LIST OF ABBREVIATIONS.

Bs. *Begriffsschrift.* Eine der arithmetischen nachgebildete Formelsprache des reinen Denkens. Halle a/S, 1879.

Gl. *Grundlagen der Arithmetik.* Eine logisch-mathematische Untersuchung über den Begriff der Zahl. Breslau, 1884.

FT. *Ueber formale Theorien der Arithmetik.* Sitzungsberichte der Jenaischen Gesellschaft für Medicin und Naturwissenschaft, 1885.

FuB. *Function und Begriff.* Vortrag gehalten in der Sitzung vom 9. Januar, 1891, der Jenaischen Gesellschaft für Medicin und Naturwissenschaft. Jena, 1891.

BuG. *Ueber Begriff und Gegenstand.* Vierteljahrschrift für wiss. Phil., XVI 2 (1892).

SuB. *Ueber Sinn und Bedeutung.* Zeitschrift für Phil. und phil. Kritik, vol. 100 (1892).

KB. *Kritische Beleuchtung einiger Punkte in E. Schröder's Vorlesungen über die Algebra der Logik.* Archiv für syst. Phil., Vol. I (1895).

BP. *Ueber die Begriffsschrift des Herrn Peano und meine eigene.* Berichte der math.-physischen Classe der Königl. Sächs. Gesellschaft der Wissenschaften zu Leipzig (1896).

Gg. *Grundgesetze der Arithmetik.* Begriffsschriftlich abgeleitet. Vol. I. Jena, 1893. Vol. II. 1903.

APPENDIX A.

THE LOGICAL AND ARITHMETICAL DOCTRINES OF FREGE.

475. THE work of Frege, which appears to be far less known than it deserves, contains many of the doctrines set forth in Parts I and II of the present work, and where it differs from the views which I have advocated, the differences demand discussion. Frege's work abounds in subtle distinctions, and avoids all the usual fallacies which beset writers on Logic. His symbolism, though unfortunately so cumbrous as to be very difficult to employ in practice, is based upon an analysis of logical notions much more profound than Peano's, and is philosophically very superior to its more convenient rival. In what follows, I shall try briefly to expound Frege's theories on the most important points, and to explain my grounds for differing where I do differ. But the points of disagreement are very few and slight compared to those of agreement. They all result from difference on three points: (1) Frege does not think that there is a contradiction in the notion of concepts which cannot be made logical subjects (see § 49 *supra*); (2) he thinks that, if a term a occurs in a proposition, the proposition can always be analysed into a and an assertion about a (see Chapter VII); (3) he is not aware of the contradiction discussed in Chapter X. These are very fundamental matters, and it will be well here to discuss them afresh, since the previous discussion was written in almost complete ignorance of Frege's work.

Frege is compelled, as I have been, to employ common words in technical senses which depart more or less from usage. As his departures are frequently different from mine, a difficulty arises as regards the translation of his terms. Some of these, to avoid confusion, I shall leave untranslated, since every English equivalent that I can think of has been already employed by me in a slightly different sense.

The principal heads under which Frege's doctrines may be discussed are the following: (1) meaning and indication; (2) truth-values and judgment; (3) Begriff and Gegenstand; (4) classes; (5) implication and symbolic logic; (6) the definition of integers and the principle of abstraction; (7) mathematical induction and the theory of progressions. I shall deal successively with these topics.

476. *Meaning and indication.* The distinction between meaning (*Sinn*) and indication (*Bedeutung*)* is roughly, though not exactly, equivalent to my distinction between a concept as such and what the concept denotes (§ 96). Frege did not possess this distinction in the first two of the works under consideration (the *Begriffsschrift* and the *Grundlagen der Arithmetik*); it appears first in BuG. (cf. p. 198), and is specially dealt with in SuB. Before making the distinction, he thought that identity has to do with the names of objects (Bs. p. 13): "A is identical with B" means, he says, that the sign A and the sign B have the same signification (Bs. p. 15)—a definition which, verbally at least, suffers from circularity. But later he explains identity in much the same way as it was explained in § 64. "Identity," he says, "calls for reflection owing to questions which attach to it and are not quite easy to answer. Is it a relation? A relation between Gegenstände? or between names or signs of Gegenstände?" (SuB. p. 25). We must distinguish, he says, the meaning, in which is contained the way of being given, from what is indicated (from the *Bedeutung*). Thus "the evening star" and "the morning star" have the same indication, but not the same meaning. A word ordinarily stands for its indication; if we wish to speak of its meaning, we must use inverted commas or some such device (pp. 27–8). The indication of a proper name is the object which it indicates; the presentation which goes with it is quite subjective; between the two lies the meaning, which is not subjective and yet is not the object (p. 30). A proper name *expresses* its meaning, and *indicates* its indication (p. 31).

This theory of indication is more sweeping and general than mine, as appears from the fact that *every* proper name is supposed to have the two sides. It seems to me that only such proper names as are derived from concepts by means of *the* can be said to have meaning, and that such words as *John* merely indicate without meaning. If one allows, as I do, that concepts can be objects and have proper names, it seems fairly evident that their proper names, as a rule, will indicate them without having any distinct meaning; but the opposite view, though it leads to an endless regress, does not appear to be logically impossible. The further discussion of this point must be postponed until we come to Frege's theory of Begriffe.

477. *Truth-values and Judgment.* The problem to be discussed under this head is the same as the one raised in § 52†, concerning the difference between asserted and unasserted propositions. But Frege's position on this question is more subtle than mine, and involves a more radical analysis of judgment. His *Begriffsschrift*, owing to the absence of the distinction between meaning and indication, has a simpler theory than his later works. I shall therefore omit it from the discussions.

There are, we are told (Gg. p. x), three elements in judgment: (1) the recognition of truth, (2) the Gedanke, (3) the truth-value (*Wahrheitswerth*).

* I do not translate *Bedeutung* by *denotation*, because this word has a technical meaning different from Frege's, and also because *bedeuten*, for him, is not quite the same as *denoting* for me.

† This is the logical side of the problem of *Annahmen*, raised by Meinong in his able work on the subject, Leipzig, 1902. The logical, though not the psychological, part of Meinong's work appears to have been completely anticipated by Frege.

Here the Gedanke is what I have called an unasserted proposition—or rather, what I called by this name covers both the Gedanke alone and the Gedanke together with its truth-value. It will be well to have names for these two distinct notions; I shall call the Gedanke alone a *propositional concept*; the truth-value of a Gedanke I shall call an *assumption**. Formally at least, an assumption does not require that its content should be a propositional concept: whatever x may be, "the truth of x" is a definite notion. This means the true if x is true, and if x is false or not a proposition it means the false (FuB. p. 21). In like manner, according to Frege, there is "the falsehood of x"; these are not assertions and negations of propositions, but only assertions of truth or of falsity, *i.e.* negation belongs to what is asserted, and is not the opposite of assertion†. Thus we have first a propositional concept, next its truth or falsity as the case may be, and finally the assertion of its truth or falsity. Thus in a hypothetical judgment, we have a relation, not of two judgments, but of two propositional concepts (SuB. p. 43).

This theory is connected in a very curious way with the theory of meaning and indication. It is held that every assumption indicates the true or the false (which are called truth-values), while it means the corresponding propositional concept. The assumption "$2^2 = 4$" indicates the true, we are told, just as "2^2" indicates 4‡ (FuB. p. 13; SuB. p. 32). In a dependent clause, or where a name occurs (such as Odysseus) which indicates nothing, a sentence may have no indication. But when a sentence has a truth-value, this is its indication. Thus every assertive sentence (*Behauptungssatz*) is a proper name, which indicates the true or the false (SuB. pp. 32—4; Gg. p. 7). The sign of judgment (*Urtheilstrich*) does not combine with other signs to denote an object; a judgment indicates nothing, but asserts something. Frege has a special symbol for judgment, which is something distinct from and additional to the truth-value of a propositional concept (Gg. pp. 9—10).

