The Theory of Relativity and A Priori Knowledge

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ing back and forth in his large office. When I told Einstein the next day about our experience and asked him whether he was able to have discussions with Bohr, he said: "No, I have given that up long ago. Either he talks too much or he does not listen."

My last anecdote concerns our visiting Einstein when, unknown to us, he was ill in bed. We wanted to leave right away, but he called down to his housekeeper to let us come up to his bedroom. There he lay, in a sky-blue T-shirt, his feet sticking out from the blankets and the bed covered with sheets of notes and formulas. Although we had made this appointment beforehand, we apologized, but he reassured us: "Only the belly is sick, the head is all right," and immediately plunged into a scientific discussion.

Einstein and Reichenbach had differences of opinion on the logical foundations of quantum physics, a topic that my husband on later occasions always strenuously tried to avoid and to which Einstein always came around by one ruse or another, but they got along amiably, perhaps because both of them were so natural and unpretentious.

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I

*Introduction**

Einstein's theory of relativity has greatly affected the fundamental principles of epistemology. It will not serve any purpose to deny this fact or to pretend that the physical theory changed only the concepts of physics while the philosophical truths remained inviolate. Even though the theory of relativity concerns only relations of *physical* measurability and *physical* magnitudes, it must be admitted that these physical assertions contradict general *philosophical* principles. The philosophical axioms, even in their critical form, were always formulated in such a way that they remained invariant with respect to specific interpretations but definitely excluded certain kinds of physical statements. Yet the theory of relativity selected exactly

* In regard to notes: the author's explanatory notes, which are not numbered, are printed as footnotes; so are the editor's notes, but the latter are numbered, the numbers being set in brackets. Finally, the author's reference notes, which are numbered consecutively, not chapter by chapter, will be found at the end of the book.

those statements that had been regarded as inadmissible and made them the guiding principles of its physical assumptions.

The special theory of relativity already made difficult demands upon the tolerance of a critical philosopher. It deprived time of its character of an irreversible process and asserted that events exist whose temporal succession may be assumed in the opposite direction. This interpretation contradicts previous concepts, including the concept of time held by Kant. Occasionally, philosophers have attempted to eliminate these difficulties through a distinction between "physical time" and "phenomenal time," by pointing out that time as *subjective experience* always remains an irreversible sequence. But this distinction is not in the Kantian tradition. For Kant, an essential trait of the a priori type of knowledge is that it constitutes a *presupposition of scientific knowledge* and not merely a subjective property of our sensations. Even though he speaks occasionally of the manner in which the objects "affect" our perceptions, he always believes that this subjective form is simultaneously an objective form of knowledge because the subjective component is necessarily contained in the concept of object. He would not have conceded that one could apply a time order to physical events which was different from that inherent in the nature of the knowing subject. It was, therefore, consistent when certain philosophical circles were already attacking the special theory of relativity by objections that had their roots in the logical constructions of Kant's philosophy.

The general theory of relativity has greatly increased these difficulties. It has asserted nothing less than that *Euclidean geometry is not applicable to physics*. One should clearly understand the far-reaching implications of this statement. Actually, for the last hundred years the a priori character of Euclidean geometry had no longer been taken for granted. The construction of non-Euclidean geometries had shown the possibility of conceptual systems contradicting the well-known, intuitively evident axioms of Euclid. Riemann had developed a general theory of manifold in analytic form which contained "plane" space as a special case. After Euclidean geometry had been deprived of its necessary character, its privileged character could be justified only if its *intuitive evidence* distinguished it from the other manifolds. This distinction became the only basis—in conformity with Kant—for the requirement that specifically this geometry ought to be applied to the description of reality, that is, in physics. Thus the refutation of Euclidean geometry was reduced to an objection to its purely *conceptual* justification. At the same time the empiricists expressed their doubt anew; from the possibility of constructing other geometries they wanted to derive that the theorems of Euclidean geometry had received their intuitively evident character merely through experience and familiarity. In the third place, mathematicians asserted that a geometrical system was established according to conventions and represented an empty schema that did not contain any statements about the physical world. It was chosen on purely formal grounds and might equally well be replaced by a non-Euclidean schema.[1] In the face of these criticisms the objection of the general theory of relativity embodies a completely new idea. This theory asserts simply and clearly that the theorems of Euclidean geometry do not apply to our physical space. This statement differs essentially from the other three points of view, which have in common that they do not question the validity of the Euclidean axioms and differ only with respect to the justification of this validity and its epistemological interpretation. It is obvious that thereby critical philosophy, too, is faced with a brand-new question. There is no doubt that Kant's transcendental aesthetics starts from the self-evident validity of the Euclidean axioms. Even though one might dispute whether Kant sees in their intuitive evidence the proof of his theory of a priori space, or, conversely, in the a priori character of space the proof of their evidence, it remains quite certain that his theory is incompatible with the *invalidity* of these axioms.

Therefore, there are only two possibilities: either the theory of relativity is false, or Kant's philosophy needs to be modified in those parts which contradict Einstein.[2] The present study is devoted to the investigation of this question. The first possibility appears to be very doubtful because of the tremendous success of the theory of relativity, its repeated empirical confirmation and its fertility for the formation of theoretical concepts. Yet we do not want to accept this physical theory unconditionally, especially since the epistemological interpretation of its statements is

still so much under discussion. We shall, therefore, choose the following procedure. First, we shall establish the contradictions existing between the theory of relativity and critical philosophy and indicate the assumptions and empirical data that the theory of relativity adduces for its assertions.[3] Subsequently, starting with an analysis of the concept of knowledge, we shall investigate what assumptions are inherent in Kant's theory of knowledge. By confronting these assumptions with the results of our analysis of the theory of relativity, we shall decide in what sense Kant's theory has been refuted by experience. Finally, we shall modify the concept of a priori in such a way that it will no longer contradict the theory of relativity, but will, on the contrary, be confirmed by it on the basis of the theory's own concept of knowledge. The method of this investigation is called the method of logical analysis.

II

The Contradictions Asserted by the Special Theory of Relativity

In the present as well as in the following chapter we shall use the term "a priori" in Kant's sense; that is, we shall call a priori what the forms of intuition or the concept of knowledge require as self-evident. We are doing this in order to arrive at exactly those contradictions that occur with respect to a priori principles; for, of course, the theory of relativity contradicts many other principles of traditional physics. This characterization as a priori is, however, not supposed to function as a proof of the *validity* of these principles.[4]

In the special theory of relativity—which may still be held to be valid for homogeneous gravitational fields—Einstein states that the Newtonian-Galilean relativity principle of mechanics is incompatible with the principle of the constancy of the velocity of light

unless, in addition to the transformation of the spatial coördinates, a time transformation is performed which in turn leads to the relativization of simultaneity and the partial reversibility of time. This contradiction certainly exists. We ask: What assumptions support Einstein's principles?

Galileo's principle of inertia is an empirical statement. It is not intuitively obvious why a body that is not affected by a force should move uniformly. If we had not become so accustomed to this idea, we would at first probably assert the opposite. According to Galileo, the stationary state is also free of forces; but this implies the far-reaching assertion that uniform motion is mechanically equivalent to the state of rest. A force is defined in terms of physical relations. It is not a priori evident, however, that a force occurs only if it is accompanied by a *change* of velocity, that is, that the phenomena which we call the effects of a force are dependent upon the occurrence of *acceleration*. With this interpretation Galileo's principle of inertia is undoubtedly an empirical statement.

But this principle can be formulated in another way: a certain group of coördinate systems, that is, all those moving uniformly relative to one another, are equivalent descriptions of the mechanical process. The laws of mechanics do not change their form when transformations are made from one system to another. But in this form the statement is much more general than in its first form. The laws of mechanics can retain their form even when the dynamic magnitudes change. The preservation of the form requires merely that the forces in the new system be derived from the coördinates in the same way as in the old system, that is, that the *functional connection* remain unchanged. This assertion is more fundamental than Galileo's statement. The principle of inertia, the equal status of uniformly moving systems, appears now as a special case, because those coördinate transformations are indicated in view of which the preservation of the functional relationship is obtained specifically by means of the preservation of the *dynamic magnitudes.* Only experience can teach whether such transformations exist and what they are. But the fact that the physical *law,* and not only the *force,* is supposed to be invariant relative to the coördinate transformations is justified more fundamentally. This principle requires, in other words, that space have no physical properties, that the law be a function of the distribution and the nature of *masses,* and that the choice of the reference system have no influence upon the process. From the Kantian point of view, according to which space and time are only forms of order and not part of nature such as matter and forces, this principle is actually obvious. It is strange that philosophers have not long ago pointed out in objection to Galileo's and Newton's laws and also to the special theory of relativity, that the postulated invariance is not sufficient. There is no reason for the philosopher to single out the uniform translation. As soon as space is characterized as a scheme of order and not as a physical entity, all arbitrarily moving coördinate systems become equivalent for the description of events. Mach seems

to have been the only one who expressed this idea clearly. But he was not able to translate it into a physical theory. And nobody protested that Einstein's special theory of relativity was not radical enough. Only Einstein himself made this objection against his own theory, afterward showing the way to carry through a genuine, general covariance. According to its fundamental principles, Kantian philosophy would always have required the relativity of the coördinates. The reason that it did not do so and did not anticipate the consequences that were implicitly contained in this requirement lies in the fact that experimental physics had to make the discovery of a second fundamental requirement that was too far removed from speculative philosophy to be detected by it.

The constancy of the velocity of light represents the physical form of the second requirement. The physicists had discovered it by observation; but when Einstein made it the fundamental principle of his special theory of relativity in his famous first publication,[5] he could already show its significance in a more profound respect.

Einstein suggested that the definition of synchronous time at every point of a chosen coördinate system necessitates a physical process spreading with a certain velocity and permitting a comparison of clocks at different points. Subsequently a hypothesis must be formulated about the state of motion of this process relative to the coördinate system. The time of the coördinate system and the simultaneity at distant points depend on this hypothesis. Yet it is impossible to determine this state of motion; such a determination presupposes a time definition. Experiments either would show which time definition had been used or would lead to contradictions with the consequences of the hypothesis. These experiments would thus make a negative selection. There is, therefore, a certain arbitrariness contained in any "coördinate time." This arbitrariness is reduced to a minimum if the speed of propagation of the process is assumed to be constant, independent of the direction, and equal for all coördinate systems.

It is not necessarily the case that this *simplest* assumption is also *physically admissible.* For instance, if the irreversibility of causal processes is retained (principle of irreversible causality), the assumption leads to the result that there is no velocity higher than the chosen one; among all known velocities, therefore, the highest should be chosen if it is to be suitable for a definition of time. This is the reason that the speed of light was suitable to take the role of this particular velocity. Furthermore, it had to be determined whether the time defined by this velocity coincides with the time defined by the mechanical laws of the celestial bodies, that is, whether the simple formulas of mechanics representing fundamental laws do not suggest the existence of an even higher unknown velocity. The Michelson experiment that demonstrated the constancy of the velocity of light for all systems could be regarded as decisive in this respect. Nevertheless, it remained an open question whether some day ob-

servations might be made that would make it impossible to base the definition of time on such a simple assumption as the constancy of a velocity. Such observations actually occurred, yet only after theoretical considerations had rejected the special theory of relativity: the deflection of light in the gravitational field of the sun observed during the last eclipse of the sun shows that the simplest definition of time cannot always be carried through. The special theory of relativity was thereby reduced to the special case of a homogeneous gravitational field.

These considerations show the empirical foundations of the concept of time in the special theory of relativity. But beyond the empirical foundation stands Einstein's profound idea *that a definition of time is impossible without a physical hypothesis concerning certain velocities of propagation.* Even the traditional definition of absolute time appears only as a special case of this view: it contains the hypothesis that there exists an action that spreads with infinite velocity.

This relation is particularly noteworthy. An objection to Einstein was that his considerations merely show that the physicist can never arrive at a precise "absolute" time with his restricted means; the idea of such a time, however, and its progressively approximate measurement would nevertheless have to be retained. This objection is false. "Absolute" time requires a process spreading with infinite velocity. Such a process would contradict our concept of causal action. Many philosophers have made the requirement that action at a distance may not be assumed. Action at a distance is equivalent to an infinitely fast action between two distant points. If it is assumed that the propagation of a force takes a finite time and that this time increases with distance, that propagation can be imagined as traveling from point to point, that is, as action by contact. Whether one speaks of an ether medium in this context is a matter of terminology. The principle of action by contact can just as well be called an a priori principle as Kant could call the principle of the permanence of substance a priori. In any case, the exact determination of absolute time is excluded by an a priori principle. At best one might want to retain the possibility of a successive approximation to absolute time. But in such a case there can be no upper limit for physically possible velocities. This is a purely physical question, and nothing can be said about it a priori. If the energy needed for the production of a certain determined finite velocity would have to be infinite in the first place, however —and all experimental investigations concerning the theory of relativity have shown that—then the production of arbitrarily selected higher velocities is certainly impossible. This fact is not derivable from the old formulas; these formulas were discovered empirically, and the theory of relativity could justifiably replace them by others in which, say, the kinetic energy of a mass point becomes infinite when approaching the velocity of light. Just as it is physically impossible to increase the energy of a closed system or to go beyond a certain lower limit of temperature by in-

creasing refrigeration,* so an unlimited increase of velocity beyond a certain point may be physically impossible. Both are *logically* possible, but here we are concerned with what is *physically* possible. If a physical law exists that prescribes an upper limit to velocities, then even an approximation to "absolute" time is impossible, let alone the attainment of the ideal state. It no longer makes sense to assume an "ideal time," for we ought to establish only those ideal requirements that are at least attainable through increasing approximation and so may have some significance for the physical world.[6]

Let us summarize our discussion. The principle of the relativity of all coördinate systems, even if restricted to a certain class of coördinates (i. e., to systems moving uniformly relative to one another), and the principle of action by contact admit an absolute time only if no upper limit exists for physically attainable velocities. According to the traditional meaning of the term, both principles may rightly be called a priori. However, the question of an upper limit for physically attainable velocities is an empirical problem of physics. Therefore, the definition of time is also dependent upon empirical facts so long as the principle

* One should not make the objection that a lower limit of temperature is intuitively necessary because the motion of the molecules must eventually cease. How do I know that the zero point of kinetic energy has been reached at a finite negative temperature rather than at an infinite negative temperature? Only from experience. It is, therefore, also possible that an infinite kinetic energy may be reached at a finite velocity.

is retained that a yardstick may be chosen as norm only if it can be approximated empirically (principle of the approximate ideal). Einstein's discovery that the time of a coördinate system can be defined only by means of a physical process of propagation constitutes the connecting link between these considerations.

If the requirement of absolute time is also called an a priori principle, the result is a contradiction of several a priori principles, or more precisely, a contradiction of these principles in their entirety with experience. The assumption of an absolute time, however it be defined, always implies the possibility of arbitrarily high, physically attainable velocities. It is probably impossible to give an exact experimental demonstration for the fact that the velocity of light cannot be exceeded. We must infer from observations of smaller velocities that the velocity of light represents the limiting velocity. We observe, for instance, that the kinetic energy becomes infinite when the motion of electrons approaches the velocity of light. We cannot make observations of the velocity of light itself and must, therefore, always rely on extrapolations. Even the Michelson experiment is a proof only if very intricate theories for the retention of the familiar theorem concerning the addition of velocities are rejected. Any extrapolation has only a certain degree of probability. Let us call the principle of using the most probable extrapolation of observational data the *principle of normal induction.* However, the concept of most probable extrapolation contains an indeterminateness. It might be contended that extrapolations

leading to contradictions with certain general assumptions are impossible and ought to be excluded from the selection of the most probable extrapolation. Yet there are borderline cases in which such a procedure contradicts the requirement of evidence. Let us assume that the kinetic energy of the electron is experimentally determined for velocities from 0–99 per cent of the velocity of light and graphically represented by a curve that at 100 per cent will obviously fit an asymptote. No one will maintain that the curve will make a salient point between 99 per cent and 100 per cent, and go to infinity only at infinitely high velocities. Actually, the constancy of the velocity of light, based on existing experimental data, including the Michelson experiment, is not less probable than the given example. At this point, we restrict ourselves to a mere illustration of the principle of normal induction in order to show its a priori character in the sense of the criterion of self-evidence. In chapter IV we shall consider in more detail the epistemological status of this principle.

