) . a = (\iota x)(\psi x) . \supset . (\iota x)(\phi x) = (\iota x)(\psi x)$

Dem.

$\vdash . $ *14·1 . $\supset \vdash :. a = (\iota x)(\phi x) . \equiv : (\exists b) : \phi x . \equiv_x . x = b : a = b :$

[*13·195] $\equiv : \phi x . \equiv_x . x = a$ (1)

$\vdash . (1) . $ *14·1 . $\supset \vdash :: \mathrm{Hp} . \equiv :. \phi x . \equiv_x . x = a :. (\exists b) : \psi x . \equiv_x . x = b : a = b :.$

[*10·35] $\equiv :. (\exists b) :. \phi x . \equiv_x . x = a : \psi x . \equiv_x . x = b : a = b :.$

[*14·111] $\supset :. (\iota x)(\phi x) = (\iota x)(\psi x) :: \supset \vdash . \mathrm{Prop}$

***14·15.** $\vdash :. (\iota x)(\phi x) = b . \supset : \psi\{(\iota x)(\phi x)\} . \equiv . \psi b$

Dem.

$\vdash . $ *14·1 . $\supset$

$\vdash :: \mathrm{Hp} . \supset :. (\exists c) : \phi x . \equiv_x . x = c : c = b :.$

[*13·195] $\supset :. \phi x . \equiv_x . x = b$ (1)

$\vdash . (1) . $ *14·1 . $\supset$

$\vdash :: \mathrm{Hp} . \supset :. \psi\{(\iota x)(\phi x)\} . \equiv : (\exists c) : x = b . \equiv_x . x = c : \psi c :$

[*13·192] $\equiv : \psi b :: \supset \vdash . \mathrm{Prop}$

***14·16.** $\vdash :. (\iota x)(\phi x) = (\iota x)(\psi x) . \supset : \chi\{(\iota x)(\phi x)\} . \equiv . \chi\{(\iota x)(\psi x)\}$

Dem.

$\vdash . $ *14·1 . $\supset \vdash :. \mathrm{Hp} . \supset : (\exists b) : \phi x . \equiv_x . x = b : b = (\iota x)(\psi x)$ (1)

$\vdash . $ *14·1 . $\supset \vdash :: \phi x . \equiv_x . x = b : \supset :.$

$\chi\{(\iota x)(\phi x)\} . \equiv : (\exists c) : x = b . \equiv_x . x = c : \chi c :$

[*13·192] $\equiv : \chi b$ (2)

$\vdash . *14\cdot13\cdot15 . \supset \vdash :. b = (\iota x)(\psi x) . \supset : \chi b . \equiv . \chi\{(\iota x)(\psi x)\}$ (3)

$\vdash . (2) . (3) . \quad \supset \vdash :. \phi x . \equiv_x . x = b : b = (\iota x)(\psi x) :$

$\supset : \chi\{(\iota x)(\phi x)\} . \equiv . \chi\{(\iota x)(\psi x)\}$ (4)

$\vdash . (1) . (4) . *10\cdot1\cdot23 . \supset \vdash . \text{Prop}$

∗14·17. $\vdash :. (\iota x)(\phi x) = b . \equiv : \psi ! (\iota x)(\phi x) . \equiv_\psi . \psi ! b$

Dem.

$\vdash . *14\cdot15 . *10\cdot11\cdot21 . \supset$

$\vdash :. (\iota x)(\phi x) = b . \supset : \psi ! (\iota x)(\phi x) . \equiv_\psi . \psi ! b$ (1)

$\vdash . *10\cdot1 . *4\cdot22 . \supset \vdash :: \chi ! x . \equiv_x . x = b : \psi ! (\iota x)(\phi x) . \equiv_\psi . \psi ! b :$

$\supset : (\iota x)(\phi x) = b . \equiv . b = b :$

[∗13·15] $\supset : (\iota x)(\phi x) = b$ (2)

$\vdash . (2) . \text{Exp} . *10\cdot11\cdot23 . \supset$

$\vdash :: (\exists \chi) : \chi ! x . \equiv_x . x = b : \supset :. \psi ! (\iota x)(\phi x) . \equiv_\psi . \psi ! b : \supset . (\iota x)(\phi x) = b$ (3)

$\vdash . *12\cdot1 . \quad \supset \vdash : (\exists \chi) : \chi ! x . \equiv_x . x = b$ (4)

$\vdash . (3) . (4) . \supset \vdash :. \psi ! (\iota x)(\phi x) . \equiv_\psi . \psi ! b : \supset . (\iota x)(\phi x) = b$ (5)