478. There are some difficulties in the above theory which it will be well to discuss. In the first place, it seems doubtful whether the introduction of truth-values marks any real analysis. If we consider, say, "Caesar died," it would seem that what is asserted is the propositional concept "the death of Caesar," not "the truth of the death of Caesar." This latter seems to be merely another propositional concept, asserted in "the death of Caesar is true," which is not, I think, the same proposition as "Caesar died." There is great difficulty in avoiding psychological elements here, and it would seem that Frege has allowed them to intrude in describing judgment as the recognition of truth (Gg. p. x). The difficulty is due to the fact that there is a psychological sense of assertion, which is what is lacking to Meinong's *Annahmen*, and that this does not run parallel with the logical sense. Psychologically, any proposition, whether true or false, may be merely thought of, or may be actually asserted: but for this possibility, error would be impossible. But logically, true propositions only are asserted,

* Frege, like Meinong, calls this an *Annahme*: FuB. p. 21.

† Gg. p. 10. Cf. also Bs. p. 4.

‡ When a term which indicates is itself to be spoken of, as opposed to what it indicates, Frege uses inverted commas. Cf. § 56.

though they may occur in an unasserted form as parts of other propositions. In "*p* implies *q*," either or both of the propositions *p*, *q* may be true, yet each, in this proposition, is unasserted in a logical, and not merely in a psychological, sense. Thus assertion has a definite place among logical notions, though there is a psychological notion of assertion to which nothing logical corresponds. But assertion does not seem to be a constituent of an asserted proposition, although it is, in some sense, contained in an asserted proposition. If *p* is a proposition, "*p*'s truth" is a concept which has being even if *p* is false, and thus "*p*'s truth" is not the same as *p* asserted. Thus no concept can be found which is equivalent to *p* asserted, and therefore assertion is not a constituent in *p* asserted. Yet assertion is not a term to which *p*, when asserted, has an external relation; for any such relation would need to be itself asserted in order to yield what we want. Also a difficulty arises owing to the apparent fact, which may however be doubted, that an asserted proposition can never be part of another proposition: thus, if this be a fact, where any statement is made about *p* asserted, it is not really about *p* asserted, but only about the assertion of *p*. This difficulty becomes serious in the case of Frege's one and only principle of inference (Bs. p. 9): "*p* is true and *p* implies *q*; therefore *q* is true*." Here it is quite essential that there should be three actual assertions, otherwise the assertion of propositions deduced from asserted premisses would be impossible; yet the three assertions together form one proposition, whose unity is shown by the word *therefore*, without which *q* would not have been deduced, but would have been asserted as a fresh premiss.

It is also almost impossible, at least to me, to divorce assertion from truth, as Frege does. An asserted proposition, it would seem, must be the same as a true proposition. We may allow that negation belongs to the content of a proposition (Bs. p. 4), and regard every assertion as asserting something to be true. We shall then correlate *p* and not-*p* as unasserted propositions, and regard "*p* is false" as meaning "not-*p* is true." But to divorce assertion from truth seems only possible by taking assertion in a psychological sense.

479. Frege's theory that assumptions are proper names for the true or the false, as the case may be, appears to me also untenable. Direct inspection seems to show that the relation of a proposition to the true or the false is quite different from that of (say), "the present King of England" to Edward VII. Moreover, if Frege's view were correct on this point, we should have to hold that in an asserted proposition it is the meaning, not the indication, that is asserted, for otherwise, all asserted propositions would assert the very same thing, namely the true, (for false propositions are not asserted). Thus asserted propositions would not differ from one another in any way, but would be all strictly and simply identical. Asserted propositions have no indication (FuB. p. 21), and can only differ, if at all, in some way analogous to meaning. Thus the meaning of the unasserted proposition together with its truth-value must be what is asserted, if the meaning simply is rejected. But there seems no purpose in introducing the truth-value here: it seems quite sufficient to say that an asserted proposition is one whose meaning is true, and that to say the meaning is true is the same as to say the meaning is asserted. We might then conclude that true propositions, even when they occur as parts of others, are always and essentially asserted, while false propositions are always unasserted, thus escaping the difficulty about *therefore* discussed above. It may also be objected to Frege that "the true" and "the false," as opposed to truth and falsehood, do not denote single definite things, but rather the classes of true and false propositions respectively. This objection, however, would be met by his theory of ranges, which correspond approximately to my classes; these, he says, are things, and the true and the false are ranges (*v. inf.*).

* Cf. *supra*, § 18, (4) and § 38.

480. *Begriff and Gegenstand. Functions.* I come now to a point in which Frege's work is very important, and requires careful examination. His use of the word *Begriff* does not correspond exactly to any notion in my vocabulary, though it comes very near to the notion of an assertion as defined in § 43, and discussed in Chapter VII. On the other hand, his *Gegenstand* seems to correspond exactly to what I have called a *thing* (§ 48). I shall therefore translate *Gegenstand* by *thing*. The meaning of *proper name* seems to be the same for him as for me, but he regards the range of proper names as confined to things, because they alone, in his opinion, can be logical subjects.

Frege's theory of functions and *Begriffe* is set forth simply in FuB. and defended against the criticisms of Kerry* in BuG. He regards functions—and in this I agree with him—as more fundamental than predicates and relations; but he adopts concerning functions the theory of subject and assertion which we discussed and rejected in Chapter VII. The acceptance of this view gives a simplicity to his exposition which I have been unable to attain; but I do not find anything in his work to persuade me of the legitimacy of his analysis.

An arithmetical function, *e.g.* $2x^3 + x$, does not denote, Frege says, the result of an arithmetical operation, for that is merely a number, which would be nothing new (FuB. p. 5). The essence of a function is what is left when the x is taken away, *i.e.*, in the above instance, $2(\ \ )^3 + (\ \ )$. The argument x does not belong to the function, but the two together make a whole (*ib.* p. 6). A function may be a proposition for every value of the variable; its value is then always a truth-value (p. 13). A proposition may be divided into two parts, as "Caesar" and "conquered Gaul." The former Frege calls the *argument*, the latter the *function*. Any thing whatever is a possible argument for a function (p. 17). (This division of propositions corresponds exactly to my *subject* and *assertion* as explained in § 43, but Frege does not restrict this method of analysis as I do in Chapter VII.) A thing is anything which is not a function, *i.e.* whose expression leaves no empty place. The two following accounts of the nature of a function are quoted from the earliest and one of the latest of Frege's works respectively.

(1) "If in an expression, whose content need not be propositional

* Vierteljahrschrift für wiss. Phil., vol. XI, pp. 249–307.

(*beurtheilbar*), a simple or composite sign occurs in one or more places, and we regard it as replaceable, in one or more of these places, by something else, but by the same everywhere, then we call the part of the expression which remains invariable in this process a *function*, and the replaceable part we call its argument" (Bs. p. 16).

(2) "If from a proper name we exclude a proper name, which is part or the whole of the first, in some or all of the places where it occurs, but in such a way that these places remain recognizable as to be filled by one and the same arbitrary proper name (as argument positions of the first kind), I call what we thereby obtain the name of a function of the first order with one argument. Such a name, together with a proper name which fills the argument-places, forms a proper name" (Gg. p. 44).