According to the special theory of relativity, we assert that the following principles in their totality are incompatible with experimental observations:

the principle of the relativity of uniformly moving coördinates;
the principle of irreversible causality;
the principle of action by contact;
the principle of the approximate ideal;
the principle of normal induction;
the principle of absolute time.

All of these principles can be called a priori with justification even though Kant did not call all of them a priori, for they all possess the criterion of self-evidence to a high degree and represent fundamental assumptions that have always been made in physics. We mention this property only to show that the stated contradiction changes from a physical to a philosophical problem. If there should be any resistance to our view and should the self-evidence of some of these principles, for instance that of action by contact, be disputed, the justification of our assertion will not be affected. These principles may also be regarded as empirical statements, in which case the principle of normal induction, which we mention separately in the above list, will be implied by them.

It should be noted that the assumptions of the special theory of relativity do not contradict the *principle of causality*. On the contrary, causality attains a special distinction: those temporal sequences that are to be regarded as causal chains are irreversible. In this way causality orders time sequences objectively, whereas by itself the time schema has no objective order relations.

Minkowski has formulated Einstein's idea in a way that makes it much clearer. He defines an x_4 coördinate by $x_4 = ict$ and derives the Lorentz transformation from the requirement that the line element of the four-dimensional manifold

$$ds^2 = \sum_1^4 dx_\nu^2$$

is to be invariant, that is, that the transformations are not to destroy this simple expression for the line element. This assertion contains the principle of the relativity of all uniformly moving systems as well as the principle of the constancy of the velocity of light. The two requirements can therefore be combined in the requirement of the *relativity of all orthogonal transformations of the Minkowski-world.* The constancy of the velocity of light will automatically be contained in it. This velocity is the factor of the unit of measure by which the time measured in seconds must be multiplied in order to become equivalent to the spatial axes measured in centimeters and to be combined with them in a symmetrical fourfold system. It would contradict the four-dimensional relativity if this factor were different for the individual systems.

It should be noted, however, that Minkowski's principle is only a more elegant and fruitful formulation of Einstein's idea. The principle does not change the physical and philosophical content of Einstein's idea. It does not require a modification of our view of space, because the introduction of the fourth coördinate is a purely formal device. Nor does it assert the interchangeability of space and time, as has occasionally been suggested. On the contrary, spacelike and timelike vectors in Minkowski's world are fundamentally different and cannot be transformed into one another by any physically possible transformation.

Still to be investigated are to what extent the general theory of relativity has changed the assumptions of the special theory and whether our formulations can be maintained if the discoveries of the general theory are assumed to be known. The principle of the constancy of light, which played such an important role in our considerations, has been displaced by the new theory.

According to Einstein's second theory, special relativity holds only for the special case of homogeneous gravitational fields; for all other fields, for instance the central fields of our planetary system, such a simple assumption as that of the constancy of the velocity of light cannot be used. Consequently the special theory is limited to extremely limited domains, for fields in which the field strength is approximately homogeneous and equidirectional throughout are realized only in small dimensions and will hardly extend beyond the range of human vision. If the simultaneity of two events in a larger coördinate system characterized by central gravitational fields is to be defined, a more complicated assumption for the propagation of light must be made. According to this assumption, the light ray describes a curved path the various parts of which it travels with different velocities. Again simultaneity will depend on the choice of the coördinates and will have merely relative significance; thus the contradiction with the old view remains. But if velocities higher than $c = 3 \cdot 10^{10}$cm/sec are admitted for light itself, the question arises whether the character of this velocity as an upper limit has not been abandoned.

This is not the case. Even in gravitational fields the velocity of light is the limiting velocity although its numerical value is different. There are no physical

processes that travel with a velocity greater than that of light. For every element of volume of space, the velocity c has a certain numerical value that cannot be exceeded by any physical process. This numerical value has all the properties of the previously used constant $c = 3 \cdot 10^{10}$ if the inertial system is determined for the element of volume. Even though the upper limit of all velocities changes its numerical value from place to place, there always remains an *upper limit.* Our previous considerations and the asserted contradiction of a priori principles apply, therefore, to every element of volume. A time definition according to the model of the special theory of relativity can be carried through only for such elements.

Nevertheless, one more objection can be made. It was essential to our considerations that one cannot even speak of a *gradual approximation* to absolute time, that this concept cannot be retained in the sense of an ideal that though unattained is progressively satisfiable. Is it not at least possible, from the point of view of the general theory, to coördinate an arbitrarily large number $c > 3 \cdot 10^{10}$ to the element of volume so that absolute time can be approximated to an arbitrary degree of exactness?

This is not possible. The number c for the chosen element of volume depends on the distribution of the masses in the universe, and it would increase its value only if the total mass density in the universe should increase. However, we do not want to exclude such a change from physical possibility. The essential fact is, rather, that with such a change the state of the element of volume would also change; all clocks and measuring rods in the element of volume would experience a non-Euclidean deformation with the result that the earlier measurement of time could not be compared with the later one. It would make no sense, even if we could carry out such a change of mass density, to regard the measurement of time with the larger constant c as an increase in exactness relative to the previous one. The fact that the constant c has a greater value always expresses a relation to a unit clock; but if the clock itself is affected by a change, the comparison with the earlier state has lost its meaning. It appears to be convenient to hold the value of c constant, for instance, to set $c = 1$ for all inertial systems (as is frequently done) and to determine by means of this definition the change of the clocks.

We note the difference of these relations from other physical processes. If precision is increased in a physical arrangement, this is always possible without a fundamental change in the arrangement itself; only certain parts of the arrangement are changed. If a projectile is used as a signal, then for the purpose of increasing precision, its velocity can be increased by an increase in the powder charge; this change has no influence upon the state of space. The magnitude c, however, is not a function of certain individual processes, but the expression of a *universal state,* and all measuring methods are comparable only within this state. There remains the fact that within every universal state there exists an upper limit c for every element of volume. Therefore, the contradiction men-

tioned above prevails even if the special theory of relativity is incorporated as a special case into the general theory.

We add this analysis only to show that the general theory did not give up the epistemological principle of the special theory. The *validity* of the general theory is a special problem to be analyzed in the following chapter.

III

The Contradictions Asserted by the General Theory of Relativity

We shall now consider the general theory of relativity. It asserts that physical reality must not be assumed to be Euclidean. We ask: What are the principles and experiences used to justify the theory? Why is the assumption of Euclidean space called false?

Einstein says in his fundamental work: "I do not intend in this treatise to present the general theory of relativity in its simplest logical form with a minimum of axioms. My main aim is to develop this theory in such a way that the reader will find the reasoning intuitive, and that the fundamental presuppositions are grounded as far as possible in experience." [7]

This kind of justification is natural for the physicist because he aims not at a rigid preservation of philosophical principles but at a close correspondence of his logical constructions to physical reality. The

philosopher, on the other hand, must demand justification for the abandonment of principles so fundamental as those contained in Euclidean geometry. By following this maxim of justifying the theory we shall discover that Einstein's presentation actually gives a more profound justification than that claimed in the above quotation.

We have already stressed in our discussion of the special theory of relativity that the general relativity of all coördinate systems is an obvious requirement of critical philosophy, and thus there is no need to consider it again. We ask, however: Why does this requirement lead to a rejection of Euclidean space?

Let us imagine a large, homogeneous gravitational field containing an inertial system. In this coördinate system the gravitational field is equal to zero at every point. We know that the four-dimensional line element

$$ds^2 = \sum_1^4 dx_\nu^2$$

is expressed as the sum of squares of the coördinate differential. If we now introduce new coördinates by means of an arbitrary substitution, say a system accelerating relative to the inertial system, the line element will not preserve its simple form but will change into a mixed quadratic expression:

$$ds^2 = \sum_1^4 g_{\mu\nu} dx_\mu dx_\nu.$$

According to Gauss and Riemann, this expression is characteristic of a non-Euclidean geometry.* The coefficients $g_{\mu\nu}$ occurring in it manifest themselves in the acceleration of the second coördinate system relative to the inertial system; since this acceleration directly characterizes the gravitational field of the second system, we may regard it as a measure of this gravitational field. We notice, therefore, that the transition from a gravity-free field to a gravitational field is connected with a transition to non-Euclidean coördinates, and that the metric of these coördinates is a measure of the gravitational field. Einstein inferred from this that *every* gravitational field, not only that produced by transformation, manifests itself by a deviation from Euclidean geometry.

We are dealing, therefore, with an extrapolation. Extrapolations can always be performed in different ways, and we shall ask what specific principles have led to the Einsteinian extrapolation.

Let us have a closer look at the gravitational field described above. Our example demonstrates that the requirement of general relativity leads to non-Euclidean coördinates that must be accepted on an equal

* We are retaining the conventional meaning of "Euclidean" for the four-dimensional manifold. Although the following considerations apply to the four-dimensional space-time manifold, they apply also to the three-dimensional space defined by this manifold; if the former shows a Riemannian curvature, the latter is necessarily curved, and if the former is Euclidean, the latter can always be given a Euclidean form. For the analogy between these two manifolds cf. note 3 (Erwin Freundlich, *Die Grundlagen der Einsteinschen Gravitationstheorie* [Berlin: Julius Springer, 1920], pp. 29 ff.).

basis with Euclidean ones. But the non-Euclidean space-time manifold originating in this way has a special distinction: coördinates can be chosen in this manifold in such a way that the line element will be Euclidean at every point. This result represents a far-reaching restriction for the non-Euclidean coördinate system; it follows, for instance, that the Riemannian measure of curvature of this system will be zero at every point. Such a space is only apparently non-Euclidean; actually it does not differ structurally from Euclidean space. On the other hand, the three-dimensional Euclidean space can be expressed in terms of non-Euclidean coördinates. One need only choose any curved oblique coördinates, and the line element will become a mixed quadratic expression. Even the ordinary polar coördinates furnish an expression differing from the pure quadratic sum for the line element. If their intuitive aspect is disregarded and if they are treated as a three-axial manifold similar to the three axes of space, they represent a non-Euclidean space. The representation of Euclidean space by means of polar coördinates can be conceived as a projection upon a non-Euclidean space. The measure of the curvature remains zero.

The chosen example shows merely the equivalence of pseudo-non-Euclidean spaces to Euclidean ones. Since Einstein's theory asserts the need of genuine non-Euclidean coördinates upon transition from homogeneous gravitational fields to arbitrary inhomogeneous fields, his theory transcends in essential respects the content of our example. His theory states that in general it is not possible to make the coördinates Euclidean. We are dealing, therefore, with a far-reaching extrapolation. A theory that would permit a transformation upon Euclidean coördinates even in the general case, that is, one in which a mass-filled space would retain a zero curvature, might seem more plausible.

Einstein's own example of a rotating circular disk[8] does not show the necessity for the far-reaching extrapolation. It is true that an observer stationed on and rotating with the disk would obtain a value larger than π for the quotient of circumference and diameter of the disk; for him and the co-rotating coördinate system, the geometry would be non-Euclidean. But the observer would soon discover that the metrical results could be simplified if he would introduce a (seen from him) rotating system—that is, a system rotating with equal velocity in the direction opposite to that of the disk and therefore remaining stationary relative to the surrounding plane—and that relative to this reference system, he could describe all events in Euclidean geometry. He could also define a synchronous time for this system (which is not possible for the disk itself). For him this reference system could play a role similar to the inertial system of the sun system assumed by the astronomers for the Newtonian equations. The geometry of the rotating circular disk is, therefore, also pseudo-non-Euclidean; its measure of curvature is equal to zero.

The question is whether a theory of gravitation with a less far-reaching extrapolation than Einstein's is possible. We shall make the following requirements for it:

(a) for homogeneous fields, the theory should become equivalent to the special theory of relativity;

(b) the theory should permit under all circumstances the choice of Euclidean coördinates.

Such a theory is indeed possible; the two requirements do not contradict each other. For instance, the coördinate system defined according to requirement (b) could be produced by means of measuring the field strength at every point of the field, of calculating the mean value of all field strengths, and of determining that system in which the mean becomes a minimum. For a constant field strength, that is, for a homogeneous field, the mean would be equal to the constant field strength. It would thus be a minimum in that system in which the field strength is equal to zero. This system would be the inertial system. In this way the general theory would be connected with the special case of the homogeneous field and the special theory of relativity. Of course, the hypothesis assumed for the special system would have to be tested by experience. It should be noted that such a system so distinguished does not contradict the relativity of the coördinates. It is a matter of course and not a physical singularity that space is expressed differently in different systems. The homogeneous gravitational field is also characterized by the Euclidean system.

However, requirement (a) is not the one chosen by Einstein, although he also insists on a successive approximation of his theory to the special theory. Requirement (a) achieves this approximation by letting the field strengths become equal to one another at the different points while *keeping the spatial domains constant*. There exists, however, another form of approximation. The field strength is regarded as a continuous function of space; in such a case, infinitesimal domains of the field are homogeneous. We can, therefore, attain a transition to the homogeneous field by letting the spatial domain become smaller and smaller while *retaining the strength of the field*. We can achieve this transition at every point of the field and shall, therefore, make the following Einsteinian assumption for the extrapolation:

(c) at every point of the field, the theory should pass into the special theory of relativity for infinitesimal domains.

We ask: are requirements (b) and (c) compatible?

Let us imagine a small domain G_1 in an inhomogeneous gravitational field that may be regarded as sufficiently homogeneous. In this domain we can choose an inertial system K_1 in which the field strength disappears. The system which according to requirement (b) is Euclidean at every point of the field must therefore belong to the family of systems moving uniformly in translatory fashion relative to K_1, since otherwise it could not be Euclidean for G_1. I shall apply the same consideration to a second distant domain G_2 in which the field strength has a value different from that in G_1. The inertial systems K_2 in G_2 must have an accelerated motion relative to K_1 and therefore do not belong to the family of the inertial systems in G_1. For

the system according to requirement (b) to become Euclidean at both points, it would have to belong to the family K_1 as well as to the family K_2; but that is a contradiction. Therefore, requirement (c) is incompatible with requirement (b).

This analysis shows that the Euclidean character of space must be given up if, by extrapolation according to Einstein's requirement (c), a transition is made from the special theory to the general theory of relativity. It is therefore not possible in a given gravitational field to choose the coördinates in such a way that the line element becomes Euclidean at all points simultaneously; the degree of curvature of a mass-filled space is not equal to zero.

As mentioned above, requirement (c) depends, on the one hand, upon the continuity of the gravitational field. Since continuity is not only a property of gravitation, but is generally presupposed for physical magnitudes, we can speak of a principle of continuity of physical magnitudes. On the other hand, requirement (c) depends on the fact that the properties of small spatial domains are not different from those of large domains, that is, that space is *homogeneous*. Only on this assumption may we require the special theory of relativity to hold for arbitrarily small domains if the strength of the gravitational field remains approximately constant. If we did not presuppose the homogeneity of space, the error stemming from the reduction of the domain might just compensate the influence of the reduced fluctuation of the field strength in the domain and prevent an approximation to the special theory of relativity. In this case, passing to the limit would be admissible only according to requirement (a). In the third place, requirement (c) depends upon Einstein's principle of equivalence, because (c) says that *every* homogeneous gravitational field, whether a field of gravity or a field of inertia, can be transformed into a force-free field. This foundation of requirement (c) is purely empirical. The principle of equivalence asserts the equivalence of gravitational and inertial mass for *every* gravitational field, and this assertion can be tested only experimentally. Until now this experiment could be made only in the field of the earth. But the general equivalence can be inductively inferred from this experiment.