$\vdash . (1) . (5) . \supset \vdash . \text{Prop}$

It should be observed that we do *not* have

$$(\iota x)(\phi x) = b . \equiv : \psi ! (\iota x)(\phi x) . \supset_\psi . \psi ! b$$

for, if $\sim E ! (\iota x)(\phi x)$, $\psi ! (\iota x)(\phi x)$ is always false, and therefore

$$\psi ! (\iota x)(\phi x) . \supset_\psi . \psi ! b$$

holds for all values of b. But we do have

∗14·171. $\vdash :. (\iota x)(\phi x) = b . \equiv : \psi ! b . \supset_\psi . \psi ! (\iota x)(\phi x)$

Dem.

$\vdash . *14\cdot17 . \quad \supset \vdash :. (\iota x)(\phi x) = b . \supset : \psi ! b . \supset_\psi . \psi ! (\iota x)(\phi x)$ (1)

$\vdash . *10\cdot1 . *12\cdot1 . \supset \vdash :. \psi ! b . \supset_\psi . \psi ! (\iota x)(\phi x) : \supset : b = b . \supset . (\iota x)(\phi x) = b :$

[∗13·15] $\supset : (\iota x)(\phi x) = b$ (2)

$\vdash . (1) . (2) . \quad \supset \vdash . \text{Prop}$

∗14·18. $\vdash :. E ! (\iota x)(\phi x) . \supset : (x) . \psi x . \supset . \psi (\iota x)(\phi x)$

Dem.

$\vdash . *10\cdot1 . \quad \supset \vdash : (x) . \psi x . \supset . \psi b :$

[Fact] $\supset \vdash :. \phi x . \equiv_x . x = b : (x) . \psi x : \supset : \phi x . \equiv_x . x = b : \psi b :$

[∗10·11·28] $\supset \vdash :. (\exists b) : \phi x . \equiv_x . x = b : (x) . \psi x : \supset : (\exists b) : \phi x . \equiv_x . x = b : \psi b :.$

[∗10·35] $\supset \vdash :: (\exists b) : \phi x . \equiv_x . x = b :. (x) . \psi x :. \supset : (\exists b) : \phi x . \equiv_x . x = b : \psi b :.$

[∗14·1·11] $\supset \vdash :. E ! (\iota x)(\phi x) : (x) . \psi x : \supset : \psi (\iota x)(\phi x) :. \supset \vdash . \text{Prop}$

The above proposition shows that, provided $(\iota x)(\phi x)$ exists, it has (speaking formally) all the logical properties of symbols which directly represent objects. Hence when $(\iota x)(\phi x)$ exists, the fact that it is an incomplete symbol becomes irrelevant to the truth-values of logical propositions in which it occurs.

∗14·2. $\vdash . (\iota x)(x = a) = a$

Dem.

$\vdash . *14\cdot101 . \supset \vdash :. (\iota x)(x = a) = a . \equiv : (\exists b) : x = a . \equiv_x . x = b : b = a :$

[∗13·195] $\equiv : x = a . \equiv_x . x = a$ (1)

$\vdash . (1) . \text{Id} . \quad \supset \vdash . \text{Prop}$

∗14·201. $\vdash : E ! (\iota x)(\phi x) . \supset . (\exists x) . \phi x$

Dem.

$\vdash . *14\cdot11 . \supset \vdash :. \text{Hp} . \supset : (\exists b) : \phi x . \equiv_x . x = b :$

[∗10·1] $\supset : (\exists b) : \phi b . \equiv . b = b :$

[∗13·15] $\supset : (\exists b) . \phi b :. \supset \vdash . \text{Prop}$

∗14·202. $\vdash :. \phi x . \equiv_x . x = b : \equiv : (\iota x)(\phi x) = b : \equiv : \phi x . \equiv_x . b = x : \equiv : b = (\iota x)(\phi x)$

Dem.

$\vdash . *14\cdot1 . \supset \vdash :. (\iota x)(\phi x) = b . \equiv : (\exists c) : \phi x . \equiv_x . x = c : c = b :$

[∗13·195] $\equiv : \phi x . \equiv_x . x = b :. \supset \vdash . \text{Prop}$

[The second half is proved in the same way as the first half.]

∗14·203. $\vdash :. E ! (\iota x)(\phi x) . \equiv : (\exists x) . \phi x : \phi x . \phi y . \supset_{x,y} . x = y$

Dem.