The latter definition may become plainer by the help of some examples. "The present king of England" is, according to Frege, a proper name, and "England" is a proper name which is part of it. Thus here we may regard England as the argument, and "the present king of" as function. Thus we are led to "the present king of x." This expression will always have a meaning, but it will not have an indication except for those values of x which at present are monarchies. The above function is not propositional. But "Caesar conquered Gaul" leads to "x conquered Gaul"; here we have a propositional function. There is here a minor point to be noticed: the *asserted* proposition is not a proper name, but only the assumption is a proper name for the true or the false (*v. supra*); thus it is not "Caesar conquered Gaul" as asserted, but only the corresponding assumption, that is involved in the genesis of a propositional function. This is indeed sufficiently obvious, since we wish x to be able to be any thing in "x conquered Gaul," whereas there is no such asserted proposition except when x did actually perform this feat. Again consider "Socrates is a man implies Socrates is a mortal." This (unasserted) is, according to Frege, a proper name for the true. By varying the proper name "Socrates," we can obtain three propositional functions, namely "x is a man implies Socrates is a mortal," "Socrates is a man implies x is a mortal," "x is a man implies x is a mortal." Of these the first and third are true for all values of x, the second is true when and only when x is a mortal.

By suppressing in like manner a proper name in the name of a function of the first order with one argument, we obtain the name of a function of the first order with two arguments (Gg. p. 44). Thus *e.g.* starting from "$1 < 2$," we get first "$x < 2$," which is the name of a function of the first order with one argument, and thence "$x < y$," which is the name of a function of the first order with two arguments. By suppressing a function in like manner, Frege says, we obtain the name of a function of the second order (Gg. p. 44). Thus *e.g.* the assertion of existence in the mathematical sense is a function of the second order: "There is at least one value of x satisfying ϕx" is not a function of x, but may be regarded as a function of ϕ. Here ϕ must on no account be a thing, but may be any function. Thus this proposition, considered as a function of ϕ, is quite different from functions of the first order, by the fact that the possible arguments are different. Thus given any proposition, say $f(a)$, we may consider either $f(x)$, the function of the first order resulting from varying a and keeping f constant, or $\phi(a)$, the function of the second order got by varying f and keeping a fixed; or, finally, we may consider $\phi(x)$, in which both f and a are separately varied. (It is to be observed that such notions as $\phi(a)$, in which we consider any proposition concerning a, are involved in the identity of indiscernibles as stated in § 43.) Functions of the first order with two variables, Frege points out, express relations (Bs. p. 17); the referent and the relatum are both subjects in a relational proposition (Gl. p. 82). Relations, just as much as predicates, belong, Frege rightly says, to pure logic (*ib.* p. 83).

481. The word *Begriff* is used by Frege to mean nearly the same thing as *propositional function* (*e.g.* FuB. p. 28)*; when there are two variables, the Begriff is a relation. A thing is anything not a function, *i.e.* anything whose expression leaves no empty place (*ib.* p. 18). To Frege's theory of the essential cleavage between things and Begriffe, Kerry objects (*loc. cit.* p. 272 ff.) that Begriffe also can occur as subjects. To this Frege makes two replies. In the first place, it is, he says, an important distinction that some terms can only occur as subjects, while others can occur also as concepts, even if Begriffe can also occur as subjects (BuG. p. 195). In this I agree with him entirely; the distinction is the one employed in §§ 48, 49. But he goes on to a second point which appears to me mistaken. We can, he says, have a concept falling under a higher one (as Socrates falls under man, he means, not as Greek falls under man); but in such cases, it is not the concept itself, but its name, that is in question (BuG. p. 195). "The concept horse," he says, is not a concept, but a thing; the peculiar use is indicated by inverted commas (*ib.* p. 196). But a few pages later he makes statements which seem to involve a different view. A concept, he says, is essentially predicative even when something is asserted of it: an assertion which can be made of a concept does not fit an object. When a thing is said to fall under a concept, and when a concept is said to fall under a higher concept, the two relations involved, though similar, are not the same (*ib.* p. 201). It is difficult to me to reconcile these remarks with those of p. 195; but I shall return to this point shortly.

Frege recognizes the unity of a proposition: of the parts of a propositional concept, he says, not all can be complete, but one at least must be incomplete (*ungesättigt*) or predicative, otherwise the parts would not cohere (*ib.* p. 205). He recognizes also, though he does not discuss, the oddities resulting from *any* and *every* and such words: thus he remarks that every positive integer is the sum of four squares, but "every positive integer" is not a possible value of x in "x is the sum of four squares." The meaning of "every positive integer," he says, depends upon the context (Bs. p. 17)—a remark which is doubtless correct, but does not exhaust the subject. Self-contradictory notions are admitted as concepts: F is a concept if "a falls under the concept F" is a proposition whatever thing a may be (Gl. p. 87). A concept is the indication of a predicate; a thing is what can never be

* "We have here a function whose value is always a truth-value. Such functions with one argument we have called Begriffe; with two, we call them relations." Cf. Gl. pp. 82–3.

the whole indication of a predicate, though it may be that of a subject (BuG. p. 198).

482. The above theory, in spite of close resemblance, differs in some important points from the theory set forth in Part I above. Before examining the differences, I shall briefly recapitulate my own theory.

Given any propositional concept, or any unity (see § 136), which may in the limit be simple, its constituents are in general of two sorts: (1) those which may be replaced by anything else whatever without destroying the unity of the whole; (2) those which have not this property. Thus in "the death of Caesar," anything else may be substituted for Caesar, but a proper name must not be substituted for *death*, and hardly anything can be substituted for *of*. Of the unity in question, the former class of constituents will be called *terms*, the latter *concepts*. We have then, in regard to any unity, to consider the following objects:

(1) What remains of the said unity when one of its terms is simply removed, or, if the term occurs several times, when it is removed from one or more of the places in which it occurs, or, if the unity has more than one term, when two or more of its terms are removed from some or all of the places where they occur. This is what Frege calls a function.

(2) The class of unities differing from the said unity, if at all, only by the fact that one of its terms has been replaced, in one or more of the places where it occurs, by some other terms, or by the fact that two or more of its terms have been thus replaced by other terms.

(3) Any member of the class (2).

(4) The assertion that every member of the class (2) is true.

(5) The assertion that some member of the class (2) is true.

(6) The relation of a member of the class (2) to the value which the variable has in that member.

The fundamental case is that where our unity is a propositional concept. From this is derived the usual mathematical notion of function, which might at first sight seem simpler. If $f(x)$ is not a propositional function, its value for a given value of x ($f(x)$ being assumed to be one-valued) is the term y satisfying the propositional function $y = f(x)$, *i.e.* satisfying, for the given value of x, some relational proposition; this relational proposition is involved in the definition of $f(x)$, and some such propositional function is required in the definition of any function which is not propositional.