One might call the continuity of physical magnitudes and the homogeneity of space a priori evident principles in the Kantian sense. Reversing the relation we might say that these two a priori principles permit a renunciation of requirement (c) only if inertial and gravitational mass are generally not equivalent. This idea would be equivalent to rejecting normal induction in the interpretation of the relevant observations made up to now. Since requirement (c) contradicts the Euclidean nature of space, the Euclidean nature of space, in combination with the other principles, demands the rejection of normal induction in connection with the problem of equivalence. If we call the requirement that the general theory converge toward the special theory for the special case, *the continuity of laws*, and understand by the principle of special relativity the total content of the special theory of relativity as a theory of the force-free field, we can say that the general theory of relativity has shown the following principles *in their totality to be incompatible with experience:*

the principle of special relativity;
the principle of normal induction;
the principle of general covariance;
the principle of the continuity of laws;
the principle of the continuity of physical magnitudes;
the principle of the homogeneity of space;
the principle of the Euclidean character of space.

The totality of these principles is incompatible with the observational fact that in the gravitational field of the earth, inertial and gravitational mass are equal. Yet all these principles, with the exception of the first, are a priori in the Kantian sense. But it is the first principle that solves the contradiction represented in the corresponding list of the previous chapter.[1]

[1] A later publication by the author contains a correction of these considerations. Reichenbach writes: ". . . the aprioristic philosopher cannot be prevented from retaining Euclidean geometry, a consequence which follows from the relativity of geometry. However, under the circumstances mentioned he faces a great difficulty. He can still retain Euclidean geometry, but he must renounce normal causality as a general principle. Yet for this philosopher causality is another *a priori* principle; he will thus be compelled to renounce one of his *a priori* principles. He cannot deny that facts of the kind we described could actually occur. We made it explicit that in such a case we would deal with perceptions which no *a priori* principle could change. Hence there are conceivable circumstances under which two *a priori* requirements postulated by philosophy would contradict each other. This is the strongest refutation of the philos-

We have, therefore, discovered the basis for rejecting a Euclidean interpretation of space. Finally, we have to say something about the special character possessed even by the Einsteinian space.

It is not correct to say that Euclidean space is no longer singled out in Einstein's theory. A preference still lies in the assumption that infinitesimal domains are Euclidean. Riemann calls this property "planeness in the smallest elements." Analytically it is expressed in the mixed quadratic form of the line element. It follows from this form that it is always possible to choose coördinates in such a way that in a single point the line element appears as a pure quadratic sum. A coördinate system, therefore, always can be chosen in such a way that it will be Euclidean for an arbitrarily given domain of points. This means, physically speaking, that for an infinitesimal domain the gravitational field can always be "transformed away," whatever the character of the field may be in other respects; there exists no essential difference between static gravitational fields and those produced by transformation. This is the content of Einstein's hypothesis of the equivalence of inertial and gravitational mass. Conversely, this hypothesis is the reason for the quadratic form of the line element and the *physical* basis for the planeness in the smallest elements. If the physical relations were different, a different differential expression, perhaps of fourth degree,

ophy of the *a priori*." H. Reichenbach, *The Philosophy of Space and Time* (New York: Dover Publications, 1958), p. 67; cf. also note on p. 67.

would have to be chosen for the line element. Under these circumstances Euclidean space would lose any privileged position.

The special position of the mixed quadratic form of the line element can also be characterized in the following way. The ten functions $g_{\mu\nu}$ determining the metric are not absolutely fixed, but depend on the choice of the coördinates. They are not independent of one another, however, and if four of them are given, the coördinates as well as the other six functions are determined. This dependence expresses the absolute character of the curvature of space. The metric functions $g_{\mu\nu}$ are *not* relative; that is, their choice is not arbitrary. Another relativity can be indicated, however. If ten arbitrary numbers are given, a coördinate system can always be chosen in such a way that the metric coefficients at any arbitrarily given point will exactly equal these ten numbers. (At the other points they, of course, will not be arbitrary.) This property may be called "relativity of the metric coefficients"; it says that for a given point the metric coefficients are not absolute. It easily can be shown that this relativity holds only for the mixed quadratic line element; for other forms, for instance, the differential expression of the fourth degree, an arbitrary choice of numbers is not possible. With the relativity of the metric coefficients, Einstein's theory has introduced an additional arbitrary element into the description of nature. We are stressing this fact because this principle of relativity, in particular, exhibits an empirical foundation, the equivalence of inertial and gravitational mass.

IV

Cognition as Coördination

Before we offer an analysis of the contradictions between Kant's conception of physics and the theory of relativity, we shall develop a theory of the physical concept of cognition and try to formulate the meaning of "a priori."

It is characteristic of modern *physics* to represent all processes in terms of *mathematical* equations. But the close connection between the two sciences must not blur their essential difference. The truth of mathematical propositions depends upon internal relations among their terms; the truth of physical propositions, on the other hand, depends on relations to something external, on a connection with experience. Usually, this fact is expressed by the ascription of absolute certainty to mathematical propositions and of probability to physical propositions. This distinction is due to the difference in the objects of knowledge of the two sciences.

The *mathematical object* of knowledge is uniquely determined by the axioms and definitions of mathematics. The definitions indicate how a term is related to

previously defined terms. The mathematical object receives meaning and content within this framework of definitions through an analysis of its differences from and equivalences to other mathematical objects. The axioms indicate the mathematical rules according to which concepts are to be defined. Even the fundamental concepts occurring in the axioms are defined through such relations. When Hilbert[9] includes among his axioms of geometry the proposition: "Among any three points of a straight line there is always one and only one point lying between the two other points," he is defining the properties of points as well as of those of straight lines and of the relation "between." Hilbert's proposition is not an *exhaustive* definition; it is made complete by the totality of the axioms. Hilbert's points and straight lines are those entities possessing the properties stated in the axioms. If the symbols *a, b, c,* . . . were substituted for the words "point," "straight line," "between," and so forth, the geometry would not change. This fact is most clearly expressed in projective geometry whose theorems for the plane remain correct if the concepts, point and straight line, are interchanged. Their axiomatically defined relations are symmetrical for the two concepts. Although our intuition invests the two concepts with different content and consequently ascribes different contents to the axioms, the conceptual symmetry is expressed in the fact that the theorem resulting from the interchange is also correct, even intuitively, although its intuitive meaning has changed. This peculiar mutuality of mathematical definitions, in which one concept always defines another without the need of referring to "absolute definitions," has been clearly stated by Schlick[10] in the theory of implicit definitions. This method of giving definitions is to be distinguished from the scholastic method of giving definitions in terms of higher class and specific difference.

Under these circumstances it is not surprising that mathematical propositions are absolutely certain. They merely represent new combinations of known concepts according to known rules. The only surprising thing perhaps is that the human mind, a very imperfect instrument, can make the inferences. But this is a different problem. Schlick invented the instructive example of the calculating machine that can make logical inferences, yet is a physical machine with all the imperfections of a physical thing.

The *physical object* cannot be determined by axioms and definitions. It is a thing of the real world, not an object of the logical world of mathematics. Offhand it looks as if the method of representing physical events by mathematical equations is the same as that of mathematics. Physics has developed the method of defining one magnitude in terms of others by relating them to more and more general magnitudes and by ultimately arriving at "axioms," that is, the fundamental equations of physics. Yet what is obtained in this fashion is just a system of mathematical relations. What is lacking in such a system is a statement regarding the significance of physics, the assertion that the system of equations is *true for reality*. This relation is totally different from the internal coherence of mathematics. The physical relation can be conceived as a coördination: physical things are coördinated to equations. Not only the totality of real things is coördinated to the total system of equations, but *individual* things are coördinated to *individual* equations. The real must always be regarded as given by some perception. By calling the earth a sphere, we are coördinating the mathematical figure of a sphere to certain visual and tactile perceptions that we call "perceptual images of the earth," according to a coördination on a more primitive level. If we speak of Boyle's gas law, we coördinate the formula $p \cdot V = R \cdot T$ to certain perceptions, some of which we call direct perceptions of gases (such as the feeling of air on the skin) and some of which we call indirect perceptions (such as the position of the pointer of a manometer). The fact that our sense organs mediate between concepts and reality is inherent in human nature and cannot be refuted by any metaphysical doctrine.

The coördination performed in a physical proposition is very peculiar. It differs distinctly from other kinds of coördination. For example, if two sets of points are given, we establish a correspondence between them by coördinating to every point of one set a point of the other set. For this purpose, the elements of each set must be *defined;* that is, for each element there must exist another definition in addition to that which determines the coördination to the other set. Such definitions are lacking on one side of the coördination dealing with the cognition of reality. Although the equations, that is, the conceptual side of the coördination, are uniquely defined, the "real" is not. On the contrary, the "real" is defined by coördinations to the equations.

This kind of coördination might be compared to the mathematical case in which a discrete set is coördinated to a subset of the continuum. Let us consider as an example the coördination of the rational fractions to the points of a straight line. We note that all the points of the straight line are well defined; we can say of every point of the plane whether or not it belongs to the straight line. More than that: the points of the straight line are ordered; we can say of any two points which of them lies "on the right," which of them lies "on the left." But the coördination does not refer to all the points of the straight line. An infinite set of points corresponding to the irrational numbers remains unaffected, and the selection of the points corresponding to the rational fractions is determined only by the coördination. Offhand we cannot say of a point of the straight line whether or not it belongs to the coördinated subset; to do so requires an analysis according to a method given by the construction of rational fractions. In this sense does the coördination to the other set determine the selection of the subset of the continuum. We notice that even so the problem has not been precisely defined, since such a coördination can be accomplished in an infinite number of ways. For instance, if the segment chosen as unit were to be increased, the required coördination could be achieved; but under these circumstances a different point of the straight line would correspond to a certain rational

fraction. Moreover, points which previously corresponded to an irrational number might now be coördinated to a rational fraction so that the selected subset would consist of quite different elements. Other coördinations result if the straight line is divided into segments corresponding to the integers, and if the coördination is carried out backwards within each segment, or if arbitrary finite segments are excluded from the coördination altogether—there is an infinite number of possibilities. It is obvious that the subset to be selected is defined only if certain additional conditions are specified. It might be specified, for instance, that of any two fractions the larger is always to be coördinated to the point farther to the right, or that a fraction twice as large is always to be coördinated to a point twice as far to the right, and so forth. The question is: when are the additional conditions sufficiently specified to make the coördination unique? Only when these have been found will a unique selection among the points of the continuum be possible by means of the discrete set and the additional conditions. The selection is still a mathematical problem, but one that can be solved uniquely; "to solve" it means to find other relations that also hold between the points but are not explicitly given in the additional conditions.

Yet even this example is still different from the coördination carried out in the *cognitive process*. In our example every element of the universal set was defined and even a direction given. The additional conditions were dependent on these properties, not only on the direction but also on the fact that the individual elements were defined. This fact requires, for instance, that to a fraction twice as large, a straight line segment twice as long is to correspond. This requirement presupposes that the distance from the zero point can be indicated for every point. Yet all such specifications fail with regard to coördinations in the cognitive process, where one side is completely undefined. It is not delimited, it contains no direction, and it does not even give a clue as to what constitutes an individual element of the set. What is the length of a physical rod? It is defined by a large number of physical equations that are interpreted as "length" with the help of readings on geodetic instruments. The definition results from a coördination of things to equations. Thus we are faced with the strange fact that in the realm of cognition two sets are coördinated, one of which not only attains its order through this coördination, but whose elements are *defined by means of this coördination.*

The attempt to regard an individual perception as a defined element of reality is not successful either. The content of every perception is far too complex to serve as an element of coördination. For instance, if we interpreted the perception of the pointer of the manometer in the above example as such an element, we would get into difficulties because this perception contains much more than the position of the pointer. Should the factory label be on the manometer, it would be part of the perception. Two perceptions different with respect to this label may still be equivalent for the coördination to Boyle's equation. Before a per-

ception is coördinated, its relevant components must be distinguished from the irrelevant ones; that is, it must be ordered. But such a coördination presupposes the equations or the laws expressed in them. Nor is a direction given by perceptions. It might be supposed that the *temporal sequence* of perceptions furnishes a direction for the physical side of the coördination. Yet this is not true, because the temporal sequence asserted in a cognitive judgment may well contradict that of the perceptions. If during an observation of two coincidences the stop watches are read in the opposite direction, a judgment about the "real" temporal sequence is made independently of these readings. This judgment is based on physical knowledge, that is, on coördinations; the physical nature of the watches, for example, their correction, must be known. The time order of perceptions is irrelevant for the time order asserted in cognitive judgments; it does not furnish a direction suitable for the coördination.

A perception does not contain even a sufficient criterion to decide whether or not a given phenomenon belongs to the class of real things. Optical illusions and hallucinations demonstrate this fact. Only a cognitive judgment, that is, an act of coördination, can decide whether the sensation of a tree corresponds to a real tree or owes its existence merely to the delirium of a desert wanderer parched with thirst. Of course, *every* perception, even a hallucinated one, represents something real—a hallucination points to physiological changes—and we shall have to indicate later what this peculiarity involves. However, perceptions do not furnish *definitions* of what is real.

If we compare this fact with the above example of a coördination, we discover that, since perceptions do not define the elements of the universal set, one side of the cognitive process contains an undefined class. Thus it happens that individual things and their order will be defined by physical laws. The coördination itself creates one of the sequences of elements to be coördinated.

One might be inclined to dismiss this difficulty simply by declaring that only the ordered set is real, while the undefined one is fictitious, a hypostatized thing-in-itself. Berkeley's solipsism and, in a certain sense, modern positivism may perhaps be interpreted in this way. But such a view is certainly false. There remains the peculiarity that the defined side does not carry its justification within itself; its structure is determined from outside. Although there is a coördination to undefined elements, it is restricted, not arbitrary. This restriction is called "the determination of knowledge by experience." We notice the strange fact that it is the defined side that determines the individual things of the undefined side, and that, vice versa, it is the undefined side that prescribes the order of the defined side. *The existence of reality is expressed in this mutuality of coördination.* It is irrelevant in this context whether one speaks of a thing-in-itself or denies its existence. This mutuality attests to what is real. In this

way existence can be conceptually apprehended and formulated.

Here the questions arise: what characterizes the "correct" coördination? how does it differ from an "incorrect one"? The answer is: by the fact that it is consistent. Contradictions are discovered by observation. For instance, if from Einstein's theory a deflection of light of 1.7″ near the sun were predicted, but 10″ were observed instead, there would arise a contradiction, and such contradictions are always used to test the correctness of a theory. The value 1.7″ has been obtained on the basis of equations and experiences concerning other data; but the value 10″ has in principle not been ascertained in a different way since it is not read off directly. Rather, it has been constructed from the recorded data with the help of complicated theories concerning the measuring instruments. It can be maintained therefore that *one* chain of reasoning and experience coördinates the value 1.7 to the physical event, the *other,* the value 10, and here lies the contradiction. That theory which continuously leads to consistent coördinations is called true. Schlick is therefore right when he defines *truth in terms of unique coördination.*[11] We always call a theory true when all chains of reasoning lead to the same number for the same phenomenon. This is the only criterion of truth; it is that criterion which, since the discovery of exact empirical science by Galileo and Newton and of its philosophical justification by Kant, has been regarded as an indispensable test. And we notice that we can now point out the role played by perceptions in the cognitive process. *Perceptions furnish the criterion for the uniqueness of the coördination.* We saw previously that they cannot define the elements of reality; but they can always be used to judge uniqueness. So-called optical illusions are not different from normal perceptions in this respect. Optical illusions are due not to a deception of our *senses* but to false *interpretations* of our perceptions; even the impressions in hallucinations are real, although the inferences from these impressions to external causes are false. When I press my finger on the optical nerve, I see a light flash; this is a sense datum, and merely the inference that there was a light flash in the room is false. Were I to order this perception along with others, say, with the observation of a photographic plate exposed simultaneously in the room, a contradiction would result from explaining the perception by a light process; for there is no blackening of the photographic plate. If I ordered the perception within another conceptual context, for example, within that of a physiological theory, *no* contradiction would result. On the contrary, the perception of the light flash serves to confirm assumptions concerning the location of the optical nerve. We see that so-called optical illusions represent, like any normal perception, a criterion for the uniqueness of a coördination, that is, a criterion of truth. Every perception has this property, and this is its only epistemological significance.