$\vdash . *14\cdot12\cdot201 . \quad \supset \vdash :. E ! (\iota x)(\phi x) . \supset : (\exists x) . \phi x : \phi x . \phi y . \supset_{x,y} . x = y$ (1)

$\vdash . *10\cdot1 . \quad \supset \vdash :. \phi b : \phi x . \phi y . \supset_{x,y} . x = y : \supset : \phi b : \phi x . \phi b . \supset_x . x = b :$

[∗5·33] $\supset : \phi b : \phi x . \supset_x . x = b :$

[∗13·191] $\supset : x = b . \supset_x . \phi x :$

$\phi x . \supset_x . x = b :$

[∗10·22] $\supset : \phi x . \equiv_x . x = b$ (2)

$\vdash . (2) . *10\cdot1\cdot28 . \supset \vdash :. (\exists b) : \phi b : \phi x . \phi y . \supset_{x,y} . x = y : \supset : (\exists b) : \phi x . \equiv_x . x = b :.$

[∗10·35] $\supset \vdash :. (\exists b) . \phi b : \phi x . \phi y . \supset_{x,y} . x = y : \supset : (\exists b) : \phi x . \equiv_x . x = b :$

[∗14·11] $\supset : E ! (\iota x)(\phi x)$ (3)

$\vdash . (1) . (3) . \quad \supset \vdash . \text{Prop}$

∗14·204. $\vdash :. E ! (\iota x)(\phi x) . \equiv : (\exists b) . (\iota x)(\phi x) = b$

Dem.

$\vdash . *14\cdot202 . *10\cdot11 . \supset$

$\vdash :. (b) :. \phi x . \equiv_x . x = b : \equiv : (\iota x)(\phi x) = b :. \supset$

[∗10·281] $\vdash :. (\exists b) : \phi x . \equiv_x . x = b : \equiv : (\exists b) . (\iota x)(\phi x) = b$ (1)

$\vdash . (1) . *14\cdot11 . \supset \vdash . \text{Prop}$

∗14·205. $\vdash : \psi (\iota x)(\phi x) . \equiv . (\exists b) . b = (\iota x)(\phi x) . \psi b$ [∗14·202·1]

∗14·21. $\vdash : \psi (\iota x)(\phi x) . \supset . E ! (\iota x)(\phi x)$

Dem.

$\vdash . *14\cdot1 . \supset$

$\vdash :. \psi \{(\iota x)(\phi x)\} . \supset : (\exists b) : \phi x . \equiv_x . x = b : \psi b :$

[∗10·5] $\supset : (\exists b) : \phi x . \equiv_x . x = b :$

[∗14·11] $\supset : E ! (\iota x)(\phi x) :. \supset \vdash . \text{Prop}$

This proposition shows that if any true statement can be made about $(\iota x)(\phi x)$, then $(\iota x)(\phi x)$ must exist. Its use throughout the remainder of the work will be very frequent.

When $(\iota x)(\phi x)$ does not exist, there are still true propositions in which "$(\iota x)(\phi x)$" occurs, but it has, in such propositions, a *secondary* occurrence, in the sense explained in Chapter III of the Introduction, *i.e.* the asserted proposition concerned is not of the form $\psi(\iota x)(\phi x)$, but of the form $f\{\psi(\iota x)(\phi x)\}$, in other words, the proposition which is the scope of $(\iota x)(\phi x)$ is only part of the whole asserted proposition.

***14·22.** $\vdash : E\,!(\iota x)(\phi x) . \equiv . \phi(\iota x)(\phi x)$

Dem.

$\vdash . *14·122 . \quad \supset \vdash :. \phi x . \equiv_x . x = b : \supset . \phi b \qquad (1)$

$\vdash . (1) . *4·71 . \supset \vdash :. \phi x . \equiv_x . x = b : \equiv : \phi x . \equiv_x . x = b : \phi b :.$

$[*10·11·281] \quad \supset \vdash :. (\exists b) : \phi x . \equiv_x . x = b : \equiv : (\exists b) : \phi x . \equiv_x . x = b : \phi b :.$

$[*14·11·101] \quad \supset \vdash : E\,!(\iota x)(\phi x) . \equiv . \phi(\iota x)(\phi x) : \supset \vdash . \text{Prop}$

As an instance of the above proposition, we may take the following: "The proposition 'the author of Waverley existed' is equivalent to 'the man who wrote Waverley wrote Waverley.'" Thus such a proposition as "the man who wrote Waverley wrote Waverley" does not embody a logically necessary truth, since it would be false if Waverley had not been written, or had been written by two men in collaboration. For example, "the man who squared the circle squared the circle" is a false proposition.