As regards (1), confining ourselves to one variable, it was maintained in Chapter VII that, except where the proposition from which we start is predicative or else asserts a fixed relation to a fixed term, there is no such entity: the analysis into argument and assertion cannot be performed in the manner required. Thus what Frege calls a function, if our conclusion was sound, is in general a non-entity. Another point of difference from Frege, in which, however, he appears to be in the right, lies in the fact that I place no restriction upon the variation of the variable, whereas Frege, according to the nature of the function, confines the variable to things, functions of the first order with one variable, functions of the first order with two variables, functions of the second order with one variable, and so on. There are thus for him an infinite number of different kinds of variability. This arises from the fact that he regards as distinct the concept occurring as such and the concept occurring as term, which I (§ 49) have identified. For me, the functions, which cannot be values of variables in functions of the first order, are non-entities and false abstractions. Instead of the rump of a proposition considered in (1), I substitute (2) or (3) or (4) according to circumstances. The ground for regarding the analysis into argument and function as not always possible is that, when one term is removed from a propositional concept, the remainder is apt to have no sort of unity, but to fall apart into a set of disjointed terms. Thus what is fundamental in such a case is (2). Frege's general definition of a function, which is intended to cover also functions which are not propositional, may be shown to be inadequate by considering what may be called the identical function, *i.e.* x as a function of x. If we follow Frege's advice, and remove x in hopes of having the function left, we find that nothing is left at all; yet nothing is not the meaning of the identical function. Frege wishes to have the empty places where the argument is to be inserted indicated in some way; thus he says that in $2x^3 + x$ the function is $2(\ \ )^3 + (\ \ )$. But here his requirement that the two empty places are to be filled by the same letter cannot be indicated: there is no way of distinguishing what we mean from the function involved in $2x^3 + y$. The fact seems to be that we want the notion of any term of a certain class, and that this is what our empty places really stand for. The function, as a single entity, is the relation (6) above; we can then consider any relatum of this relation, or the assertion of all or some of the relata, and any relation can be expressed in terms of the corresponding referent, as "Socrates is a man" is expressed in terms of Socrates. But the usual formal apparatus of the calculus of relations cannot be employed, because it presupposes propositional functions. We may say that a propositional function is a many-one relation which has all terms for the class of its referents, and has its relata contained among propositions*: or, if we prefer, we may call the class of relata of such a relation a propositional function. But the air of formal definition about these statements is fallacious, since propositional functions are presupposed in defining the class of referents and relata of a relation.

Thus by means of propositional functions, propositions are collected into classes. (These classes are not mutually exclusive.) But we may also collect them into classes by the terms which occur in them: all propositions containing a given term a will form a class. In this way we obtain propositions concerning variable propositional functions. In the notation $\phi(x)$, the ϕ is essentially variable; if we wish it not to be so, we must take some particular proposition about x, such as "x is a class" or "x implies x." Thus $\phi(x)$ essentially contains two variables. But, if we have decided that ϕ is not a separable entity, we cannot regard ϕ itself as the second variable. It will be necessary to take as our variable either the relation of x to $\phi(x)$, or else the class of propositions $\phi(y)$ for different values of y but for constant ϕ. This does not matter formally, but it is important for logic to be clear as to

* Not all relations having this property are propositional functions; *v. inf.*

the meaning of what appears as the variation of ϕ. We obtain in this way another division of propositions into classes, but again these classes are not mutually exclusive.

In the above manner, it would seem, we can make use of propositional functions without having to introduce the objects which Frege calls functions. It is to be observed, however, that the kind of relation by which propositional functions are defined is less general than the class of many-one relations having their domain coextensive with terms and their converse domain contained in propositions. For in this way any proposition would, for a suitable relation, be relatum to any term, whereas the term which is referent must, for a propositional function, be a constituent of the proposition which is its relatum*. This point illustrates again that the class of relations involved is fundamental and incapable of definition. But it would seem also to show that Frege's different kinds of variability are unavoidable, for in considering (say) $\phi(2)$, where ϕ is variable, the variable would have to have as its range the above class of relations, which we may call *propositional relations*. Otherwise, $\phi(2)$ is not a proposition, and is indeed meaningless, for we are dealing with an indefinable, which demands that $\phi(2)$ should be the relatum of 2 with regard to some propositional relation. The contradiction discussed in Chapter x seems to show that some mystery lurks in the variation of propositional functions; but for the present, Frege's theory of different kinds of variables must, I think, be accepted.

483. It remains to discuss afresh the question whether concepts can be made into logical subjects without change of meaning. Frege's theory, that when this appears to be done it is really the name of the concept that is involved, will not, I think, bear investigation. In the first place, the mere assertion "not the concept, but its name, is involved," has already made the concept a subject. In the second place, it seems always legitimate to ask: "what is it that is named by this name?" If there were no answer, the name could not be a name; but if there is an answer, the concept, as opposed to its name, can be made a subject. (Frege, it may be observed, does not seem to have clearly disentangled the logical and linguistic elements of naming: the former depend upon denoting, and have, I think, a much more restricted range than Frege allows them.) It is true that we found difficulties in the doctrine that everything can be a logical subject: as regards "any *a*," for example, and also as regards plurals. But in the case of "any *a*," there is ambiguity, which introduces a new class of problems; and as regards plurals, there are propositions in which the many behave like a logical subject in every respect except that they are many subjects and not one only (see §§ 127, 128). In the case of concepts, however, no such escapes are possible. The case of asserted propositions is difficult, but is met, I think, by holding that an asserted proposition is merely a true proposition, and is therefore asserted wherever it occurs, even when grammar would lead to the opposite conclusion. Thus, on the whole, the doctrine of concepts which cannot be made subjects seems untenable.

484. *Classes.* Frege's theory of classes is very difficult, and I am not

* The notion of a constituent of a proposition appears to be a logical indefinable.

sure that I have thoroughly understood it. He gives the name *Werthverlauf** to an entity which appears to be nearly the same as what I call the class as one. The concept of the class, and the class as many, do not appear in his exposition. He differs from the theory set forth in Chapter VI chiefly by the fact that he adopts a more intensional view of classes than I have done, being led thereto mainly by the desirability of admitting the null-class and of distinguishing a term from a class whose only member it is. I agree entirely that these two objects cannot be attained by an extensional theory, though I have tried to show how to satisfy the requirements of formalism (§§ 69, 73).

The extension of a *Begriff*, Frege says, is the range of a function whose value for every argument is a truth-value (FuB. p. 16). Ranges are things, whereas functions are not (*ib.* p. 19). There would be no null-class, if classes were taken in extension; for the null-class is only possible if a class is not a collection of terms (KB. pp. 436–7). If x be a term, we cannot identify x, as the extensional view requires, with the class whose only member is x; for suppose x to be a class having more than one member, and let y, z be two different members of x; then if x is identical with the class whose only member is x, y and z will both be members of this class, and will therefore be identical with x and with each other, contrary to the hypothesis†. The extension of a *Begriff* has its being in the *Begriff* itself, not in the individuals falling under the *Begriff* (*ib.* p. 451). When I say something about all men, I say nothing about some wretch in the centre of Africa, who is in no way indicated, and does not belong to the indication of *man* (p. 454). *Begriffe* are prior to their extension, and it is a mistake to attempt, as Schröder does, to base extension on individuals; this leads to the calculus of regions (*Gebiete*), not to Logic (p. 455).

What Frege understands by a range, and in what way it is to be conceived without reference to objects, he endeavours to explain in his *Grundgesetze der Arithmetik*. He begins by deciding that two propositional functions are to have the same range when they have the same value for every value of x, *i.e.* for every value of x both are true or both false (pp. 7, 14) This is laid down as a primitive proposition. But this only determines the equality of ranges, not what they are in themselves. If $X(\xi)$ be a function which never has the same value for different values of ξ and if we denote by ϕ' the range of ϕx, we shall have $X(\phi') = X(\psi')$ when and only when ϕ' and ψ' are equal, *i.e.* when and only when ϕx and ψx always have the same value. Thus the conditions for the equality of ranges do not of themselves decide what ranges are to be (p. 16). Let us decide arbitrarily —since the notion of a range is not yet fixed—that the true is to be the range of the function "x is true" (as an assumption, not an asserted proposition), and the false is to be the range of the function "x = not every term is identical with itself." It follows that the range of ϕx is the true when and only when the true and nothing else falls under the *Begriff* ϕx; the range of ϕx is the false when and only when the false and nothing else falls under the *Begriff* ϕx; in other cases, the range is neither the true nor

* I shall translate this as *range*.

† *Ib.* p. 444. *Cf. supra*, § 74.

the false (pp. 17—18). If only one thing falls under a concept, this one thing is distinct from the range of the concept in question (p. 18, note)—the reason is the same as that mentioned above.