It should be noticed that the concept of uniqueness

used in this context is quite different from that used in the context of our set-theory examples. In set theory we called a coördination unique if to every element of one set it coördinated always one and the same element of the other set, independently of the manner in which the required coördination was carried out. For this purpose, the elements of the other set must also be defined; it must be possible to determine whether or not a given element is the same as a previously coördinated one. Such a determination is not possible for reality. The only fact that can be determined is whether two numerical values derived from two different measurements are the same. We cannot know whether a coördination with this result always refers to the same element in the real world. The question is therefore meaningless; but if the values obtained by the measurements are consistently the same, then the coördination possesses that property which we call truth or objective validity. Therefore, we define: *Uniqueness* of a cognitive coördination means that a physical variable of state is represented by the *same value* resulting from *different empirical data.*

This definition does not assert that this variable of state must have the same value at every space-time point so long as all physical factors remain constant. Rather, the assumption that the four coördinates do not explicitly occur in the physical equations is included in the principle of causality.* Even if this as-

* Causality, which has often been called an a priori principle of natural science, cannot be conceived on closer analysis as *one* principle, but must be regarded as a complex of principles

sumption were not satisfied, uniqueness would still hold. Uniqueness does not concern the repetition of processes; it merely requires that with respect to an individual process the value of the constants be completely determined by all factors, including, in a given case, the coördinates. This requirement must be satisfied; otherwise the numerical value of the variable of state cannot be calculated by a chain of reasoning and experience. Such a determination is expressed not only in the comparison of two equal events at different space-time points, but in the relation as well of quite different events by means of the connecting equations.

How is it possible to achieve such a coördination in a consistent manner? This question belongs in critical philosophy, for it is equivalent to Kant's question: "How is natural science possible?" It will be our task to compare Kant's answer with the results of the theory of relativity and to investigate whether his answer can still be defended. We should like to stress that the question is meaningful independently of any

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the individual components of which have not been previously formulated precisely. One of them seems to be the assumption that the coördinates do not occur explicitly in the equations, that is, that equal causes have equal effects at different space-time points. Another one is the previously mentioned assertion of the existence of irreversible physical processes. In addition, the uniqueness of physical relations belongs in this complex. It would be better to dispense with the collective word "causality" altogether and to replace it by individual principles.

given answer and that there can be no epistemology that ignores it.

What does "possible" mean in the above question? Obviously not that an individual human being will achieve such a coördination. That he cannot do, and the concept of knowledge must not be defined in such a way that it will depend on the intellectual capacity of an average person. "Possible" is not meant in a psycho-physical, but in a logical sense: it pertains to the logical conditions of a coördination. We have seen in our example that conditions specifying a coördination must exist; these conditions are principles of a general sort such as those of direction, metric relations, and so forth. Analogous principles must exist for cognitive coördinations; they must have the specific property of rendering the coördination defined by them unique according to our criterion. We may therefore formulate the critical question in the following way: *By means of which principles will a coördination of equations to physical reality become unique?*

Before we answer this question, we must characterize the epistemological position of the principles of coördination. They are equivalent to Kant's synthetic a priori judgments.

V

Two Meanings of "A Priori" and Kant's Implicit Presupposition[2]

Kant's concept of a priori has two different meanings. First, it means "necessarily true" or "true for all times," and secondly, "constituting the concept of object."

The second meaning must be clarified. According to Kant, the object of knowledge, the thing of appearance, is not immediately given. Perceptions do not give the object, only the material of which it is constructed. Such constructions are achieved by an act of judgment. The judgment is the synthesis constructing the object from the manifold of the perception. For this purpose it orders the perceptions according to a

[2] Cf. H. Reichenbach, *Modern Philosophy of Science* (London: Routledge & Kegan Paul, 1959), p. 129, note, for a reference to § V—§ VII of the present study for a discussion of the two aspects of Kant's concept of a priori.

certain schema; depending on the choice of the schema, either an object or a certain type of relation will result. Intuition is the form in which perceptions present the material—thus performing another synthesis. But the conceptual schema, the category, creates the object; the object of science is therefore not a "thing-in-itself" but a reference structure based on intuition and constituted by categories.

Our previous analysis confirms the fundamental principle of this theory. We saw that perception does not define reality, but that a coördination to mathematical concepts determines the element of reality, the real object. We saw, furthermore, that there must exist certain principles of coördination in order to make the coördination unique. Indeed, the principles must be of such a kind that they determine how the coördinated concepts combine into structures and processes; they ultimately define real objects and real events. We may call them constitutive principles of experience. Kant's schemata are space, time, and the categories. We shall have to investigate whether they are suitable additional conditions for unique coördinations.

The second meaning of the concept of a priori is the more important one. It lends to this concept the central position in epistemology which it has held since Kant. It was Kant's great discovery that the object of knowledge is not immediately given but constructed, and that it contains conceptual elements not contained in pure perception. Such a construction is not a mere fiction; if it were, its structure could not be so strictly prescribed from outside by repeated perceptions. Kant therefore relates the construction to a thing-in-itself which, though not knowable itself, manifests itself so that it fills the empty schema of the categories with positive content.

All this sounds quite metaphorical, and we must return to more precise formulations in order to find valid results. Yet it is not impractical to imagine Kant's doctrine more intuitively, because in this way one can grasp its essential ideas more rapidly. The metaphorical aspect has its reason partly in the fact that Kant's conceptual constructions belong to an era distinguished more by grammatical than by mathematical precision, and that therefore only the formal structure, not the objective content of these concepts, is expressible. It may well be that a later era will call *our* concepts metaphorical.

The coördinated categories are not of course part of the object in the same sense as its material parts. The real thing is the thing confronting us; there is no point in trying to define its existence more closely, because what is meant by "real" can only be experienced. All attempts to describe it remain analogies or they characterize the *logical structure* of the experience. The reality of things must be distinguished from the reality of concepts which, insofar as one wishes to call them real, have a mere psychological existence. But there remains a strange relation between the real thing and the concept, because only the coördination of the concept defines the individual thing in the "continuum" of reality; and only the conceptual connection decides

on the basis of perceptions whether a conceived individual thing "is there in reality."

If a set of real functions of two variables is coördinated to the plane in terms of a coördinate cross, then each function determines a figure in the continuum of the plane. The individual figure is therefore defined by the function. It can also be defined in a different way, for instance, by means of a curve actually drawn on paper. But which actual curve of the plane will be coördinated to a certain function depends on the way the coördinate cross is arranged in the plane, how the metric relations are chosen, and so forth. In this connection two kinds of coördinating principles must be distinguished: those the elements of which are defined on *both* sides and those the elements of which are defined on one side only. The determination of the coördinate cross is of the first kind, because it results from a coördination of certain defined points to the coördinate numbers; it is itself a coördination. The following example illustrates the second kind. The coördination of a function $f(x,y,z) = 0$ of three variables to the plane is achieved by a one-parameter family of curves. The determination of the coördinate cross defines which variables correspond to the axes; this determination indicates that such and such points of the plane correspond to the values x and such and such points of the plane to the values y. Additionally it is determined which variable occurs as parameter. Nevertheless, there exists an arbitrariness. In general, the family of curves is obtained by the method of constructing a curve $f(x,y,p) = 0$

for every value $z = p = constant$. It is also possible to assume an arbitrary function $\varphi(x,z) = p' = constant$ and to choose p' as parameter; under these circumstances a very different family of curves is obtained. Yet this family of curves is just as adequate a picture of the function $f(x,y,z)$ as the first one. One family of curves is not better fitted than the other; the first one is merely more intuitive and better adapted to our psychological faculties. Which set of actual curves is selected by the coördination to $f(x,y,z)$ depends therefore on the choice of the parameter. In spite of this fact, the choice of the parameter is a prescription for only the analytical side of the coördination; this choice does not use any properties of the geometrical side for its formulation. We notice that there are principles of coördination referring only to *one* side of the coördination and yet having a decisive influence upon the selection of the other side.

We have seen that with respect to knowledge of the physical world the elements on one side of the coördination are not defined; therefore, there cannot exist coordinating principles of the first kind concerning such knowledge, only principles referring to the conceptual side of the coördination. These may justifiably be called order principles. It seems very strange that it should be possible to get along with the second kind of order principles alone; I do not know of any other case except that of empirical knowledge. But this result is no more surprising than the experience of reality as such; it is connected with the fact that "uniqueness" for this kind of coördination means

something other than a reference to the "same" element on the side of reality and the fact that it is determined by perception, a criterion independent of the coördination. This is the reason that the principles of coördination are much more significant for the cognitive process than for any other coördination. By determining the coördination, they define the individual elements of reality and in this sense *constitute* the real object. In Kant's words: "because only through them can an object of experience be thought." [12]

The principle of probability may serve as an example of coördinating principles; it defines when a class of measured values is to be regarded as pertaining to the same constants.[13] (Imagine, for instance, a distribution according to the Gaussian law of errors.) This principle refers solely to the conceptual side of the coördination. Yet compared to other physical principles, it has the distinction of serving directly as a definition of something real; it defines the physical constant. Another example is the principle of genidentity,[14] which indicates how physical concepts are to be connected in sequences in order to define "the same thing remaining identical with itself in time." Other coördinating principles are time and space, since they indicate, for example, that four numbers are necessary to define a single real point. For traditional physics the Euclidean metric was such a coördinating principle, because it indicated relations according to which space points combine to form extended structures independently of their physical quality. The metric did not define a physical state as do temperature and pressure, but constituted part of the concept of physical object, the ultimate carrier of all states. Although these principles are prescriptions for the conceptual side of the coördination and may precede it as *axioms of coördination,* they differ from those principles generally called axioms of physics. The individual laws of physics can be combined into a deductive system so that all of them appear as consequences of a small number of fundamental equations. These fundamental equations still contain special mathematical operations; thus Einstein's equations of gravitation indicate the special mathematical relation of the physical variable R_{ik} to the physical variables T_{ik} and g_{ik}. We shall call them, therefore, *axioms of connection.*[15] The axioms of coördination differ from them in that they do not connect certain variables of state with others but contain general rules according to which connections take place. In the equations of gravitation, the axioms of arithmetic are presupposed as rules of connection and are therefore coördinating principles of physics.

Although the cognitive coördination can be achieved only by experience and may not be sufficiently characterized by abstract relations, it is, nevertheless, dependent in a special way upon the application of those coördinating principles. For instance, if a certain mathematical symbol is coördinated to a physical force, the properties of the mathematical vector must be ascribed to it in order to enable us to think of this force as an object. In this case the axioms of arithmetic referring

to vector operations are constitutive principles, that is, categories of a physical concept.* When we speak of the path of an electron, we must think of the electron as a thing remaining identical with itself; that is, we must make use of the principle of genidentity as a constitutive category. This connection between the conceptual category and the experience of coördination remains as an ultimate, not as an analyzable, residue. But this connection clearly defines a class of principles that precede the most general laws of connection as presuppositions of knowledge though they hold as conceptual formulas only for the conceptual side of the coördination. These principles are so important because they define the otherwise completely undefined problem of cognitive coördination.

We must now connect the two meanings of the concept of a priori mentioned above. Let us define for the moment "a priori" in the sense of the second meaning, "constituting the object." How does it follow that a priori principles are necessarily true, that is, forever independent of experience?

Kant gives the following justification for this inference: Human reason [*Vernunft*], the essence of understanding and intuition, has a certain structure. This structure prescribes the general laws according to which perceptual material is ordered to result in knowledge. All empirical knowledge has become

* This is the reason that the theorems of the parallelogram of forces appear so evident to us and that we do not see their empirical character. They are evident, too, if the force is a vector; but that is just the problem.

knowledge by means of such ordering and can never represent a disproof of the ordering principles. They are therefore absolutely necessary. They hold so long as human reason does not change, and in this sense forever. Anyway, *experiences* cannot effect a change of human reason, because experience presupposes reason. It is a moot question and irrelevant for Kant whether some day reason will change because of internal causes. He does not want to deny that other beings may exist who use constitutive principles different from ours.[16] This concession leaves the possibility that there may exist transitional biological forms between these beings and us, and that a biological development of our reason into such different rational beings is taking place. Kant never speaks of such a possibility, but it would not contradict his theory. All that his theory excludes is a change of reason and its order principles by *experience:* "necessarily true" must be understood in this sense.

If we transfer these considerations to our previous formulations, they read as follows: If perceptual data are to be ordered to result in knowledge, there must exist principles defining this coördination more precisely. We called these principles of coördination and discovered in them those principles that define the object of knowledge. If we inquire after these principles, we must turn to reason, not to experience, for experience is constituted by reason. Kant's method of answering the critical question consists therefore in an analysis of reason. In Chapters II and III we called a number of principles a priori. We want to express

thereby that, according to Kant's analysis, they would turn out to be principles of coördination. We could use the criterion of self-evidence, because this criterion is also introduced by Kant as characteristic for his principles. It seems obvious that these principles, which originate in reason, must be self-evident.[17]

We had established previously that the coördinating principles must be distinguished by the fact that they permit unique coördinations; this appeared to be the significance of the critical question. But there is no guarantee that those principles that originate in reason possess this property, because the criterion of uniqueness, that is, perception, is independent of reason. It would be a strange accident of nature if those principles originating in reason were also those determining uniqueness. There is only one possibility to explain this coincidence: if the principles of coördination are irrelevant for the requirement of uniqueness; if, in other words, a unique coördination is always possible for any arbitrary system of coördinating principles.

In our previous examples of coördination this requirement was by no means satisfied. Among them there is only one class of systems of conditions defining unique coördinations. Thus we mentioned that the rational fractions can be coördinated to the points of a straight line in different ways depending upon the choice of the additional conditions. Not all the different systems of additional conditions lead to different coördinations; rather, there are systems that can be substituted for one another because they define the same coördination. Such systems will be called equivalent; only those systems that lead to different coördinations will be called different. On the other hand, there are systems that contradict each other in their requirements. To show this one need only combine a principle and its contradictory in one system. Such explicitly contradictory systems are to be excluded in principle. With respect to the example of the rational fractions, we can say that their coördination to points of a straight line is made unique by different systems of additional conditions. It is easy to indicate systems that do not achieve this result. It is merely necessary to omit an essential principle from a system of this class; the result is an incomplete system, not capable of achieving uniqueness.

The same simple inference cannot be drawn concerning cognitive coördination. If, for instance, the system of principles were incomplete, it could be completed easily by empirical statements so that a unique system would result. The position of those philosophers (but not of Kant) maintaining the a priori character of philosophy can perhaps be given the interpretation that the system of self-evident principles is incomplete. Until now no attempt has been made to demonstrate this fact. It is true that this system does not contain explicit contradictions. Still, the system may belong to the large class of those systems that result in an implicit contradiction for the coördination. Since the criterion of uniqueness, that is, perception, is determined independently by the system from without, it is possible that contradictions will be noticed only after the system has achieved a certain

expansion. We may refer to the non-Euclidean geometries in which the axiom of the parallels has been changed but which otherwise use the Euclidean system. Only after *all consequences have been derived from these geometries,* can it be ascertained that the resulting systems do not contain any contradictions. Of course, the cognitive system is not a mathematical one, and therefore only the consequences of *experimental physics* will be decisive. This is the reason that the theory of relativity, which originated as a purely physical theory, has become so important for the theory of knowledge.

In the literature the problem of consistency has usually been discussed only with regard to individual principles. It was believed that the principle of causality could never encounter contradictions and that the interpretation of experiences would always be sufficiently arbitrary to retain this principle. But in this way the question is not formulated correctly. The problem is not whether one individual principle can be retained but whether the whole *system* of principles can always be preserved. Knowledge requires a *system* and cannot be based on an individual principle; Kant's philosophy is also a system. It seems probable, although by no means certain, that an individual principle can always be carried through. Sometimes a principle contains a *complex* of ideas and is thus equivalent to a system. It would be difficult to prove that a principle is always equivalent to an *incomplete* system.