***14·23.** $\vdash : E\,!(\iota x)(\phi x . \psi x) . \equiv . \phi\{(\iota x)(\phi x . \psi x)\}$

Dem.

$\vdash . *14·22 . \supset \vdash :. E\,!(\iota x)(\phi x . \psi x) .$

$\equiv : [(\iota x)(\phi x . \psi x)] : \phi\{(\iota x)(\phi x . \psi x)\} . \psi\{(\iota x)(\phi x . \psi x)\}$

$[*10·5 . *3·26] \quad \supset : \phi\{(\iota x)(\phi x . \psi x)\} \qquad (1)$

$\vdash . *14·21 . \supset \vdash : \phi\{(\iota x)(\phi x . \psi x)\} . \supset . E\,!(\iota x)(\phi x . \psi x) \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

Note that in the second line of the above proof $*10·5$, not only $*3·26$, is required. For the scope of the descriptive symbol $(\iota x)(\phi x . \psi x)$ is the whole product $\phi\{(\iota x)(\phi x . \psi x)\} . \psi\{(\iota x)(\phi x . \psi x)\}$, so that, applying $*14·1$, the proposition on the right in the first line becomes

$$(\exists b) : \phi x . \psi x . \equiv_x . x = b : \phi b . \psi b$$

which, by $*10·5$ and $*3·26$, implies

$$(\exists b) : \phi x . \psi x . \equiv_x . x = b : \phi b,$$

i.e. $\qquad \phi\{(\iota x)(\phi x . \psi x)\}.$

***14·24.** $\vdash :. E\,!(\iota x)(\phi x) . \equiv : [(\iota x)(\phi x)] : \phi y . \equiv_y . y = (\iota x)(\phi x)$

Dem.

$\vdash . *14·1 . \supset \vdash :. [(\iota x)(\phi x)] : \phi y . \equiv_y . y = (\iota x)(\phi x) :$

$\equiv : (\exists b) : \phi y . \equiv_y . y = b : \phi y . \equiv_y . y = b :$

$[*4·24 . *10·281] \quad \equiv : (\exists b) : \phi y . \equiv_y . y = b :$

$[*14·11] \quad \equiv : E\,!(\iota x)(\phi x) :. \supset \vdash . \text{Prop}$

This proposition should be compared with $*14·241$, where, in virtue of the smaller scope of $(\iota x)(\phi x)$, we get an implication instead of an equivalence.

***14·241.** $\vdash :. E\,!(\iota x)(\phi x) . \supset : \phi y . \equiv_y . y = (\iota x)(\phi x)$

Dem.

$\vdash . *14·203 . \supset \vdash :: \text{Hp} . \supset :. \phi y . \phi x . \supset . y = x :.$

$[\text{Exp}] \quad \supset :. \phi y . \supset : \phi x . \supset . y = x ::$

$[*10·11·21] \quad \supset \vdash :: \text{Hp} . \supset :. \phi y . \supset : \phi x . \supset_x . y = x :.$

$[*4·71] \quad \supset :. \phi y . \equiv : \phi y : \phi x . \supset_x . y = x :$

$[*13·191] \quad \equiv : y = x . \supset_x . \phi x : \phi x . \supset_x . y = x :$

$[*10·22] \quad \equiv : \phi x . \equiv_x . y = x :$

$[*14·202] \quad \equiv : y = (\iota x)(\phi x) :: \supset \vdash . \text{Prop}$

***14·242.** $\vdash :. \phi x . \equiv_x . x = b : \supset : \psi b . \equiv . \psi(\iota x)(\phi x) \quad [*14·202·15]$

***14·25.** $\vdash :. E\,!(\iota x)(\phi x) . \supset : \phi x \supset_x \psi x . \equiv . \psi(\iota x)(\phi x)$

Dem.

$\vdash . *4·84 . *10·27·271 . \supset \vdash :: \phi x . \equiv_x . x = b : \supset :. \phi x \supset_x \psi x . \equiv : x = b . \supset_x . \psi x :$

$[*13·191] \quad \equiv : \psi b :$

$[*14·242] \quad \equiv . \psi(\iota x)(\phi x) \qquad (1)$

$\vdash . (1) . *10·11·23 . \supset \vdash :. (\exists b) : \phi x . \equiv_x . x = b :$

$\supset : \phi x \supset_x \psi x . \equiv . \psi(\iota x)(\phi x) \qquad (2)$

$\vdash . (2) . *14·11 . \quad \supset \vdash . \text{Prop}$

***14·26.** $\vdash :. E\,!(\iota x)(\phi x) . \supset : (\exists x) . \phi x . \psi x . \equiv . \psi\{(\iota x)(\phi x)\} . \equiv . \phi x \supset_x \psi x$

Dem.