There is an argument (p. 49) to prove that the name of the range of a function always has an indication, *i.e.* that the symbol employed for it is never meaningless. In view of the contradiction discussed in Chapter X, I should be inclined to deny a meaning to a range when we have a proposition of the form $\phi[f(\phi)]$, where f is constant and ϕ variable, or of the form $f_x(x)$, where x is variable and f_x is a propositional function which is determinate when x is given, but varies from one value of x to another—provided, when f_x is analyzed into things and concepts, the part dependent on x does not consist only of things, but contains also at least one concept. This is a very complicated case, in which, I should say, there is no class as one, my only reason for saying so being that we can thus escape the contradiction.

485. By means of variable propositional functions, Frege obtains a definition of the relation which Peano calls ϵ, namely the relation of a term to a class of which it is a member*. The definition is as follows: "$a \epsilon u$" is to mean the term (or the range of terms if there be none or many) x such that there is a propositional function ϕ which is such that u is the range of ϕ and ϕa is identical with x (p. 53). It is observed that this defines $a \epsilon u$ whatever things a and u may be. In the first place, suppose u to be a range. Then there is at least one ϕ whose range is u, and any two whose range is u are regarded by Frege as identical. Thus we may speak of *the* function ϕ whose range is u. In this case, $a \epsilon u$ is the proposition ϕa, which is true when a is a member of u, and is false otherwise. If, in the second place, u is not a range, then there is no such propositional function as ϕ, and therefore $a \epsilon u$ is the range of a propositional function which is always false, *i.e.* the null-range. Thus $a \epsilon u$ indicates the true when u is a range and a is a member of u; $a \epsilon u$ indicates the false when u is a range and a is not a member of u; in other cases, $a \epsilon u$ indicates the null-range.

It is to be observed that from the equivalence of $x \epsilon u$ and $x \epsilon v$ for all values of x we can only infer the identity of u and v when u and v are ranges. When they are not ranges, the equivalence will always hold, since $x \epsilon u$ and $x \epsilon v$ are the null-range for all values of x; thus if we allowed the inference in this case, any two objects which are not ranges would be identical, which is absurd. One might be tempted to doubt whether u and v must be identical even when they are ranges: with an intensional view of classes, this becomes open to question.

Frege proceeds (p. 55) to an analogous definition of the propositional function of three variables which I have symbolised as $x R y$, and here again he gives a definition which does not place any restrictions on the variability of R. This is done by introducing a *double range*, defined by a propositional function of two variables; we may regard this as a class of couples with sense†. If then R is such a class of couples, and if $(x ; y)$ is a member of this

* Cf. §§ 21, 76, *supra*.

† Neglecting, for the present, our doubts as to there being any such entity as a couple with sense, cf. § 98.

class, $x R y$ is to hold; in other cases it is to be false or null as before. On this basis, Frege successfully erects as much of the logic of relations as is required for his Arithmetic; and he is free from the restrictions on the variability of R which arise from the intensional view of relations adopted in the present work (cf. § 83).

486. The chief difficulty which arises in the above theory of classes is as to the kind of entity that a range is to be. The reason which led me, against my inclination, to adopt an extensional view of classes, was the necessity of discovering some entity determinate for a given propositional function, and the same for any equivalent propositional function. Thus "x is a man" is equivalent (we will suppose) to "x is a featherless biped," and we wish to discover some one entity which is determined in the same way by both these propositional functions. The only single entity I have been able to discover is the class as one—except the derivative class (also as one) of propositional functions equivalent to either of the given propositional functions. This latter class is plainly a more complex notion, which will not enable us to dispense with the general notion of *class*; but this more complex notion (so we agreed in § 73) must be substituted for the class of terms in the symbolic treatment, if there is to be any null-class and if the class whose only member is a given term is to be distinguished from that term. It would certainly be a very great simplification to admit, as Frege does, a range which is something other than the whole composed of the terms satisfying the propositional function in question; but for my part, inspection reveals to me no such entity. On this ground, and also on account of the contradiction, I feel compelled to adhere to the extensional theory of classes, though not quite as set forth in Chapter VI.

487. That some modification in that doctrine is necessary, is proved by the argument of KB. p. 444. This argument appears capable of proving that a class, even as one, cannot be identified with the class of which it is the only member. In § 74, I contended that the argument was met by the distinction between the class as one and the class as many, but this contention now appears to me mistaken. For this reason, it is necessary to re-examine the whole doctrine of classes.

Frege's argument is as follows. If a is a class of more than one term, and if a is identical with the class whose only term is a, then to be a term of a is the same thing as to be a term of the class whose only term is a, whence a is the only term of a. This argument *appears* to prove not merely that the extensional view of classes is inadequate, but rather that it is wholly inadmissible. For suppose a to be a collection, and suppose that a collection of one term is identical with that one term. Then, if a can be regarded as one collection, the above argument proves that a is the only term of a. We cannot escape by saying that ϵ is to be a relation to the class-concept or the concept of the class or the class as many, for if there is any such entity as the class as one, there will be a relation, which we may call ϵ, between terms and their classes as one. Thus the above argument leads to the conclusion that either (α) a collection of more than one term is not identical with the collection whose only term it is, or (β) there is no collection as one term at all in the case of a collection of many terms, but the collection is strictly and only many. One or other of these must be admitted in virtue of the above argument.

488. (*a*) To either of these views there are grave objections. The former is the view of Frege and Peano. To realize the paradoxical nature of this view, it must be clearly grasped that it is not only the collection as many, but the collection as one, that is distinct from the collection whose only term it is. (I speak of collections, because it is important to examine the bearing of Frege's argument upon the possibility of an extensional standpoint.) This view, in spite of its paradox, is certainly the one which seems to be required by the symbolism. It is quite essential that we should be able to regard a class as a single object, that there should be a null-class, and that a term should not (in general, at any rate) be identical with the class of which it is the only member. It is subject to these conditions that the *symbolic* meaning of *class* has to be interpreted. Frege's notion of a range may be identified with the collection as one, and all will then go well. But it is very hard to see any entity such as Frege's range, and the argument that there must be such an entity gives us little help. Moreover, in virtue of the contradiction, there certainly are cases where we have a collection as many, but no collection as one (§ 104). Let us then examine (β), and see whether this offers a better solution.

(β) Let us suppose that a collection of one term is that one term, and that a collection of many terms is (or rather are) those many terms, so that there is not a single term at all which is the collection of the many terms in question. In this view there is, at first sight at any rate, nothing paradoxical, and it has the merit of admitting universally what the Contradiction shows to be sometimes the case. In this case, unless we abandon one of our fundamental dogmas, ϵ will have to be a relation of a term to its class-concept, not to its class; if a is a class-concept, what appears symbolically as the class whose only term is a will (one might suppose) be the class-concept under which falls only the concept a, which is of course (in general, if not always) different from a. We shall maintain, on account of the contradiction, that there is not always a class-concept for a given propositional function ϕx, *i.e.* that there is not always, for every ϕ, some class-concept a such that $x \epsilon a$ is equivalent to ϕx for all values of x; and the cases where there is no such class-concept will be cases in which ϕ is a quadratic form.

So far, all goes well. But now we no longer have one definite entity which is determined equally by any one of a set of equivalent propositional functions, *i.e.* there is, it might be urged, no meaning of *class* left which is determined by the extension alone. Thus, to take a case where this leads to confusion, if a and b be different class-concepts such that $x \epsilon a$ and $x \epsilon b$ are equivalent for all values of x, the class-concept under which a falls and nothing else will not be identical with that under which falls b and nothing else. Thus we cannot get any way of denoting what should symbolically correspond to the class as one. Or again, if u and v be similar but different classes, "similar to u" is a different concept from "similar to v"; thus, unless we can find some extensional meaning for *class*, we shall not be able to say that the number of u is the same as that of v. And all the usual elementary problems as to combinations (*i.e.* as to the number of classes of specified kinds contained in a given class) will have become impossible and even meaningless. For these various reasons, an objector might contend, something like the class as one must be maintained; and Frege's range fulfils the conditions required. It would seem necessary therefore to accept ranges by an act of faith, without waiting to see whether there are such things.