Under all circumstances chance must be excluded; it must not become a presupposition of a scientific theory of knowledge that there exists a preëstablished harmony between reality and reason. Therefore, if the system of the principles of reason is to belong to the class of uniquely determining systems or to the class of incomplete systems, there must not exist any implicitly contradictory (overdetermining) systems for knowledge.

We have reached the conclusion that the validity of Kant's theory of knowledge can be made dependent upon the validity of a clearly formulated hypothesis. Kant's theory contains the hypothesis *that there are no implicitly contradictory systems of coördinating principles for the knowledge of reality.* Since this hypothesis is equivalent to the statement that any arbitrary, explicitly consistent system of coördinating principles can arrive at a unique coördination of equations to reality, we shall call it the *hypothesis of the arbitrariness of coördination.* Only if this hypothesis is correct, are the two meanings of the concept of a priori compatible: only then are the constitutive principles independent of experience and necessary, that is, true for all times. We shall investigate how the theory of relativity answers this question.

VI

Refutation of Kant's Presupposition by the Theory of Relativity

Let us reconsider the results of Chapters II and III. They stated that the theory of relativity has ascertained a contradiction between principles hitherto regarded as a priori and experience. How is this possible? Does not Kant's proof of the unrestricted validity of constitutive principles exclude such a contradiction?

On page 15 were listed the principles the incompatibility of which with experience has been asserted by the special theory of relativity. It was explained there in what sense the incompatibility must be understood. If absolute time is retained, it is necessary to abandon the normal procedure in extrapolating the empirical data. Within certain limits this is always possible because of the vagueness of the term "normal"; but there are cases—and one of them occurs here—where the extrapolation becomes decidedly anomalous. One has, therefore, the choice of either retaining absolute time, thereby giving up normal induction, or of retaining normal induction and giving up absolute time. Only in this sense can a contradiction with experience be asserted. But all of the above principles are a priori in Kant's sense. We may therefore say that the special theory of relativity has demonstrated the incompatibility of a system of a priori principles with the normal inductive interpretation of empirical data.

The situation is essentially the same for the general theory of relativity. The principles that result in a contradiction according to the general theory of relativity are listed on page 31. This list differs from the first merely by containing in addition to the a priori principles a non-evident one, the principle of special relativity. But this principle is internally consistent and not explicitly inconsistent with the rest of the principles; the result is an explicitly consistent system incompatible with the normal inductive interpretation of empirical data. A special feature must be mentioned. The non-evident principle is just that principle which has the distinction of solving the contradiction in the first list of principles. The second system is therefore also a system characterized by the fact that it contradicts experience.

With the help of these lists of principles, the answer to the hypothesis of the arbitrariness of coördination, which we presupposed for the validity of Kant's theory of knowledge, has been reduced to the problem of

normal induction. It is therefore necessary to analyze the significance of this principle for epistemology.

It is quite understandable that the problem of induction belongs in this context. The inductive inference, above all others, is characterized by the uncertainty and vagueness of its results. Offhand, the hypothesis of the arbitrariness of coördination appears quite improbable. If it were to be justified, it would have to be reducible to the uncertainty on the empirical side of the coördination. But this uncertainty is exactly the crux of the problem of induction. The inductive inference results in a statement going beyond the immediate data of experience. Such a statement must be made because experience provides only data, no relations, because experience furnishes only a criterion for the uniqueness of the coördination—and not the coördination itself. We spoke of normal induction. But is not an induction "normal" only if it excludes in principle interpretations that contradict the principles of coördination? Kant's proof of the independence of the coördinating principles from experience is based on this idea. We shall therefore keep this proof in mind when we investigate the problem.

Kant gives the following proof: Every experience presupposes the validity of the constitutive principles. If, therefore, laws are to be inferred from empirical data, then those interpretations of the empirical data that contradict the presupposed principles must be excluded at the outset. An induction can be called normal only when such an exclusion has taken place beforehand. Therefore, no empirical result can refute the constitutive principles.

The analysis of this proof can be reduced to the answers of two questions.

Is it logically *inconsistent* to make inductive interpretations of empirical data which constitute a contradiction to the coördinating principles?

Is it logically *admissible* to exclude before the inductive interpretation of empirical data those interpretations contradicting a certain coördinating principle?

In order to clarify the terminology, we should like to mention that by "normal inductive procedure" we shall assume the usual method of physics described in Chapter II, not the procedure developed in Kant's proof.

Let us answer the first question. Why should such a procedure be inconsistent? The implicit principle is tested by means of the question whether or not a unique coördination is achieved by the continuous application of a certain principle and the normal inductive procedure. This is a frequently used method of physics: one formulates a theory by means of which the empirical data are interpreted and then checks for uniqueness. If uniqueness is not obtained, the theory is abandoned. The same procedure can be used for coördinating principles. It does not matter that the principle to be tested is already presupposed in the totality of experiences used for the inductive inferences. It is not inconsistent to assert a contradiction between the system of coördination and experience.

The answer to the second question is more difficult. We wish to prove that its affirmation leads to the renunciation of the uniqueness of coördination.

Let us show first that the method characterized in the question and applied to any arbitrary individual law deprives the coördination of its uniqueness. Let us imagine that measurements concerning Boyle's law have been carried out and that a number of data for the product and volume have been recorded for various values of the two variables. Let us require these numerical values to be interpreted in such a way that they do not contradict a fictitious formula $pV^2 = constant$ and at the same time do not violate the physical laws used for the establishment of the data, such as the relations between pressure and the height of the mercury column.* This interpretation of the values is possible since the values are not exactly equal because of observational errors and since they always represent only a selection from the infinitely different possible values of the variables. The normal procedure is such that the numerical values are interpreted as the values of a constant showing small variations because of errors of measurement if their deviations are small, and that for the intermediate values not measured, and even for a part beyond the ends of the measured sequence, the same value of the constant is

* Such a restriction must be added because otherwise the logical consquence of the requirement would lead to a definition of "volume" that would give the meaning of "volume" as the square root of the value customarily used. This would be not a change of the laws, only a change in terminology.

assumed. This is normal induction. But if the formula $pV^2 = constant$ is dogmatically retained and any contradictory induction is excluded, the metrical values will be interpreted differently. It might be assumed, for instance, that disturbances in the apparatus have influenced the measured values; by simply omitting the most contradictory values, one interpolates and extrapolates in such a way that with increasing volume a descending curve results. Such a procedure is *possible* even though it contradicts ordinary scientific method. But it does not lead to a unique coördination. In order to characterize a coördination as unique, a hypothesis concerning the dispersion of the numerical values must be made because of the always occurring errors of measurement; and this hypothesis requires that a mean continuous curve be drawn through the measured values. If in spite of the inexactness of any measuring device a unique coördination is assumed, the principle of normal induction must be retained.[18]

The situation does not change if the investigation is extended to a principle of coördination. If empirical data have been collected the inductive interpretation of which contradicts a principle of coördination, normal induction must not be abandoned. In this case, too, the uniqueness of the coördination would be given up thereby; if this uniqueness is to be ascertained at all, probability assumptions concerning the measured values must be made. The principle of normal induction, above all other coördinating principles, is distinguished by the fact that it defines the

uniqueness of the coördination. If uniqueness is to be retained, then all other coördinating principles rather than the principle of induction mut be abandoned.

Kant's proof is, therefore, false. It is quite possible to discover a contradiction between the constitutive principles and experience. Since the theory of relativity, supported by the evidence of empirical physics, has demonstrated this contradiction, we can summarize its answer to Kant's hypothesis concerning the arbitrariness of coördination as follows: *There exist systems of coördinating principles which make the uniqueness of the coördination impossible; that is, there exist implicitly inconsistent systems.* We stress again that this result is not self-evident, but a consequence of the consistent elaboration of empirical physics. If such a scientific system is not available, then the arbitrariness in the interpretation of the few, immediate empirical data is far too great to speak of a contradiction to the principle of induction.

The answer given by the theory of relativity has a special significance. This theory has shown that the system of coördination which is distinguished by *self-evidence* results in a contradiction, and that if the first contradiction is resolved by the elimination of one of the self-evident principles, immediately a second contradiction arises because of the occurrence of additional self-evident principles. This fact has important consequences. Until now all results of physics have been obtained by means of the self-evident system. We discovered that this fact does not exclude a contradiction the existence of which can be ascertained

—but how shall we obtain a new system? With respect to individual laws, this aim is easily reached because only those presuppositions that contain the individual law have to be changed. But we have seen that *all* laws contain coördinating principles, and if we wish to test new coördinating principles inductively, we must first change every physical law. It would indeed be nonsensical to test new principles by means of experiences still presupposing the old principles. If, for instance, space were tentatively assumed to be four-dimensional, to test the assumption, all methods of measuring lengths used until now would have to be abandoned and to be replaced by a measurement compatible with four-dimensionality. Furthermore, all laws concerning the behavior of the material used in the measuring instrument, concerning the velocity of light, and so forth, would have to be given up. Such a procedure would be *technically impossible*. We cannot start physics all over again.

We are therefore in a dilemma. We admit that the principles used until now have led to a contradiction, but we do not see a way to replace them by new ones. This dilemma is resolved by the theory of relativity. It not only has refuted the old system of coördination but it has also constructed a new one, and the method used by Einstein for this purpose represents a brilliant solution of this problem.

The contradiction that arises if experiences are made with the old coördinating principle by means of which a new coördinating principle is to be proved disappears on one condition: if the old principle can

be regarded as an approximation for certain simple cases. Since all experiences are merely approximate laws, they may be established by means of the old principles; this method does not exclude the possibility that the totality of experiences inductively confirms a more general principle. *It is logically admissible and technically possible to discover inductively new coördinating principles that represent a successive approximation of the principles used until now.* We can call such a generalization "successive," because for certain approximately realized cases the new principle is to converge toward the old principle with an exactness corresponding to the approximation of these cases. We shall call this inductive procedure the *method of successive approximations*.

We notice that this is the method used by the theory of relativity. When Eötvös confirmed experimentally the equivalence of inertial and gravitational mass, he had to presuppose the validity of Euclidean geometry for the interpretation of his observations within the dimensions of his torsion balance. Nevertheless, the result of his inductions could support the validity of Riemannian geometry in stellar dimensions. The corrections of the theory of relativity with respect to the measurements of distance and time are all of such a kind that they can be neglected for ordinary experimental conditions. When an astronomer transfers from one table to another a watch used for his observations of the stars, he need not introduce Einstein's time correction for moving watches, but can establish with its help a position of Mercury that constitutes a shift of the perihelion and thus a confirmation of the theory of relativity. When the theory of relativity asserts a curving of the light rays in the gravitational field of the sun, the interpretations of the pictures of the stars nevertheless can presuppose the light segment within the telescope to be straight and calculate the aberration correction according to the usual method. This assumption is valid not only for the inference from small to large dimensions. If physics should arrive at the conclusion that there exists a strong curvature for the electron within its gravitational field, such a curvature could be discovered indirectly by means of instruments with measurements that lie within the usual order of magnitude and can be assumed therefore to be Euclidean.

It seems to me that this method of successive approximations represents the essential point in the refutation of Kant's doctrine of the a priori. It shows not only a way of refuting the old principles, but also a way of justifying new ones. This method is therefore capable of eliminating not only all theoretical reservations, but all practical ones.

In this connection it must be noted that the hypothesis of the arbitrariness of coördination formulated by us, and its refutation through experience, are not so alien to Kant's own ideas as it may at first appear. Kant based his theory of the a priori upon the possibility of knowledge; but he was well aware of the fact that he could *not demonstrate this possibility.* He did not exclude the idea that *knowledge might be impossible;* he regarded it as an accident that nature's

properties are so simple and regular that they can be ordered according to the principles of human reason. The conceptual difficulties that he encountered in this context were analyzed in his *Critique of Judgment.* "The understanding is, no doubt, in possession a priori of universal laws of nature, without which nature could not be an object of experience, but it needs in addition a certain order of nature. . . . This harmony of nature with our cognitive faculty is presupposed a priori by the judgment . . . while the understanding at the same time cognizes it objectively as contingent. . . . For it might easily be thought that it would be impossible for our understanding to detect in nature a comprehensible order." [19] It seems strange that Kant clung to his dogmatic theory of the a priori with such tenacity in spite of his clear insight into the accidental character of the affinity of nature and reason. The case that he anticipated, namely, that it may become impossible for reason to establish an intelligible order in nature by means of its inherent system, has indeed occurred: the theory of relativity has shown that a unique order of experience is no longer possible by means of Kant's "self-evident" system of reason. Whereas the theory of relativity drew the conclusion that the constitutive principles have to be changed, Kant believed that in such a case all knowledge would come to an end. He deemed such a change impossible, because only so far as that compatibility of nature with reason exists, can we "make any progress with the use of our understanding in experience and achieve knowledge." Only the method of successive approximations unknown to Kant overcomes this difficulty; therefore, his rigid a priori could be refuted only after this method was discovered by physics.

We wish to add some general remarks to the resolution of Kant's doctrine of the a priori. It seems to have been Kant's mistake that he who had discovered the essence of epistemology in his critical question confused two aims in his answers to this question. If he searched for the conditions of knowledge, he should have analyzed *knowledge;* but what he analyzed was *reason.* He should have searched for *axioms* instead of *categories.* It is correct that the nature of knowledge is determined by reason; but how this influence of reason manifests itself can be expressed only by knowledge, not by reason. There cannot be a logical analysis of reason, because reason is not a system of fixed propositions but a faculty that becomes fruitful in application to concrete problems. Thus his method always leads him back to the criterion of self-evidence. He makes use of it in his philosophy of space and refers to the self-evidence of the axioms of geometry. Even for the validity of the categories, he has essentially no other arguments. He tries to establish them as necessary conditions of knowledge. But that precisely his categories are necessary he can justify only by maintaining that they are contained in our rational thinking and ascertained by a kind of intuition of concepts. The logical analysis of the judgments from which the table of the categories is derived did not result from immediate contact with the cognitive process,

but represents a speculative order-schema of reason adopted for the cognitive process in virtue of its self-evidence. Essentially, the system of his a priori principles represents merely a canonization of "common sense," of that naïve affirmation of reason which he himself occasionally rejects with sober incisiveness.

Kant's methodological mistake seems to lie in this procedure, and had the effect that the grandiose plan of the system of critical philosophy did not lead to results that can stand up to the advancing sciences. However illuminating the critical question, "How is knowledge possible?" stands at the beginning of all epistemology—it cannot lead to valid answers before the method of answering it has been freed from the narrowness of psychological speculation.

VII

The Answer to the Critical Question by the Method of Logical Analysis

The refutation of the positive part of Kant's theory of knowledge does not free us from the obligation to resume the critical part of this theory in its essential form. We had found that the question "How is knowledge possible?" is justified independently of Kant's answer, and we could give it a precise form within our conceptual framework. After rejection of Kant's answer our task will now be to show a way to answer the critical question: "What coördinating principles make a unique coördination of equations to reality possible?"

We see such a way in the application of the *method of logical analysis* to epistemology. The results discovered by the positive sciences in continuous contact with experience presuppose principles the detection of which by means of logical analysis is the task of

philosophy. Fundamental contributions have been made through constructions of axiomatic theories that since Hilbert's axioms of geometry have applied modern mathematical and logical concepts to science. It must be realized that there is no other method for epistemology *than to discover the principles actually employed in knowledge.* Kant's attempt to detect these principles in reason must be regarded as a failure; an inductive method must replace his deductive method. The method is inductive insofar as it is tied to the actual empirical data. Of course, the analytic method as such is not equivalent to inductive inference. In order to avoid confusion we shall call it the method of logical analysis.

The author was able to carry through such an analysis for a special domain of physics, for the theory of probability.[20] It led to the discovery of an axiom that has fundamental significance for our understanding of physics, and as a principle of distribution finds its place next to causality, a principle of connection. The analysis of the theory of relativity has essentially been carried through by Einstein himself. In all of his works Einstein has formulated the fundamental principles from which he deduced his theory. However, the point of view according to which the physicist establishes his principles is different from that of the philosopher. The physicist aims at the simplest and most comprehensive fundamental assumptions; the philosopher wants to order these assumptions and classify them as special and general principles, and as principles of connection and coördination. In this respect some work will still have to be done for the theory of relativity. Chapters II and III of this investigation may be regarded as a contribution to this task.