$\vdash . *14·11 . \supset$

$\vdash :. \text{Hp} . \supset : (\exists b) : \phi x . \equiv_x . x = b \qquad (1)$

$\vdash . *10·311 . \supset \vdash :: \phi x . \equiv_x . x = b : \supset :. \phi x . \psi x . \equiv_x . x = b . \psi x :.$

$[*10·281] \quad \supset :. (\exists x) . \phi x . \psi x . \equiv . (\exists x) . x = b . \psi x .$

$[*13·195] \quad \equiv . \psi b .$

$[*14·242] \quad \equiv . \psi\{(\iota x)(\phi x)\} \qquad (2)$

$\vdash . (2) . *10·11·23 . \supset$

$\vdash :. (\exists b) : \phi x . \equiv_x . x = b : \supset : (\exists x) . \phi x . \psi x . \equiv . \psi\{(\iota x)(\phi x)\} \qquad (3)$

$\vdash . (1) . (3) . *14·25 . \supset \vdash . \text{Prop}$

***14·27.** $\vdash :. E\,!(\iota x)(\phi x) . \supset : \phi x \equiv_x \psi x . \equiv . (\iota x)(\phi x) = (\iota x)(\psi x)$

Dem.

$\vdash . *4·86·21 . \quad \supset \vdash :: \phi x . \equiv \; . x = b : \supset :. \phi x . \equiv . \psi x : \equiv : \psi x . \equiv . x = b \quad (1)$

$\vdash . (1) . *10·11·27 . \supset \vdash :: \phi x . \equiv_x . x = b : \supset :. (x) :. \phi x . \equiv . \psi x : \equiv : \psi x . \equiv . x = b :.$

$[*10·271] \quad \supset :. \phi x . \equiv_x . \psi x : \equiv : \psi x . \equiv_x . x = b :$

$[*14·202] \quad \equiv : b = (\iota x)(\psi x) :$

$[*14·242] \quad \equiv : (\iota x)(\phi x) = (\iota x)(\psi x) \qquad (2)$

$\vdash . (2) . *10·11·23 . *14·11 . \supset \vdash . \text{Prop}$

***14·271.** $\vdash :. \phi x . \equiv_x . \psi x : \supset : E!(\imath x)(\phi x) . \equiv . E!(\imath x)(\psi x)$

Dem.

$\vdash . *4\cdot86 . \quad \supset \vdash :: \phi x \equiv \psi x . \supset :. \phi x . \equiv . x = b : \equiv : \psi x . \equiv . x = b ::$

$[*10\cdot11\cdot27] \supset \vdash :: \mathrm{Hp} . \quad \supset :. (x) :. \phi x . \equiv . x = b : \equiv : \psi x . \equiv . x = b :.$

$[*10\cdot271] \quad \supset :. (x) : \phi x . \equiv . x = b : \equiv : (x) : \psi x . \equiv . x = b ::$

$[*10\cdot11\cdot21] \supset \vdash :: \mathrm{Hp} . \quad \supset :. (b) :. \phi x . \equiv_x . x = b : \equiv : \psi x . \equiv_x . x = b :.$

$[*10\cdot281] \quad \supset :. (\exists b) : \phi x . \equiv_x . x = b : \equiv : (\exists b) : \psi x . \equiv_x . x = b ::$

$\supset \vdash . \mathrm{Prop}$

***14·272.** $\vdash :. \phi x . \equiv_x . \psi x : \supset : \chi(\imath x)(\phi x) . \equiv . \chi(\imath x)(\psi x)$

Dem.