Nevertheless, the non-identification of the class with the class as one, whether in my form or in the form of Frege's range, appears unavoidable, and by a process of exclusion the class as many is left as the only object which can play the part of a class. By a modification of the logic hitherto advocated in the present work, we shall, I think, be able at once to satisfy the requirements of the Contradiction and to keep in harmony with common sense*.

489. Let us begin by recapitulating the possible theories of classes which have presented themselves. A class may be identified with (α) the predicate, (β) the class concept, (γ) the concept of the class, (δ) Frege's range, (ϵ) the numerical conjunction of the terms of the class, (ζ) the whole composed of the terms of the class.

Of these theories, the first three, which are intensional, have the defect that they do not render a class determinate when its terms are given. The other three do not have this defect, but they have others. (δ) suffers from a doubt as to there being such an entity, and also from the fact that, if ranges are terms, the contradiction is inevitable. (ϵ) is logically unobjectionable, but is not a single entity, except when the class has only one member. (ζ) cannot always exist as a term, for the same reason as applies against (δ); also it cannot be identified with the class on account of Frege's argument†.

Nevertheless, without a single object‡ to represent an extension, Mathematics crumbles. Two propositional functions which are equivalent for all values of the variable may not be identical, but it is necessary that there should be some object determined by both. Any object that may be proposed, however, presupposes the notion of *class*. We may define *class* optatively as follows: A class is an object uniquely determined by a propositional function, and determined equally by any equivalent propositional function. Now we cannot take as this object (as in other cases of symmetrical transitive relations) the class of propositional functions equivalent to a given propositional function, unless we already have the notion of *class*. Again, equivalent relations, considered intensionally, may be distinct: we want therefore to find some one object determined equally by any one of a set of equivalent relations. But the only objects that suggest themselves are the class of relations or the class of couples forming their common range; and these both presuppose *class*. And without the notion of class, elementary problems, such as "how many combinations can be formed of m objects n at a time?" become meaningless. Moreover, it appears immediately evident that there is some sense in saying that two class-concepts have the *same*

* The doctrine to be advocated in what follows is the direct denial of the dogma stated in § 70, note.

† Archiv I. p. 444.

‡ For the use of the word *object* in the following discussion, see § 58, note.

extension, and this requires that there should be some object which can be called the extension of a class-concept. But it is exceedingly difficult to discover any such object, and the contradiction proves conclusively that, even if there be such an object sometimes, there are propositional functions for which the extension is not one term.

The class as many, which we numbered (ϵ) in the above enumeration, is unobjectionable, but is many and not one. We may, if we choose, represent this by a single symbol: thus $x \epsilon u$ will mean "x is one of the u's." This must not be taken as a relation of two terms, x and u, because u as the numerical conjunction is not a single term, and we wish to have a meaning for $x \epsilon u$ which would be the same if for u we substituted an equal class v, which prevents us from interpreting u intensionally. Thus we may regard "x is one of the u's" as expressing a relation of x to many terms, among which x is included. The main objection to this view, if only single terms can be subjects, is that, if u is a symbol standing essentially for many terms, we cannot make u a logical subject without risk of error. We can no longer speak, one might suppose, of a class of classes; for what should be the terms of such a class are not single terms, but are each many terms*. We cannot assert a predicate of many, one would suppose, except in the sense of asserting it of each of the many; but what is required here is the assertion of a predicate concerning the many as many, not concerning each nor yet concerning the whole (if any) which all compose. Thus a class of classes will be many many's; its constituents will each be only many, and cannot therefore in any sense, one might suppose, be single constituents. Now I find myself forced to maintain, in spite of the apparent logical difficulty, that this is precisely what is required for the assertion of number. If we have a class of classes, each of whose members has two terms, it is necessary that the members should each be genuinely two-fold, and should not be each one. Or again, "Brown and Jones are two" requires that we should not combine Brown and Jones into a single whole, and yet it has the form of a subject-predicate proposition. But now a difficulty arises as to the number of members of a class of classes. In what sense can we speak of two couples? This seems to require that each couple should be a single entity; yet if it were, we should have two units, not two couples. We require a sense for diversity of collections, meaning thereby, apparently, if u and v are the collections in question, that $x \epsilon u$ and $x \epsilon v$ are not equivalent for all values of x.

490. The logical doctrine which is thus forced upon us is this: The subject of a proposition may be not a single term, but essentially many terms; this is the case with all propositions asserting numbers other than 0 and 1. But the predicates or class-concepts or relations which can occur in propositions having plural subjects are different (with some exceptions) from those that can occur in propositions having single terms as subjects. Although a class is many and not one, yet there is identity and diversity among classes, and thus classes can be counted as though each were a genuine unity; and in this sense we can speak of *one* class and of the classes which are members of a

* Wherever the context requires it, the reader is to add "provided the class in question (or all the classes in question) do not consist of a single term."

class of classes. *One* must be held, however, to be somewhat different when asserted of a class from what it is when asserted of a term; that is, there is a meaning of *one* which is applicable in speaking of *one term*, and another which is applicable in speaking of *one class*, but there is also a general meaning applicable to both cases. The fundamental doctrine upon which all rests is the doctrine that the subject of a proposition may be plural, and that such plural subjects are what is meant by classes which have more than one term*.

It will now be necessary to distinguish (1) terms, (2) classes, (3) classes of classes, and so on *ad infinitum*; we shall have to hold that no member of one set is a member of any other set, and that $x \epsilon u$ requires that x should be of a set of a degree lower by one than the set to which u belongs. Thus $x \epsilon x$ will become a meaningless proposition; and in this way the contradiction is avoided.

491. But we must now consider the problem of classes which have one member or none. The case of the null-class might be met by a bare denial—this is only inconvenient, not self-contradictory. But in the case of classes having only one term, it is still necessary to distinguish them from their sole members. This results from Frege's argument, which we may repeat as follows. Let u be a class having more than one term; let ιu be the class of classes whose only member is u. Then ιu has one member, u has many; hence u and ιu are not identical. It may be doubted, at first sight, whether this argument is valid. The relation of x to u expressed by $x \epsilon u$ is a relation of a single term to many terms; the relation of u to ιu expressed by $u \epsilon \iota u$ is a relation of many terms (as subject) to many terms (as predicate)†. This is, so an objector might contend, a different relation from the previous one; and thus the argument breaks down. It is in different senses that x is a member of u and that u is a member of ιu; thus u and ιu may be identical in spite of the argument.

This attempt, however, to escape from Frege's argument, is capable of refutation. For all the purposes of Arithmetic, to begin with, and for many of the purposes of logic, it is necessary to have a meaning for ϵ which is equally applicable to the relation of a term to a class, of a class to a class of classes, and so on. But the chief point is that, if every single term is a class, the proposition $x \epsilon x$, which gives rise to the Contradiction, must be admissible. It is only by distinguishing x and ιx, and insisting that in $x \epsilon u$ the u must always be of a type higher by one than x, that the contradiction can be avoided. Thus, although we may identify the class with the numerical conjunction of its terms, wherever there are many terms, yet where there is only one term we shall have to accept Frege's range as an object distinct from its only term. And having done this, we may of course also admit a range in the case of a null propositional function. We shall differ from Frege only in regarding a range as in no case a term, but an object of a different logical type, in the sense that a propositional function $\phi(x)$, in which x may be any term, is in general meaningless if for x we substitute a

* Cf. §§ 128, 132 *supra*.