It is important to notice in this context the difference between physics and mathematics. Mathematics is indifferent with regard to the applicability of its theorems to physical things, and its axioms contain merely a system of rules according to which its concepts can be related to each other. A purely mathematical axiomatization never leads to principles of an *empirical theory*. Therefore, the axioms of geometry could not assert anything about the epistemological problem of physical space. Only a physical theory could answer the question of the validity of Euclidean space and discover at the same time the epistemological principles holding for the space of physical objects. Yet it is incorrect to conclude, like Weyl and Haas, that mathematics and physics are but one discipline.[21] The question concerning the *validity* of axioms for the physical world must be distinguished from that concerning *possible* axiomatic systems. It is the merit of the theory of relativity that it removed the question of the *truth* of geometry from mathematics and relegated it to physics. If now, from a general geometry, theorems are derived and asserted to be a necessary foundation of physics, the old mistake is repeated. This objection must be made to Weyl's generalization of the theory of relativity[22] which abandons altogether the concept of a definite length for an infinitesimal measuring rod. Such a generalization is possible, but

whether it is compatible with reality does not depend on its significance for a general local geometry. Therefore, Weyl's generalization must be investigated from the viewpoint of a physical theory, and only experience can be used for a critical analysis. Physics is not a "geometrical necessity"; whoever asserts this returns to the pre-Kantian point of view where it was a necessity given by reason. Just as Kant's analysis of reason could not teach the principles of physics, neither can considerations of a general geometry teach them; the only way is an analysis of empirical knowledge.

The *concept of the a priori* is fundamentally changed by our investigations. Because of the rejection of Kant's analysis of reason, one of its meanings, namely, that the a priori statement is to be eternally true, independently of experience, can no longer be maintained. The more important does its second meaning become: that the a priori principles constitute the world of experience. Indeed there cannot be a single physical judgment that goes beyond the state of immediate perception unless certain assumptions about the description of the object in terms of a space-time manifold and its functional connection with other objects are made. It must not be concluded, however, that the form of these principles is fixed from the outset and independent of experience. Our answer to the critical question is, therefore: there are a priori principles that make the coördination of the cognitive process unique. But it is impossible to deduce these principles from an immanent schema. We can detect them only gradually by means of logical analysis and must abandon the question of how long their specific forms will remain valid.

It is always only a specific formulation that we obtain in this manner. When ever we have discovered a coördinating principle used in physics, we can indicate a more general one of which the first is only a special case. We might now make the attempt to call the more general principle a priori in the traditional sense and to ascribe eternal validity at least to this principle. But such a procedure fails because for the more general principle an even more general one can be indicated; this hierarchy has no upper limit. Here we notice a danger of which the theory of knowledge becomes an easy victim. When the change of mass relative velocity was discovered and recognized to be a contradiction of Kant's principle of the conservation of substance, it was easy to say: mass was not yet the ultimate substance; the principle must be retained, and a new constant must be discovered. This proposal was a generalization since by "substance" Kant certainly meant "mass." [23] There is no guarantee that one day even this principle will not have to be given up. Should it turn out, for instance, that there is no substance that persists and represents what was originally meant by the "self-identical thing"—and today the motion of a material particle is interpreted as the motion of a concentration of energy similar to the motion of a water wave, so that one can no longer speak of a material particle remaining physically identical with itself—one might take refuge in the even

more general assertion: for every event there must exist a numerical value that remains constant. Such an assertion would be quite empty because the fact that the physical equations contain constants has very little to do with Kant's principle of substance. Nevertheless, even this formulation affords no protection against further contradictory experiences. If it should turn out that the totality of constants is not invariant with respect to transformations of the coördinates, the principle would have to be generalized again. It is obvious that such a procedure does not lead to precise and clear principles; if the principle is to have content, the *most general formulation attainable at a certain moment* must be accepted. After the refutation by advancing physics of Kant's theory of space, we do not want to climb the ladder to the next generalization and maintain that every physical theory of space must under all circumstances retain at least the Riemannian planeness in infinitesimal domains, and maintain that this statement will be eternally true. Nothing may prevent our grandchildren from being confronted some day by a physics that has made the transition to a line element of the fourth degree. Weyl's theory represents a possible generalization of Einstein's conception of space which, although not yet confirmed empirically, is by no means impossible. But even this generalization does not represent the most general local geometry imaginable. In this context one can easily trace the steps of progressive generalization. In Euclidean geometry a vector can be shifted parallel to itself along a closed curve so that upon its return to the point of departure it has the same direction and the same length. In the Einstein-Riemannian geometry it has merely the same length, no longer the original direction, after its return. In Weyl's theory it does not even retain the same length. This generalization can be continued. If the closed curve is reduced to an infinitely small circle, the changes disappear. The next step in the generalization would be to assume that the vector changes its length upon turning around itself. There is no "most general" geometry.

Even for the principle of causality, no eternal validity can be predicted. It was mentioned above as an essential content of this principle that the coördinates do not occur explicitly in the physical equations; that is, that equal causes will produce equal effects at different space-time points. Although this characteristic seems to be even more assured by the theory of relativity, since this theory has deprived the coördinates of all physical properties, it is conceivable that a more general theory of relativity will abandon it. In Weyl's generalization, for instance, spatial lengths and time intervals depend explicitly on the coördinates. In spite of this fact, a procedure might be found to ascertain this dependence according to the method of successive approximations. According to Weyl the frequency of a clock is dependent upon its previous history. However, if it is assumed, according to a probability hypothesis, that these influences compensate each other on the average, then the experiences made until now, according to which, say, the frequency of a spectral line under otherwise equal conditions is the same on all celestial bodies, can be interpreted as approximations. Conversely, those cases can be discovered by means of this law of approximations in which Weyl's theory causes a noticeable difference.

The principle of the probability function, formulated by the author, might also be generalized in terms of an approximation. The principle says that the fluctuations of a physical magnitude caused by the influence of always present small disturbances are distributed in such a way that the numerical values fit a *continuous* frequency function. If quantum theory were developed in terms of saying that every physical magnitude can assume only values that are whole multiples of an elementary unit, then the continuous distribution of numerical values would still hold approximately for the dimensions of our measuring instruments when the unit is small.[24] But we want to guard against hastily accepting this generalization as correct. Advancing science alone will be able to point out the *direction* in which the generalization ought to proceed and thus protect the more general principle from becoming empty. For all imaginable principles of coördination the following statement holds: For every principle, however it may be formulated, a more general one can be indicated that contains the first as a special case. According to the previously mentioned principle of successive approximations, which presupposes special formulations as approximations, empirical tests are possible; nothing can be said beforehand about the result of these tests.

One might still try the following method in defense of an a priori theory in the traditional sense. Since every special formulation of the coördinating principles may be superseded by empirical science, we shall renounce any attempt to give a most general formulation. But that there must exist principles that define ultimately a unique coördination is a fact, and this fact is eternally true and could be called "a priori" in the old sense. Is this not the essential meaning of Kant's philosophy?

This assertion, once more, makes an assumption that cannot be proved: that a *unique* coördination will always be possible. Where does the definition of knowledge as *unique* coördination come from? From an analysis of the knowledge gathered up to now. Yet nothing can prevent us from eventually confronting experiences that will make a unique coördination impossible, just as experiences show us today that Euclidean geometry is no longer adequate. The requirement of uniqueness has a definite physical significance. It says that there are constants in nature; by measuring them in various ways, we establish their uniqueness. Every physical magnitude of state can be regarded as a constant for a class of cases, and every constant can be regarded as a variable magnitude of state for another class.[25] But how do we know that there are constants? It is very convenient to use equations in which certain magnitudes may be regarded as constants, and this procedure is certainly connected with the nature of human reason, which in this way arrives at an ordered system. But it does not follow that this procedure will always be possible. Let us assume, for

instance, that every physical constant has the form: $C + k\alpha$, where α is very small and k is an integer; let us add the probability hypothesis that k is usually small and lies perhaps between 0 and 10. For constants of the usual order of magnitude the additional term would be very small, and the current conception would remain a good approximation. But for very small constants—for example, of the order of magnitude of electrons—uniqueness could no longer be asserted. The ambiguity nevertheless could be established, namely, according to the method of successive approximations. One need only use measurements carried out with constants of the ordinary order of magnitude, that is, constants in which the old law holds approximately. Under such circumstances it would be possible to speak no longer of a general uniqueness of the coördination, only of an approximate uniqueness for certain cases. Even the introduction of the new expression $C + k\alpha$ does not reëstablish the uniqueness. According to Chapter IV, it is the significance of the requirement of uniqueness that a determination of a certain magnitude on the basis of various empirical data must lead to the same value. Uniqueness cannot be defined in any other way since this is the only form in which it can be ascertained. Yet in the expression $C + k\alpha$ the magnitude k is completely independent of physical factors. Therefore, we can never anticipate the value of the magnitude $C + k\alpha$ on the basis of theoretical considerations and other empirical data; we can determine it only afterwards, for every individual case, on the basis of observational evidence. Since this magnitude never functions at the point of intersection between two chains of reasoning, uniqueness thereby has essentially been abandoned. Since k is also supposed to be independent of the coördinates, we would be confronted with the case that for two equal physical processes happening at the same place at the same time (this is to be realized approximately in terms of small space-time intervals), the physical magnitude $C + k\alpha$ assumes completely different values. Our assumption does not mean the introduction of an "individual causality" as described above and assumed as possible by Schlick,[26] where the same cause at a different space-time point would have a different effect, but an actual renunciation of the uniqueness of the coördination. Yet this is still a coördination that can be carried through. It represents the next step of approximation of the concept of unique coördination and corresponds to it just as Riemannian space corresponds to Euclidean space. Therefore, its introduction into the concept of knowledge is possible according to the method of successive approximations. Under these circumstances, knowledge no longer means "unique coördination," but something more general. This coördination does not lose thereby its practical value; should such ambiguous constants occur only in connection with individual magnitudes in statistical processes, exact laws can be established for the total process. At any rate, a consideration of practical possibilities need not disturb us in these theoretical discussions; once the results are

theoretically assured, their practical application will always be possible.

Such an approximation is perhaps not so remote as it may seem. We mentioned before that the uniqueness of the coördination cannot be *ascertained;* it is a conceptual fiction that is only approximately realized. A probability hypothesis must be added as a principle of coördination. This hypothesis defines when the measured values are to be regarded as values of the same magnitude; that is, it determines what is regarded as uniqueness in physics. However, if a probability hypothesis must be used after all, it can also differ just from that form which defines uniqueness. For the generalization of the concept of constant, we had to add a probability assumption; this assumption replaces the concept of uniqueness with regard to determining the definition. Certain assumptions of quantum theory may suggest such a generalization of the concept of coördination.[27]

For the demonstration that led to the rejection of Kant's hypothesis of the arbitrariness of coördination, we needed the concept of unique coördination. Even though we are questioning this concept now, our considerations do not become invalid. For the time being, this concept is *adequate;* and we can do nothing but make use of the principles of prevailing knowledge. We are not afraid of the next step in the generalization of this concept, because we know that this development will go on *continuously:* the old concept will therefore still hold approximately and demonstrate our views sufficiently. Besides, we did not make immediate use of the concept of uniqueness, but of the fact of its definition by means of a probability function. It is easy to see that the proof can be equally given by means of a materially different probability assumption. It is true that the method of successive approximations may ultimately lead to quite remote principles and make the approximate validity of our proof doubtful—but we do not by any means assert that *our* results will be true *forever,* for we just showed that all epistemological inferences are inductive.

Let us, therefore, relinquish uniqueness as an absolute requirement and call it a principle of coördination, just like all the others, that is obtained by means of the analysis of the concept of knowledge, and confirmed inductively by the possibility of knowledge. Then the question remains: is the concept of *coördination* above all not that most general principle which is independent of experience and presupposed by all knowledge?

This question shifts the problem from precise mathematical concepts to less precise ones. It is due to the limitations of our scientific vocabulary that we introduced the concept of coördination for the description of the cognitive process. We made use of a set-theoretical analogy. For the time being, coördination seems to us to be the most general concept that describes the relation between concepts and reality. It is possible, however, that some day a more general concept will be found for this relation of which our

concept of coördination is a special case. *There are no "most general" concepts.*

One must become accustomed to the fact that epistemological statements are significant even if they are not eternally true predictions. All statements containing references to time intervals are based on induction. Of course, every scientific statement claims validity not only for the present, but for future experiences. But that is possible only in the same sense in which a curve is extrapolated beyond the end of a measured sequence of points. It would be nonsense to project validity into infinity.

We should like to make some fundamental remarks concerning our view of epistemology. Although we have rejected Kant's analysis of reason, we do not want to deny that experience contains rational elements. Indeed, the principles of coördination are determined by the nature of reason; experience merely selects from among all possible principles. It is only denied that the rational component of knowledge *remains* independent of experience. The principles of coördination represent the rational components of empirical science at a given stage. This is their fundamental significance, and this is the criterion that distinguishes them from every particular law, even the most general one. A particular law represents the application of those conceptual methods laid down in a principle of coördination; the principles of coördination alone define the knowledge of objects in terms of concepts. Every change of the principles of coördination produces a change of the concept of object or event, that is, the object of knowledge. Whereas a change in particular laws produces only a change in the relations between particular things, the progressive generalization of the principles of coördination represents a development of the *concept of object* in physics. Our view differs from that of Kant as follows: whereas in Kant's philosophy only the determination of a *particular concept* is an infinite task, we contend *that even our concepts of the very object of knowledge, that is, of reality and the possibility of its description, can only gradually become more precise.*

In the following chapter we shall try to show how the theory of relativity has shifted these concepts because it is a theory with different principles of coördination, and has, in fact, led to a new concept of object. We can, however, derive another consequence for epistemology from this physical theory. If the system of coördination is determined by reason in its conceptual relations, but in its ultimate construction by experience, then the totality expresses the nature of reason as well as the nature of reality; therefore, the concept of physical object is equally determined by reason and by the reality that the concept is intended to formulate. It is therefore not possible, as Kant believed, to single out in the concept of object a component that reason regards as necessary. It is experience that decides which elements are necessary. The idea that the concept of object has its origin in reason can manifest itself only in the fact that this concept contains elements for which *no* selection is prescribed, that is, elements that are independent of the nature of reality. The arbitrariness of these elements shows that they owe their occurrence in the concept of knowledge altogether to reason. *The contribution of reason is not expressed by the fact that the system of coördination contains unchanging elements, but in the fact that arbitrary elements occur in the system.* This interpretation represents an essential modification compared to Kant's conception of the contribution of reason. The theory of relativity has given an adequate presentation of this modification.[3]

We previously formulated the hypothesis of the arbitrariness of coördination and discovered that there are implicitly contradictory systems; this discovery does not mean that there exists only a single system of coördinating principles that makes coördination unique. There are several such systems. The fact that they are equivalent descriptions is expressed in the existence of transformation formulas that accomplish the transition from one system to another. It cannot be maintained that a particular system has the property of being most adequate to reality, because all of the systems possess the only criterion of adequacy, uniqueness of coördination. It must be indicated, for the transformations, which principles can be chosen arbitrarily, that is, which of them represent independent variables, and which of them correspond to dependent variables and will change according to the transformation formulas. The theory of relativity teaches that

[3] Cf. H. Reichenbach, *op. cit.*, p. 56, note.

the four space-time coördinates can be chosen arbitrarily, but that the ten metric functions $g_{\mu\nu}$ may not be assumed arbitrarily; they have definite values for every choice of coördinates. Through this procedure, the subjective elements of knowledge are eliminated and its objective significance formulated independently of the special principles of coördination. Just as the invariance with respect to the transformations characterizes the objective nature of reality, the structure of reason expresses itself in the arbitrariness of admissible systems. Thus it is obviously not inherent in the nature of reality that we describe it by means of coördinates; this is the subjective form that enables our reason to carry through the description. On the other hand, the metric relations in nature have a certain property that holds our statements within certain limits. Kant's assertion of the ideality of space and time has been precisely formulated only in terms of the relativity of the coördinates. But we also notice that he asserted too much, for the metric furnished by human intuition does not belong to the admissible systems. If the metric were a purely subjective matter, then the Euclidean metric would have to be suitable for physics; as a consequence, all ten functions $g_{\mu\nu}$ could be selected arbitrarily. However, the theory of relativity teaches that the metric is subjective only insofar as it is dependent upon the arbitrariness of the choice of coördinates, and that independently of them it describes an objective property of the physical world. Whatever is subjective with respect to the metric is expressed in the relativity of the metric coefficients

for the domain of points, and this relativity is the consequence of the empirically ascertained equivalence of inertial and gravitational mass. It was the mistake of Kant's method to make statements about the subjective elements of physics which had not been tested empirically. Only now, after empirical physics has confirmed the relativity of the coördinates, may we regard the ideality of space and time as confirmed as far as this ideality is expressed as arbitrariness in the choice of the coördinates. Actually no final answer has been given to this question. If, for instance, Weyl's generalization should turn out to be correct, a new subjective element will have appeared in the metric. Then the comparison of two small measuring rods at two different space points also no longer contains the objective relation that it contains in Einstein's theory in spite of the dependence of the measured relation upon the choice of the coördinates, but is only a subjective form of description, comparable to the position of the coördinates. We notice that inasmuch as the concept of object changes, there is no final judgment concerning the contribution of reason to knowledge, only a gradual clarification, and that the recognition of this contribution cannot be formulated in terms of such vague notions as the ideality of space, but only by means of mathematical principles.[4]

The method of distinguishing the objective significance of a physical statement from the subjective form

[4] Cf. H. Reichenbach, *op. cit.*, p. 34, for a correction and clarification of this passage.

of the description through transformation formulas, by indirectly characterizing this subjective form has replaced Kant's analysis of reason. It is a much more complicated procedure than Kant's attempt at a direct formulation, and Kant's table of categories appears primitive in comparison with the modern method of the theory of invariance. But in freeing knowledge from the structure of reason, the method enables us to describe this structure; this is the only way that affords us an understanding of the contribution of our reason to knowledge.