$\vdash . *4\cdot86 . \quad \supset \vdash :: \phi x \equiv \psi x . \supset :. \phi x . \equiv . x = b : \equiv : \psi x . \equiv . x = b :.$

$[*10\cdot11\cdot414] \supset \vdash :: \mathrm{Hp} . \quad \supset :. \phi x . \equiv_x . x = b : \equiv : \psi x . \equiv_x . x = b :.$

$[\mathrm{Fact}] \quad \supset :. \phi x . \equiv_x . x = b : \chi b : \equiv : \psi x . \equiv_x . x = b : \chi b :.$

$[*10\cdot11\cdot21] \supset \vdash :: \mathrm{Hp} . \quad \supset :. (b) :. \phi x . \equiv_x . x = b : \chi b : \equiv : \psi x . \equiv_x . x = b : \chi b :.$

$[*10\cdot281] \quad \supset :. (\exists b) : \phi x . \equiv_x . x = b : \chi b : \equiv$

$: (\exists b) : \psi x . \equiv_x . x = b : \chi b :.$

$[*14\cdot101] \quad \supset :. \chi(\imath x)(\phi x) . \equiv . \chi(\imath x)(\psi x) :: \supset \vdash . \mathrm{Prop}$

The above two propositions show that $E!(\imath x)(\phi x)$ and $\chi(\imath x)(\phi x)$ are "extensional" properties of $\phi \hat{x}$, *i.e.* their truth-value is unchanged by the substitution, for $\phi \hat{x}$, of any formally equivalent function $\psi \hat{x}$.

***14·28.** $\vdash : E!(\imath x)(\phi x) . \equiv . (\imath x)(\phi x) = (\imath x)(\phi x)$

Dem.

$\vdash . *13\cdot15 . *4\cdot73 . \supset \vdash :. \phi x . \equiv_x . x = b : \equiv : \phi x . \equiv_x . x = b : b = b \quad (1)$

$\vdash . (1) . *10\cdot11\cdot281 . \supset$

$\vdash :. (\exists b) : \phi x . \equiv_x . x = b : \equiv : (\exists b) : \phi x . \equiv_x . x = b : b = b \quad (2)$

$\vdash . (2) . *14\cdot1\cdot11 . \supset \vdash . \mathrm{Prop}$

This proposition states that $(\imath x)(\phi x)$ is identical with itself whenever it exists, but not otherwise. Thus for example the proposition "the present King of France is the present King of France" is false.

The purpose of the following propositions is to show that, when $E!(\imath x)(\phi x)$, the scope of $(\imath x)(\phi x)$ does not matter to the truth-value of any proposition in which $(\imath x)(\phi x)$ occurs. This proposition cannot be proved generally, but it can be proved in each particular case. The following propositions show the method, which proceeds always by means of *14·242, *10·23 and *14·11. The proposition can be proved generally when $(\imath x)(\phi x)$ occurs in the form $\chi(\imath x)(\phi x)$, and $\chi(\imath x)(\phi x)$ occurs in what we may call a "truth-function," *i.e.* a function whose truth or falsehood depends only upon the truth or falsehood of its argument or arguments. This covers all the cases with which we are ever concerned. That is to say, if $\chi(\imath x)(\phi x)$ occurs in any of the ways which can be generated by the processes of *1—*11, then, provided $E!(\imath x)(\phi x)$, the truth-value of $f\{[(\imath x)(\phi x)] . \chi(\imath x)(\phi x)\}$ is the same as that of

$$[(\imath x)(\phi x)] . f\{\chi(\imath x)(\phi x)\}.$$

This is proved in the following proposition. In this proposition, however, the use of propositions as apparent variables involves an apparatus not required elsewhere, and we have therefore not used this proposition in subsequent proofs.

***14·3.** $\vdash :. p \equiv q . \supset_{p,q} . f(p) \equiv f(q) : E!(\imath x)(\phi x) : \supset :$

$f\{[(\imath x)(\phi x)] . \chi(\imath x)(\phi x)\} . \equiv . [(\imath x)(\phi x)] . f\{\chi(\imath x)(\phi x)\}$

Dem.

$\vdash . *14\cdot242 . \supset$

$\vdash :. \phi x . \equiv_x . x = b : \supset : [(\imath x)(\phi x)] . \chi(\imath x)(\phi x) . \equiv . \chi b \quad (1)$

$\vdash . (1) . \supset \vdash :. p \equiv q . \supset_{p,q} . f(p) \equiv f(q) : \phi x . \equiv_x . x = b : \supset :$

$f\{[(\imath x)(\phi x)] . \chi(\imath x)(\phi x)\} . \equiv . f(\chi b) \quad (2)$

$\vdash . *14\cdot242 . \supset$

$\vdash :. \phi x . \equiv_x . x = b : \supset : [(\imath x)(\phi x)] . f\{\chi(\imath x)(\phi x)\} . \equiv . f(\chi b) \quad (3)$