† The word *predicate* is here used loosely, not in the precise sense defined in § 48.

range; and if x may be any range of terms, $\phi(x)$ will in general be meaningless if for x we substitute either a term or a range of ranges of terms. Ranges, finally, are what are properly to be called *classes*, and it is of them that cardinal numbers are asserted.

492. According to the view here advocated, it will be necessary, with every variable, to indicate whether its field of significance is terms, classes, classes of classes, or so on*. A variable will not be able, except in special cases, to extend from one of these sets into another; and in $x \epsilon u$, the x and the u must always belong to different types; ϵ will not be a relation between objects of the same type, but $\epsilon \breve{\epsilon}$ or $\epsilon R \breve{\epsilon}$ † will be, provided R is so. We shall have to distinguish also among relations according to the types to which their domains and converse domains belong; also variables whose fields include relations, these being understood as classes of couples, will not as a rule include anything else, and relations between relations will be different in type from relations between terms. This seems to give the truth—though in a thoroughly extensional form—underlying Frege's distinction between terms and the various kinds of functions. Moreover the opinion here advocated seems to adhere very closely indeed to common sense.

Thus the final conclusion is, that the correct theory of classes is even more extensional than that of Chapter VI; that the class as many is the only object always defined by a propositional function, and that this is adequate for formal purposes; that the class as one, or the whole composed of the terms of the class, is probably a genuine entity except where the class is defined by a quadratic function (see § 103), but that in these cases, and in other cases possibly, the class as many is the only object uniquely defined.

The theory that there are different kinds of variables demands a reform in the doctrine of formal implication. In a formal implication, the variable does not, in general, take all the values of which variables are susceptible, but only all those that make the propositional function in question a proposition. For other values of the variable, it must be held that any given propositional function becomes meaningless. Thus in $x \epsilon u$, u must be a class, or a class of classes, or etc., and x must be a term if u is a class, a class if u is a class of classes, and so on; in every propositional function there will be some range permissible to the variable, but in general there will be possible values for other variables which are not admissible in the given case. This fact will require a certain modification of the principles of Symbolic Logic; but it remains true that, in a formal implication, all propositions belonging to a given propositional function are asserted.

With this we come to the end of the more philosophical part of Frege's work. It remains to deal briefly with his Symbolic Logic and Arithmetic; but here I find myself in such complete agreement with him that it is hardly necessary to do more than acknowledge his discovery of propositions which, when I wrote, I believed to have been new.

493. *Implication and Symbolic Logic.* The relation which Frege employs as fundamental in the logic of propositions is not exactly the same as what I have called implication: it is a relation which holds between

* See Appendix B. † On this notation, see §§ 28, 97.

p and q whenever q is true or p is not true, whereas the relation which I employ holds whenever p and q are propositions, and q is true or p is false. That is to say, Frege's relation holds when p is not a proposition at all, whatever q may be; mine does not hold unless p and q are propositions. His definition has the formal advantage that it avoids the necessity for hypotheses of the form "p and q are propositions"; but it has the disadvantage that it does not lead to a definition of *proposition* and of negation. In fact, negation is taken by Frege as indefinable; *proposition* is introduced by means of the indefinable notion of a truth-value. Whatever x may be, "the truth-value of x" is to indicate the true if x is true, and the false in all other cases. Frege's notation has certain advantages over Peano's, in spite of the fact that it is exceedingly cumbrous and difficult to use. He invariably defines expressions for all values of the variable, whereas Peano's definitions are often preceded by a hypothesis. He has a special symbol for assertion, and he is able to assert for all values of x a propositional function not stating an implication, which Peano's symbolism will not do. He also distinguishes, by the use of Latin and German letters respectively, between *any* proposition of a certain propositional function and *all* such propositions. By always using implications, Frege avoids the logical product of two propositions, and therefore has no axioms corresponding to Importation and Exportation*. Thus the joint assertion of p and q is the denial of "p implies not-q."

494. *Arithmetic.* Frege gives exactly the same definition of cardinal numbers as I have given, at least if we identify his *range* with my *class*†. But following his intensional theory of classes, he regards the number as a property of the class-concept, not of the class in extension. If u be a range, the number of u is the range of the concept "range similar to u." In the *Grundlagen der Arithmetik*, other possible theories of number are discusssed and dismissed. Numbers cannot be asserted of objects, because the same set of objects may have different numbers assigned to them (Gl. p. 29); for example, one army is so many regiments and such another number of soldiers. This view seems to me to involve too physical a view of objects: I do not consider the army to be the same object as the regiments. A stronger argument for the same view is that 0 will not apply to objects, but only to concepts (p. 59). This argument is, I think, conclusive up to a certain point; but it is satisfied by the view of the symbolic meaning of classes set forth in § 73. Numbers themselves, like other ranges, are things (p. 67). For defining numbers as ranges, Frege gives the same general ground as I have given, namely what I call the principle of abstraction‡. In the *Grundgesetze der Arithmetik*, various theorems in the foundations of cardinal Arithmetic are proved with great elaboration, so great that it is often very difficult to discover the difference between successive steps in a demonstration. In view of the contradiction of Chapter X, it is plain that some emendation is required in Frege's principles; but it is hard to believe that it can do more than introduce some general limitation which leaves the details unaffected.

* See § 18, (7), (8). † See Gl. pp. 79, 85; Gg. p. 57, Df. Z. ‡ Gl. p. 79; cf. § 111 *supra*.

495. In addition to his work on cardinal numbers, Frege has, already in the *Begriffsschrift*, a very admirable theory of progressions, or rather of all series that can be generated by many-one relations. Frege does not confine himself to one-one relations: as long as we move in only one direction, a many-one relation also will generate a series. In some parts of his theory, he even deals with general relations. He begins by considering, for any relation $f(x, y)$, functions F which are such that, if $f(x, y)$ holds, then $F(x)$ implies $F(y)$. If this condition holds, Frege says that the property F is inherited in the f-series (Bs. pp. 55—58). From this he goes on to define, without the use of numbers, a relation which is equivalent to "some positive power of the given relation." This is defined as follows. The relation in question holds between x and y if every property F, which is inherited in the f-series and is such that $f(x, z)$ implies $F(z)$ for all values of z, belongs to y (Bs. p. 60). On this basis, a non-numerical theory of series is very successfully erected, and is applied in Gg. to the proof of propositions concerning the number of finite numbers and kindred topics. This is, so far as I know, the best method of treating such questions, and Frege's definition just quoted gives, apparently, the best form of mathematical induction. But as no controversy is involved, I shall not pursue this subject any further.

Frege's works contain much admirable criticism of the psychological standpoint in logic, and also of the formalist theory of mathematics, which believes that the actual symbols are the subject-matter dealt with, and that their properties can be arbitrarily assigned by definition. In both these points, I find myself in complete agreement with him.

496. Kerry (*loc. cit.*) has criticized Frege very severely, and professes to have proved that a purely logical theory of Arithmetic is impossible (p. 304). On the question whether concepts can be made logical subjects, I find myself in agreement with his criticisms; on other points, they seem to rest on mere misunderstandings. As these are such as would naturally occur to any one unfamiliar with symbolic logic, I shall briefly discuss them.

The definition of numbers as classes is, Kerry asserts, a ὕστερον πρότερον. We must know that every concept has only *one* extension, and we must know what *one* object is; Frege's numbers, in fact, are merely convenient symbols for what are commonly called numbers (p. 277). It must be admitted, I think, that the notion of *a term* is indefinable (cf. § 132 *supra*), and is presupposed in the definition of the number 1. But Frege argues—and his argument at least deserves discussion—that *one* is not a predicate, attaching to every imaginable term, but has a less general meaning, and attaches to concepts (Gl. p. 40). Thus *a term* is not to be analyzed into *one* and *term*, and does not presuppose the notion of *one* (cf. § 72 *supra*). As to the assumption that every concept has only one extension, it is not necessary to be able to state this in language which employs the number 1: all we need is, that if ϕx and ψx are equivalent propositions for all values of x, then they have the same extension—a primitive proposition whose symbolic expression in no way presupposes the number 1. From this it follows that if a and b are both extensions of ϕx, a and b are identical, which again does not formally involve the number 1. In like manner, other objections to Frege's definition can be met.