VIII

The Concept of Knowledge of the Theory of Relativity as an Example of the Development of the Concept of Object

If it is true that the a priori principles of knowledge are only inductively determinable and can at any time be confirmed or disconfirmed by experience, traditional critical philosophy must be given up. We want to show, however, that this view is distinct from an empiricist philosophy that believes it can characterize all scientific statements indifferently by the notion "derived from experience." Such an empiricist philosophy has not noticed the great difference existing between specific physical laws and the principles of coördination and is not aware of the fact that the latter have a completely different status from the former for the *logical construction* of knowledge. The doctrine of the a priori has been transformed into the theory that the logical construction of knowledge is determined by a special class of principles, and that this logical function singles out this class, the significance of which has nothing to do with the manner of its discovery and the duration of its validity.

We do not see a better way of clarifying this exceptional status than by describing the change in the *concept of object* that the change of the coördinating principles brought about through the theory of relativity.

Physics arrives at quantitative statements by investigating the influence of physical factors upon determinations of lengths and time intervals; measurements of lengths and time intervals are the primary quantitative measurements. The physicist acertains the occurrence of gravitational forces by measuring the time that a free-falling body needs to transverse certain distances, or he measures a temperature increase by means of the change in the length of a mercury thread. For this purpose the concepts of space and time intervals must be defined. By space and time intervals, physics understands a numerical ratio connecting the interval to be measured with an interval used as unit. In these operations traditional physics made the fundamental assumption that lengths and times are independent of each other and that the synchronous time defined for a system has no influence upon the results of the measurements of length. In order to effect the transition from measured lengths to connecting rela-

tions, a system of rules for the connection of lengths must be added. In traditional physics the theorems of Euclidean geometry served this purpose. Let us imagine a rotating sphere; according to Newton's theory it experiences an ellipticity. The influence of rotation, that is, of a physical cause, is expressed in a change of geometrical dimensions. In spite of this fact, the rules concerning the connection of lengths are not changed. Even on such a sphere, the theorem that the relation between circumference and diameter of a circle (for instance of a latitude circle) is equal to π, and the theorem that a sufficiently small segment of an arc has the Pythagorean relation to the coördinate differentials (true for *all* small arc segments with respect to arbitrarily selected orthogonal coördinates) are valid. Physics had to make such assumptions if it wished to measure *any* changes of lengths and times. It was regarded as a necessary property of the physical body that it behaved according to these general relations. Only under this presupposition could something be thought of as a physical thing. To obtain quantitative knowledge meant nothing but to apply these general rules to reality and to order the numerical values in a system accordingly. These rules belonged to the *concept of object of physics.*

When the theory of relativity changed this view, serious conceptual difficulties arose. The theory said that the measured lengths and time intervals possess no absolute validity, but contain accidental elements: the chosen system of reference and the fact that a moving body will show contraction relative to a system at rest. This result was interpreted as contradicting causality, because no cause for the contraction could be indicated. Suddenly one was confronted with a physical change the cause of which could not be reconciled with any conception of the forces produced by the motion. Just recently Helge Holst[28] has made an attempt to save the principle of causality by indicating a preferred system of coördinates, in opposition to Einstein's relativity, in which the measured values alone are said to have objective significance, and the Lorentz contraction appears to be caused by the motion relative to this system. Einstein's relativity is represented as an elegant possibility of transformation which results from mere chance in nature.

We must notice that the apparent difficulty does not arise from the attempt to preserve the requirement of causality, but from the attempt to preserve a concept of object that the theory of relativity has overcome. There exists a definite cause for the contraction of length: the relative motion of the two bodies. Depending on which system of reference is assumed to be at rest, either of the two bodies can be called shorter. If this result is interpreted as a contradiction to causality because causality ought to require a statement as to which body is "really" contracted, then it is assumed that length is an absolute property of bodies. But Einstein has shown that length is a defined magnitude only relative to a certain coördinate system. Between a moving body and a measuring rod (which must, of course, also be regarded as a body) there exists a relation; but depending upon the chosen system of reference, this relation manifests itself sometimes as rest length, sometimes as a Lorentz contraction or Lorentz extension. What we measure as length is not the relation between the bodies, but merely their projection into a coördinate system. We can *formulate* this length only in the language of a coördinate system; but by indicating simultaneously the transformation formulas for every other system, our statement obtains objective significance. The new method of the theory of relativity consists in the following: It lends an objective meaning to subjective statements by indicating the transformation formulas. This method shifts the concept of physical relation. Only a length measured in a specific system can be ascertained and therefore be called objective. But this length is only *one* expression of the physical relation. What was formerly regarded as geometrical length is no absolute property of a body, but rather a reflection of such a property in the description of a single coördinate system. This conception does not constitute an interpretation of the real thing as a thing-in-itself, since we can formulate uniquely the physical relation by indicating the length in *one* coördinate system and adding the transformation formulas. But we must adjust to the fact that the physical relation cannot be formulated simply as a ratio.

We notice the change in the concept of object: what was formerly a property of *things* becomes now a property of things and their systems of reference. Only by stating the transformation formulas can we eliminate the influence of the system of reference; and only in this way do we arrive at a determination of what is real.

If Einstein's concept of length is restricted insofar as it formulates only one side of the fundamental physical relation, it is essentially extended in another respect. Since the state of motion of the bodies changes their physical lengths, length becomes, conversely, an expression of the state of motion. Instead of saying: two bodies are in motion relative to each other, I can also say: viewed from one of the bodies, the other experiences a Lorentz contraction. Both statements are different expressions of the same fundamental fact. We notice again that a physical fact cannot always be expressed in terms of a simple kinematic statement, but is sufficiently described only by means of two different statements and their mutual transformations.

This extended function of the metric, namely, the characterization of a *physical state,* has been developed to a much higher degree in the *general* theory of relativity. According to this theory, not only uniform motion but also accelerated motion leads to a change of the metric relations, and therefore the state of accelerated motion can be characterized conversely by metric statements. This leads to consequences that the special theory of relativity did not envisage. Accelerated motion is connected with the occurrence of gravitational forces, and in view of this extension even the occurrence of physical *forces* is expressed by metric statements. The concept of force which raised so many logical difficulties for traditional physics appears suddenly in a new light: it represents only one anthropomorphic side of a physical state the other side of which is a special form of metric. To be sure, such an extension of the function of the metric makes it impossible to preserve its simple Euclidean form; only the Riemannian analytic metric is able to assimilate such an increase in significance. Instead of saying: a celestial body approaches a gravitational field, I can also say: the metric dimensions of this body become curved. We are accustomed to perceive the occurrence of forces through their resistance to motion. We can just as well say: reality, also called a field of force, manifests itself in the fact that straight-line motion is impossible. It is a principle of the Einstein-Riemannian curvature of space that it makes the existence of straight lines impossible. "Impossible" must not be interpreted *technically,* as if merely a technical realization of a straight line by means of physical rods were impossible, but *logically.* Even the *concept* of straight line is impossible in Riemannian space. Applied to physics, this geometry implies that there is no point in searching for an approximation to a straight line by a physical rod; even *approximations* are impossible. Traditional physics also asserts that a celestial body entering a gravitational field adopts a curved path. But the theory of relativity asserts rather that it *does not make sense* to speak of straight lines in a gravitational field. This statement differs in physical content from that of the old view. The path of Einstein's theory has the same relation to the Newtonian path that a spatial curve has to a plane curve; Einstein's curvature is of a higher order than the Newtonian one.

This fundamental change of the metric is connected with its augmented significance in expressing a physical state.

The old view that the metric relations of a body—the manner in which its size and length, the angle between its sides, and the curvature of its surfaces are calculated from the data of measurement—are independent of nature can no longer be maintained. These metric rules have become dependent upon the totality of the surrounding world of bodies. What was formerly called a mathematical method of reason has become a special property of the object and its imbeddedness in the totality of bodies. *The metric is no longer an axiom of coördination but has become an axiom of connection.* This result expresses a much more profound shift in the concept of reality than that inherent in the special theory of relativity. We are used to thinking of matter as something hard and solid which our tactile sense feels as resistance. All theories of a mechanistic explanation of the world depend on this concept of matter, and it is characteristic of these explanations that they attempted again and again to conceive the coincidence of solid bodies as the prototype of all dynamic effects. One must definitely abandon this model in order to understand the meaning of the theory of relativity. What the physicist observes is measurements of lengths and time intervals, not resistances to the tactile sense. The presence of matter can manifest itself therefore only in length and time measurements. That there is something real, a substance, is physically expressed in the metric, in the

special form of the connection between these lengths and times. That is real which is described in terms of the curvature of space. Once more we notice a new method of description: the real is no longer described in terms of a *thing,* but in terms of a number of relations between the geometric dimensions. It is true that the metric contains a subjective element, and depending on the choice of the system of reference, the metric coefficients will vary; this indeterminacy still holds in the gravitational field. But there exist dependency relations among the metric coefficients, and if four of them are arbitrarily given for the whole space, then the other six are determined by transformation formulas. The presence of matter manifests itself in this restricting condition; this is the conceptual form for the defining of physical existence. These restricting conditions would not hold for empty space; but then the metric would not be determined. It makes no sense to speak of relations of length in empty space. Only bodies have lengths and widths and heights—but the physical state of the bodies must manifest itself in the metric relations.

Thus has been abandoned the traditional concept of substance used by Kant, a concept according to which substance was a metaphysical substratum of things about which only changes could be observed. Epistemologically, there is no difference between the assertion of Thales of Miletus that water is the ultimate constituent of things and the traditional concept of substance; a more advanced physics merely substituted hydrogen or the helium atom for water. Advancing physical discoveries were not able to change the epistemological concept, only its specific content. It was Einstein's change of the *coördinating principles* that affected the *concept* of reality. His theory must not be confronted with the question: What is real? Is it the electron? Is it radiation? This way of putting the question includes the traditional concept of substance and merely asks a new content for it. That something *exists* manifests itself in the dependency relations between the metric coefficients; since we can discover these relations by means of measurements—and *only* by means of them—we can discover the real. It is the essence of the general theory of relativity that the metric is much more than a mathematical measurement of bodies; it is the form by means of which the body is described as an element in the material world.*

It is only a consequence of this conception if the

* It does not contradict this thesis if modern physics still uses the traditional concept of substance. Recently, Rutherford has developed a theory in which he reports the decomposition of the positive nitrogen nucleus into hydrogen and helium nuclei. This most fruitful physical discovery may presuppose the traditional concept of substance, because it lends itself with sufficient approximation to such description of reality; nor does Rutherford's work exclude the possibility of conceiving the internal structure of the electron according to Einstein's theory. We may compare this survival of traditional concepts in modern science with a well-known example from astronomy: although we have known since Copernicus that the earth is not at the center of a celestial vault conceived as spherical and rotating, this view still serves as the foundation of astronomical measuring techniques.

boundaries between material bodies and environment are not sharply defined. Space is filled with the field that determines its metric; what we used to call matter is merely condensations of this field. It makes no sense to speak of traveling material particles as a transport of things; what occurs is a progressive condensation process that should be compared rather to the propagation of a wave in water.* The concept of individual thing loses its precision. Arbitrarily defined domains of the field can be selected for consideration; but they can be characterized only by the special values of general space-time functions in this domain. Just as a differential domain of an analytic function within the complex domain characterizes the trend of the function for the infinite domain, so every partial domain characterizes the total field; and it is not possible to indicate its metric determinations without describing the total field. Thus the individual thing is dissolved into the concept of the field, and with it all forces among things disappear. The *physics of forces and things* is replaced by the *physics of states of fields.*

We are offering this presentation of the concept of object of the theory of relativity—which makes no claim to exhaust the epistemological content of the theory—in order to show the significance of constitu-

* This is only a crude analogy. Usually, the "apparent" motion of a water wave is conversely explained by means of the "real" fluctuations of the water particles. However, there are no single particles that are carriers of the state of the field. Cf., the epistemologically important remarks by Weyl for this conception of matter, note 21, p. 162.

tive principles. In contrast to particular laws, they do not say *what* is known in the individual case, but *how* knowledge is obtained; they define the knowable and say what knowledge means in its logical sense. Thus far they are the answer to the critical question: how is knowledge possible? By defining what knowledge is, they show the order rules according to which knowledge is obtained and indicate the conditions the logical satisfaction of which leads to knowledge. This is the logical sense of the word "possible" in the above question. We understand that today's conditions of knowledge are no longer those of Kant's time, *because the concept of knowledge has changed, and the changed object of physical knowledge presupposes different logical conditions.* The change could occur only in connection with experience, and therefore the principles of knowledge are also determined by experience. But their validity does not depend only upon the judgment of particular experiences, but also upon the possibility of the whole system of knowledge: this is the sense of the a priori. The fact that we can describe reality by means of metric relations among four coördinates is as valid as the totality of physics; only the special form of these rules has become a problem of empirical physics. This principle is the basis for the conceptual construction of physical reality. *Every* physical experience ever made has confirmed this principle. This result does not exclude the possibility that some day experiences will occur that will necessitate another successive approximation—then physics again will have to change its concept of object and

presuppose new principles for knowledge. "A priori" means "*before* knowledge," but not "for all time" and not "independent of experience."

We do not want to close our investigation without mentioning the problem that is usually regarded as the focal point in the discussion of relativity: the possibility of visualizing Riemannian space. We must stress that the question of the *self-evidence* of a priori principles belongs in psychology and that it is certainly a psychological problem why Euclidean space possesses that peculiar evidence which leads to an intuitive acceptance of all of its axioms. The catchword "habit" does not explain this fact, because we are dealing not with ever-repeated chains of associations, but with a special psychological function. This self-evidence is the more amazing because visual space contains relations that deviate from the Euclidean ones. For instance, it is self-evident to us that the straight line is the shortest connection between two points. This psychological phenomenon is still completely unexplained.