$\vdash . (2) . (3) . \supset$

$\vdash :. p \equiv q . \supset_{p,q} . f(p) \equiv f(q) : \phi x . \equiv_x . x = b : \supset :$

$f\{[(\imath x)(\phi x)] . \chi(\imath x)(\phi x)\} . \equiv . [(\imath x)(\phi x)] . f\{\chi(\imath x)(\phi x)\} \quad (4)$

$\vdash . (4) . *10\cdot23 . *14\cdot11 . \supset \vdash . \mathrm{Prop}$

The following propositions are immediate applications of the above. They are, however, independently proved, because *14·3 introduces propositions (p, q namely) as apparent variables, which we have not done elsewhere, and cannot do legitimately without the explicit introduction of the hierarchy of propositions with a reducibility-axiom such as *12·1.

***14·31.** $\vdash :: E!(\imath x)(\phi x) . \supset :. [(\imath x)(\phi x)] . p \vee \chi(\imath x)(\phi x) .$

$\equiv : p . \vee . [(\imath x)(\phi x)] . \chi(\imath x)(\phi x)$

Dem.

$\vdash . *14\cdot242 . \supset \vdash :. \phi x . \equiv_x . x = b : \supset : [(\imath x)(\phi x)] . p \vee \chi(\imath x)(\phi x) . \equiv . p \vee \chi b \quad (1)$

$\vdash . *14\cdot242 . \supset \vdash :. \phi x . \equiv_x . x = b : \supset : [(\imath x)(\phi x)] . \chi(\imath x)(\phi x) . \equiv . \chi b :$

$[*4\cdot37] \quad \supset : p \vee [(\imath x)(\phi x)] \chi(\imath x)(\phi x) . \equiv . p \vee \chi b \quad (2)$

$\vdash . (1) . (2) . \supset \vdash :. \phi x . \equiv_x . x = b : \supset : [(\imath x)(\phi x)] . p \vee \chi(\imath x)(\phi x) .$

$\equiv . p \vee [(\imath x)(\phi x)] \chi(\imath x)(\phi x) \quad (3)$

$\vdash . (3) . *10\cdot23 . *14\cdot11 . \supset \vdash . \mathrm{Prop}$

The following propositions are proved in precisely the same way as *14·31; hence we shall merely give references to the propositions used in the proofs.

***14·32.** $\vdash :. E!(\imath x)(\phi x) . \equiv : [(\imath x)(\phi x)] . \sim \chi(\imath x)(\phi x) .$

$\equiv . \sim \{[(\imath x)(\phi x)] . \chi(\imath x)(\phi x)\}$

[*14·242 . *4·11 . *10·23 . *14·11]

The equivalence asserted here fails when $\sim E!(\imath x)(\phi x)$. Thus, for example, let ϕy be "y is King of France." Then $(\imath x)(\phi x) =$ the King of France. Let χy be "y is bald." Then $[(\imath x)(\phi x)] . \sim \chi(\imath x)(\phi x) . = .$ the King of France exists and is not bald; but $\sim \{[(\imath x)(\phi x)] . \chi(\imath x)(\phi x)\} . = .$ it is false that the King of France exists and is bald. Of these the first is false, the second true.

Either might be meant by "the King of France is not bald," which is ambiguous; but it would be more natural to take the first (false) interpretation as the meaning of the words. If the King of France existed, the two would be equivalent; thus as applied to the King of England, both are true or both false.

***14·33.** $\vdash :: E\,!\,(\iota x)(\phi x) \,.\supset :. [(\iota x)(\phi x)] \,.\, p \supset \chi(\iota x)(\phi x) \,.$
$\equiv : p \,.\supset .\, [(\iota x)(\phi x)] \,.\, \chi(\iota x)(\phi x)$
[*14·242 . *4·85 . *10·23 . *14·11]

***14·331.** $\vdash :: E\,!\,(\iota x)(\phi x) \,.\supset :. [(\iota x)(\phi x)] \,.\, \chi(\iota x)(\phi x) \supset p \,.$
$\equiv : [(\iota x)(\phi x)] \,.\, \chi(\iota x)(\phi x) \,.\supset .\, p$
[*4·84 . *14·242 . *10·23 . *14·11]

***14·332.** $\vdash :: E\,!\,(\iota x)(\phi x) \,.\supset :. [(\iota x)(\phi x)] \,.\, p \equiv \chi(\iota x)(\phi x) \,.\equiv$
$: p \,.\equiv .\, [(\iota x)(\phi x)] \,.\, \chi(\iota x)(\phi x)$
[*4·86 . *14·242 . *10·23 . *14·11]

***14·34.** $\vdash :. p : [(\iota x)(\phi x)] \,.\, \chi(\iota x)(\phi x) : \equiv : [(\iota x)(\phi x)] : p \,.\, \chi(\iota x)(\phi x)$

This proposition does not require the hypothesis $E\,!\,(\iota x)(\phi x)$.