Kerry is misled by a certain passage (Gl. p. 80, note) into the belief that Frege identifies a concept with its extension. The passage in question appears to assert that the number of u might be defined as the concept "similar to u" and not as the range of this concept; but it does not say that the two definitions are equivalent.

There is a long criticism of Frege's proof that 0 is a number, which reveals fundamental errors as to the existential import of universal propositions. The point is to prove that, if u and v are null-classes, they are similar. Frege defines similarity to mean that there is a one-one relation R such that "x is a u" implies "there is a v to which x stands in the relation R," and vice versa. (I have altered the expressions into conformity with my usual language.) This, he says, is equivalent to "there is a one-one relation R such that 'x is a u' and 'there is no term of v to which x stands in the relation R' cannot both be true, whatever value x may have, and vice versa"; and this proposition is true if "x is a u" and "y is a v" are always false. This strikes Kerry as absurd (pp. 287—9). Similarity of classes, he thinks, implies that they have terms. He affirms that Frege's assertion above is contradicted by a later one (Gl. p. 89): "If a is a u, and nothing is a v, then 'a is a u' and 'no term is a v which has the relation R to a' are both true for all values of R." I do not quite know where Kerry finds the contradiction; but he evidently does not realize that false propositions imply all propositions and that universal propositions have no existential import, so that "all a is b" and "no a is b" will both be true if a is the null-class.

Kerry objects (p. 290, note) to the generality of Frege's notion of relation. Frege asserts that any proposition containing a and b affirms a relation between a and b (Gl. p. 83); hence Kerry (rightly) concludes that it is self-contradictory to deny that a and b are related. So general a notion, he says, can have neither sense nor purpose. As for sense, that a and b should both be constituents of one proposition seems a perfectly intelligible sense; as for purpose, the whole logic of relations, indeed the whole of mathematics, may be adduced in answer. There is, however, what seems at first sight to be a formal disproof of Frege's view. Consider the propositional function "R and S are relations which are identical, and the relation R does not hold between R and S." This contains two variables, R and S; let us suppose that it is equivalent to "R has the relation T to S." Then substituting T for both R and S, we find, since T is identical with T, that "T does not have the relation T to T" is equivalent to "T has the relation T to T." This is a contradiction, showing that there is no such relation as T. Frege might object to this instance, on the ground that it treats relations as terms; but his double ranges, which, like single ranges, he holds to be things, will bring out the same result. The point involved is closely analogous to that involved in the Contradiction: it was there shown that some propositional functions with one variable are not equivalent to any propositional function asserting membership of a fixed class, while here it is shown that some containing two variables are not equivalent to the assertion of any fixed relation. But the refutation is the same in the case of relations as it was in the previous case. There is a hierarchy of relations according to the type of objects constituting their fields. Thus relations between terms are distinct from those between classes, and these again are distinct from relations between relations. Thus no relation can have itself both as referent and as relatum, for if it be of the same order as the one, it must be of a higher order than the other; the proposed propositional function is therefore meaningless for all values of the variables R and S.

It is affirmed (p. 291) that only the concepts of 0 and 1, not the objects themselves, are defined by Frege. But if we allow that the range of a Begriff is an object, this cannot be maintained; for the assigning of a concept will carry with it the assigning of its range. Kerry does not perceive that the uniqueness of 1 has been proved (*ib.*): he thinks that, with Frege's definition, there might be several 1's. I do not understand how this can be supposed: the proof of uniqueness is precise and formal.

The definition of immediate sequence in the series of natural numbers is also severely criticized (p. 292 ff.). This depends upon the general theory of series set forth in Bs. Kerry objects that Frege has defined "F is inherited in the f-series," but has not defined "the f-series" nor "F is inherited." The latter essentially ought not to be defined, having no precise sense; the former is easily defined, if necessary, as the field of the relation f. This objection is therefore trivial. Again, there is an attack on the definition: "y follows x in the f-series if y has all the properties inherited in the f-series and belonging to all terms to which x has the relation f*." This criterion, we are told, is of doubtful value, because no catalogue of such properties exists, and further because, as Frege himself proves, following x is itself one of these properties, whence a vicious circle. This argument, to my mind, radically misconceives the nature of deduction. In deduction, a proposition is proved to hold concerning *every* member of a class, and may then be asserted of a particular member: but the proposition concerning *every* does not necessarily result from enumeration of the entries in a catalogue. Kerry's position involves acceptance of Mill's objection to Barbara, that the mortality of Socrates is a necessary premiss for the mortality of all men. The fact is, of course, that general propositions can often be established where no means exist of cataloguing the terms of the class for which they hold; and even, as we have abundantly seen, general propositions fully stated hold of *all* terms, or, as in the above case, of *all* functions, of which no catalogue can be conceived. Kerry's argument, therefore, is answered by a correct theory of deduction; and the logical theory of Arithmetic is vindicated against its critics.

Note. The second volume of *Gg.*, which appeared too late to be noticed in the Appendix, contains an interesting discussion of the contradiction (pp. 253—265), suggesting that the solution is to be found by denying that two propositional functions which determine equal classes must be equivalent. As it seems very likely that this is the true solution, the reader is strongly recommended to examine Frege's argument on the point.

* Kerry omits the last clause, wrongly; for not all properties inherited in the f-series belong to all its terms; for example, the property of being greater than 100 is inherited in the number-series.

APPENDIX B.

THE DOCTRINE OF TYPES.

497. THE doctrine of types is here put forward tentatively, as affording a possible solution of the contradiction; but it requires, in all probability, to be transformed into some subtler shape before it can answer all difficulties. In case, however, it should be found to be a first step towards the truth, I shall endeavour in this Appendix to set forth its main outlines, as well as some problems which it fails to solve.

Every propositional function $\phi(x)$—so it is contended—has, in addition to its range of truth, a range of significance, *i.e.* a range within which x must lie if $\phi(x)$ is to be a proposition at all, whether true or false. This is the first point in the theory of types; the second point is that ranges of significance form *types*, *i.e.* if x belongs to the range of significance of $\phi(x)$, then there is a class of objects, the *type* of x, all of which must also belong to the range of significance of $\phi(x)$, however ϕ may be varied; and the range of significance is always either a single type or a sum of several whole types. The second point is less precise than the first, and the case of numbers introduces difficulties; but in what follows its importance and meaning will, I hope, become plainer.

A *term* or *individual* is any object which is not a range. This is the lowest type of object. If such an object—say a certain point in space—occurs in a proposition, any other individual may *always* be substituted without loss of significance. What we called, in Chapter VI, the class as one, is an individual, provided its members are individuals: the objects of daily life, persons, tables, chairs, apples, etc., are classes as one. (A person is a class of psychical existents, the others are classes of material points, with perhaps some reference to secondary qualities.) These objects, therefore, are of the same type as simple individuals. It would seem that all objects designated by single words, whether things or concepts, are of this type. Thus *e.g.* the relations that occur in actual relational propositions are of the same type as things, though relations in extension, which are what Symbolic Logic employs, are of a different type. (The intensional relations which occur in ordinary relational propositions are not determinate when their extensions are given, but the extensional relations of Symbolic Logic are classes of couples.) Individuals are the only objects of which numbers cannot be significantly asserted.