Yet we can make some fundamental remarks concerning this problem by starting from the concept of knowledge developed above. We could show that according to this concept of knowledge the metric has a function different from its previous one, that it does not furnish copies of bodies in the sense of geometrical similarity, but is the expression of their physical states. It seems clear to me that we cannot make use of our intuitive geometrical images for this much more fundamental function. Euclidean geometry fascinates us so much and appears so compelling to us because we are convinced that by means of this geometry we can arrive at true pictures of real things. When it has become apparent, however, that knowledge is something other than the production of such images, that metric relations do not have the function of copying figures, we shall no longer make an attempt to regard Euclidean geometry as necessarily applying to reality.

When the view that the earth is a globe became prevalent in the fifteenth century, it had to contend at first with great resistance and certainly encountered the objection that it is unintuitive. One had only to look around in one's spatial environment to discover that the earth was *not* a sphere. Later this objection was given up, and today it is obvious to every child of school age that the earth is a sphere. Actually, the objection was perfectly valid. One cannot *imagine* that the earth is a sphere. When we make an attempt to imagine this, we immediately visualize a small sphere and upon it a man who has his feet on the surface and his head sticking out. We cannot imagine this in the dimensions of the earth. The peculiarity that the sphere is at the same time equivalent to a plane within the domain of our visual field and that this plane accounts for all observed phenomena on the earth cannot be imagined. A sphere of the weak curvature of the surface of the earth lies outside the power of our imagination. We can comprehend this sphere only by means of very poor analogies. When we now assert that we can imagine the earth as a sphere, we actually

mean we have become used to renouncing intuitive images and to contenting ourselves with certain analogies.

I believe that the same is true for Riemannian space. The theory of relativity does not assert that what formerly was the geometric picture of things is now curved in the Riemannian sense. It asserts, rather, that there *is no* such picture and that the metric relations express something quite different from a copy of the object. It seems plausible that our intuitive geometrical images are not sufficient for the characterization of a physical state. We must only become used to the idea, not that these images are false, but that they cannot be applied to real things—then we will have achieved the same adjustment we made with respect to the so-called intuition of the spherical shape of the earth, namely, the complete renunciation of visualization. Then we shall be content with analogies—for instance, the beautiful analogy of the two-dimensionally thinking being on the spherical surface, and believe that those analogies represent physics.

It must remain the task of psychology to explain why we need images and analogies for knowledge to such a degree that we cannot achieve a conceptual understanding without them. It is the task of epistemology to explain the nature of knowledge; the present investigation hopes to have shown that we can fulfill this task by an analysis of positive science, without resort to images and analogies.

Reference Notes

[1] (p. 4). Poincaré has defended this conception. Cf. *Science and Hypothesis* (Dover Publications, 1952), pp. 48–51. It is characteristic that from the outset he excludes Riemannian geometry for his proof of equivalence, because it does not permit the shifting of a body without a change of form. If he had known that it would be this geometry which physics would choose, he would not have been able to assert the arbitrariness of geometry.

[2] (p. 4). I had not deemed it necessary to consider in detail occasionally occurring views that Einstein's theory of space might be reconciled with that of Kant. Independently of the decision whether one agrees with Kant or with Einstein, the contradictions between their theories can be clearly delineated. But I find to my great amazement that even today in circles of the Kantgesellschaft it is maintained that the theory of relativity does not touch Kant's theory of space in any way. E. Sellien writes in "Die erkenntnistheoretische Bedeutung der Relativitätstheorie," *Kantstudien,* Ergänzungsheft 48 (1919): "Since geometry concerns essentially the 'pure' intuition of space, physical experience cannot influence it at all. Conversely, such experience becomes possible only through geometry. This fact deprives the theory of relativity of the right to assert that the 'true' geometry is non-Euclidean. At most it might say: The laws of nature can easily be formulated in a very general form if non-Euclidean metric determinations are presupposed." Unfortunately Sellien misses one point: if space is non-Euclid-

ean in the Einsteinian sense, it is not possible by means of *any* coördinate transformation to represent it by Euclidean geometry. The transition to Euclidean geometry would mean a transition to a different physics; the physical laws would be materially different, and only *one* physics can be correct. We are confronted by an "either-or," and it is not understandable why Sellien does not call the theory of relativity *false* if he retains Kant's theory. It also seems strange to think that the theory of relativity has been invented by the physicists for the sake of convenience; I think that Newton's old theory was much more convenient. But when Sellien asserts, furthermore, that Einstein's space is different from the space that Kant had in mind, he contradicts Kant. Of course, experience cannot demonstrate that a space which as a purely fictive structure is imagined to be Euclidean *is* non-Euclidean. But Kant's space, exactly like Einstein's space, is that in which the things of experience, that is, the objects of physics, are located. In this idea lies the epistemological significance of Kant's doctrine and its difference from metaphysical speculation about intuitive fancies.

[3] (p. 5). Until now there exist no presentations of the theory of relativity in which these relations have been formulated with sufficient clarity. All existing presentations are more interested in convincing than in axiomatizing. The presentation of Erwin Freundlich (*Die Grundlagen der Einsteinschen Gravitations-theorie* [4th ed.; Berlin: Julius Springer, 1920]) comes closest to this aim in a fruitful combination of a systematic construction and an intuitive understanding of the principles. The distinction between fundamental requirements and particular experiences is clearly carried through in this work. We can therefore refer the reader to Freundlich's book, in particular to the notes, for the empirical justification of Chapters II and III of the present investigation.

Another good presentation of the physical content of the theory is contained in Moritz Schlick, *Raum und Zeit in der gegenwärtigen Physik* (3d ed.; Berlin: Julius Springer, 1920).

[4] (p. 6). Concerning the concept of a priori, cf. note 17.

[5] (p. 9). A. Einstein, "Elektrodynamik bewegter Körper," *Ann. d. Phys.*, ser. 4, vol. 17, pp. 891–921.

[6] (p. 13). We must make the same objection to Natrop's interpretation of the special theory of relativity which he offers in *Die logischen Grundlagen der exakten Wissenschaften* (Leipzig: Teubner, 1910), p. 402. He does not notice that the theory of relativity maintains the velocity of light to be the limiting velocity and believes that Einstein regards this velocity merely to be the highest velocity attainable for the time being. Therefore, Natrop's attempt to save absolute time and to explain the contradictions in terms of the impossibility of its "empirical realization" cannot be considered successful either.

[7] (p. 22). A. Einstein, "Grundlage der allgemeinen Relativitätstheorie," *Ann. d. Phys.*, ser. 4, vol. 49, p. 777.

[8] (p. 26). *Ibid.*, p. 774. Cf. also the excellent presentation of this example by W. Bloch, *Einführung in die Relativitätstheorie* (Leipzig: Teubner, 1918), p. 95.

[9] (p. 35). David Hilbert, *Grundlagen der Geometrie* (Leipzig: Teubner, 1913), p. 5.

[10] (p. 36). Moritz Schlick, *Allgemeine Erkenntnislehre* (Berlin: Springer, 1918), p. 30.

[11] (p. 43). *Ibid.*, p. 45.

[12] (p. 53). I. Kant, *Critique of Pure Reason, Great Books of the Western World* (Chicago, London, Toronto: Encyclopædia Britannica, Inc., 1952), XLII, 48.

[13] (p. 53). This principle is justified in my own publications mentioned in note 20.

[14] (p. 53). This principle has been analyzed by Kurt Lewin. Cf. his books mentioned in note 20.

[15] (p. 54). Arthur Haas gives a good survey of the development of the physical axioms of connection in *Naturwissenschaften*, VII (1919), p. 744. Haas believes, however, that he is dealing with the total number of axioms; he does not see the necessity for physical axioms of coördination.

[16] (p. 56). I. Kant *op. cit.*, p. 34. It is not quite clear why

Kant believes that these other creatures can differ from us only with respect to intuition, not with respect to the categories. His theory would not be impaired by the second assumption either.

[17] (p. 57). The objection might be made that Kant never used the word "self-evidence" for the characterization of a priori principles. However, it can easily be shown that the *insight into the necessary validity* of a priori propositions asserted by Kant does not differ from what we have called self-evidence. I admit that Kant's method of starting from the existence of self-evident a priori propositions as a fact and of analyzing merely their position within the concept of knowledge has been abandoned by some Neo-Kantians—even though it seems to me that in this way a fundamental principle of Kant's doctrine has been lost which until now has not been replaced by a better one. But I want to restrict myself in this investigation to a discussion of Kant's theory in its original form. I believe that this theory stands unexcelled by any other philosophy and that only it, in its precisely constructed system, is equivalent to Einstein's theory in the sense that a fruitful discussion can ensue. For the validation of my conception of Kant's concept of a priori, I cite the following passages from the *Critique of Pure Reason* (pages according to *Great Books of the Western World,* Robert Maynard Hutchins, ed., XLII [Chicago, London, Toronto: Encyclopædia Britannica, Inc., 1952], translated by I. M. D. Meiklejohn). "The question now is as to a *criterion,* by which we may securely distinguish a pure from an empirical cognition. Experience no doubt teaches us that this or that object is constituted in such and such a manner, but not that it could not possibly exist otherwise. Now, in the first place, if we have a proposition that contains the idea of necessity in its very conception, it is a judgment *a priori.* . . . If, on the other hand, a judgment carries with it strict and absolute universality, that is, admits of no possible exception, it is not derived from experience, but is valid absolutely *a priori*" (p. 14). "Now, that in the sphere of human cognition we have judgments which are necessary, and in the strictest sense universal, consequently pure *a priori,* it will be an easy matter to show. If we desire an example from the sciences, we need only take any proposition in mathematics. If we cast our eyes upon the commonest operations of the understanding, the proposition, 'Every change must have a cause,' will amply serve our purpose. In the latter case, indeed, the conception of a cause so plainly involves the conception of a necessity of connection with an effect, and of a strict universality of the law, that the very notion of a cause would entirely disappear, were we to derive it . . . from . . . the habit . . . of connecting representations . . ." (p. 15).

"The science of natural philosophy (physics) contains in itself synthetic judgments *a priori* as principles. I shall adduce two propositions. For instance, the proposition, 'In all changes of the material world, the quantity of matter remains unchanged'; or that, 'In all communication of motion, action and reaction must always be equal.' In both of these, not only is the necessity, and therefore their origin, *a priori* clear, but also that they are synthetical propositions" (p. 18).

And of pure mathematics and pure science, the prototype of a priori propositions in these sciences, he says: "Respecting these sciences, as they do certainly exist, it may with propriety be asked, *how* they are possible?—For *that* they must be possible is shown by the fact of their really existing" (p. 19). And in *Prolegomena* (I. Kant, *Prolegomena to any Future Metaphysics,* trans. Peter G. Lucas [Manchester University Press, 1953]: "It is fortunately the case . . . that certain pure synthetic knowledge *a priori* is real and given, namely *pure mathematics* and *pure science,* for both contain propositions which are everywhere recognized, partly as apodictically certain by mere reason, partly by universal agreement from experience . . ." (p. 29). "But here we cannot rightly start by looking for the *possibility* of such propositions, i. e., by asking whether they are possible. For there are plenty of them, really given with undisputed certainty" (p. 30).

It is not necessary to cite quotations for the second meaning of "a priori," which will not be disputed. I refer in particular to the transcendental deduction in the *Critique of Pure Reason.*

[18] (p. 66). For a precise justification of this hypothesis of the theory of probability, I refer to my publications mentioned in note 20.

[19] (p. 71). I. Kant, *Critique of Judgment,* trans. J. H. Bernard (New York: Hafner Publishing Co., 1951) pp. 21–23.

[20] (p. 75). H. Reichenbach, *Der Begriff der Wahrscheinlichkeit für die mathematische Darstellung der Wirklichkeit* (Ph.D. dissertation, 1915) and *Zeitschrift für Philosophie und philosophische Kritik,* CLXI, 210–239, and CLXII, 98–112, 223–253; "Die physikalischen Voraussetzungen der Wahrscheinlichkeitsrechnung," *Naturwissenschaften,* VIII, 3, pp. 46–55; "Philosophische Kritik der Wahrscheinlichkeitsrechnung," *Naturwissenschaften,* VIII, 8, pp. 146–153; "Über die physikalischen Voraussetzungen der Wahrscheinlichkeitsrechnung," *Zeitschrift der Physik,* II, 2, pp. 150–171.

The same scientific orientation is adopted in the theoretical studies of Kurt Lewin, *Die Verwandtschaftsbegriffe in Biologie und Physik und die Darstellung vollständiger Stammbäume* (Berlin: Bornträger, 1920), and *Der Ordnungstypus der genetischen Reihen in Physik, organismischer Biologie und Entwicklungsgeschichte* (Berlin: Bornträger, 1920).

Recently Ernst Cassirer has contributed an analysis of the epistemological significance of the theory of relativity (*Zur Einsteinschen Relativitätstheorie, erkenntnistheoretische Betrachtungen* [Berlin: B. Cassirer, 1920]) in which for the first time an outstanding representative of the Neo-Kantian school attempts a discussion of the general theory of relativity. The work is intended to furnish the basis for a discussion between physicists and philosophers. Indeed, nobody seems to be better qualified in the Neo-Kantian camp to start such a discussion than Cassirer, whose critical analysis of physical concepts has always tended in a direction familiar to the theory of relativity. This is especially true for the concept of substance. (Cf. E. Cassirer, *Substanzbegriff und Funktionsbegriff* [Berlin: B. Cassirer, 1910]). Unfortunately I could not consider Cassirer's contributions, for I was able to read them only after this book had gone to press.

[21] (p. 76). Hermann Weyl, *Raum-Zeit-Materie* (Berlin: Springer, 1918), p. 227; Arthur Haas, "Die Physik als geometrische Notwendigkeit," *Naturwissenschaften,* VIII, 7, pp. 121–140.

[22] (p. 76). Hermann Weyl, "Gravitation und Elektrizität," *Sitz. Ber. der Berliner Akademie* (1918), pp. 465–480.

[23] (p. 78). Cf., for instance, *Critique of Pure Reason:* "A philosopher was asked: 'What is the weight of smoke?' He answered: 'Subtract from the weight of the burnt wood the weight of the remaining ashes, and you will have the weight of smoke.' Thus he presumed it to be incontrovertible that even in fire matter (substance) does not perish, but that only the form of it undergoes a change" (*op. cit.,* p. 75). This example is chemically incorrect; however, it shows, clearly how concretely Kant thought of substance as weighable matter.

[24] (p. 81). In this sense I must now correct my assertion made in previous publications (cf. note 20) that this principle cannot be refuted by experience. A refutation in the sense of a conceptual generalization is possible according to the method of successive approximations; but so primitive a test as is occasionally made by means of counting simple probability distributions is worthless.

[25] (p. 82). Cf. my first publication, mentioned in note 20.

[26] (p. 84). Cf. p. 323 of the book mentioned in note 10.

[27] (p. 85). It is remarkable that Schlick, who makes the concept of unique coördination the center of his investigations and who shows great merit in his justification of the significance of this concept, has never seen the possibility of such a generalization. For him it is obvious that the coördination must be unique. He regards it as a necessary human constitution to obtain knowledge in this way, and he thinks that knowledge would arrive at a *non possumus* if some day a unique coördina-

tion could no longer be carried through. Yet Kant did not assert anything different when he established his categories. It is characteristic of Schlick's psychologizing method that he believes to have refuted by many proofs the correct part of Kant's theory, namely, the constitutive significance of the coördinating principles, and that he accepts the incorrect part without noticing it. The characterization of knowledge as unique coördination is Schlick's analysis of reason, and the uniqueness is his synthetic judgment a priori.

[28] (p. 96). Helge Holst, "Die kausale Relativitätsforderung und Einstein's Relativitätstheorie," *Det Kgl. Danske Vidensk, Selskab Math.-fys.* (Medd. II, 11, Copenhagen, 1919).