Dem.

$\vdash .\, *14\cdot1 \,.\supset$

$\vdash :. p : [(\iota x)(\phi x)] \,.\, \chi(\iota x)(\phi x) : \equiv : p : (\exists b) : \phi x \,.\equiv_x .\, x = b : \chi b :$

[*10·35] $\equiv : (\exists b) : p : \phi x \,.\equiv_x .\, x = b : \chi b :$

[*14·1] $\equiv : [(\iota x)(\phi x)] : p \,.\, \chi(\iota x)(\phi x) :. \supset \vdash .$ Prop

Propositions of the above type might be continued indefinitely, but as they are proved on a uniform plan, it is unnecessary to go beyond the fundamental cases of $p \lor q$, $\sim p$, $p \supset q$ and $p \,.\, q$.

It should be observed that the proposition in which $(\iota x)(\phi x)$ has the larger scope always implies the corresponding one in which it has the smaller scope, but the converse implication only holds if either (*a*) we have $E\,!\,(\iota x)(\phi x)$ or (*b*) the proposition in which $(\iota x)(\phi x)$ has the smaller scope implies $E\,!\,(\iota x)(\phi x)$. The second case occurs in *14·34, and is the reason why we get an equivalence without the hypothesis $E\,!\,(\iota x)(\phi x)$. The proposition in which $(\iota x)(\phi x)$ has the larger scope always implies $E\,!\,(\iota x)(\phi x)$, in virtue of *14·21.

SECTION C

CLASSES AND RELATIONS

*20. GENERAL THEORY OF CLASSES

Summary of *20.

The following theory of classes, although it provides a notation to represent them, avoids the assumption that there are such things as classes. This it does by merely defining propositions in whose expression the symbols representing classes occur, just as, in *14, we defined propositions containing descriptions.

The characteristics of a class are that it consists of all the terms satisfying some propositional function, so that every propositional function determines a class, and two functions which are formally equivalent (*i.e.* such that whenever either is true, the other is true also) determine the same class, while conversely two functions which determine the same class are formally equivalent. When two functions are formally equivalent, we shall say that they have the same *extension*. The incomplete symbols which take the place of classes serve the purpose of technically providing something identical in the case of two functions having the same extension; without something to represent classes, we cannot, for example, count the combinations that can be formed out of a given set of objects.

Propositions in which a function ϕ occurs may depend, for their truth-value, upon the particular function ϕ, or they may depend only upon the *extension* of ϕ. In the former case, we will call the proposition concerned an *intensional* function of ϕ; in the latter case, an *extensional* function of ϕ. Thus, for example, $(x) \,.\, \phi x$ or $(\exists x) \,.\, \phi x$ is an extensional function of ϕ, because, if ϕ is formally equivalent to ψ, *i.e.* if $\phi x \,.\equiv_x .\, \psi x$, we have $(x) \,.\, \phi x \,.\equiv .\, (x) \,.\, \psi x$ and $(\exists x) \,.\, \phi x \,.\equiv .\, (\exists x) \,.\, \psi x$. But on the other hand "I believe $(x) \,.\, \phi x$" is an *intensional* function, because, even if $\phi x \,.\equiv_x .\, \psi x$, it by no means follows that I believe $(x) \,.\, \psi x$ provided I believe $(x) \,.\, \phi x$. The mark of an extensional function f of a function $\phi\,!\,\hat{z}$ is

$$\phi\,!\,x \,.\equiv_x .\, \psi\,!\,x : \supset_{\phi,\psi} : f(\phi\,!\,\hat{z}) \,.\equiv .\, f(\psi\,!\,\hat{z}).$$

(We write "$\phi\,!\,\hat{z}$" when we wish to speak of the function itself as opposed to its argument.) The functions of functions with which mathematics is specially concerned are all extensional.

When a function of $\phi\,!\,\hat{z}$ is extensional, it may be regarded as being about the class determined by $\phi\,!\,\hat{z}$, since its truth-value remains unchanged so long as the class is unchanged. Hence we require, for the theory of classes, a method of obtaining an extensional function from any given function of a function. This is effected by the following definition: