THE WAYS OF PARADOX AND OTHER ESSAYS

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pretation of 'F_i' in θ, for each i. Yet, under the above definition, '$A(F_1, \ldots, F_n)$' is logically deducible from just the arithmetical truth '$A(K_1, \ldots, K_n)$'. (Proof: 'F_1', ..., 'F_n' are equated by the definition to 'K_1', ..., 'K_n' unless $A(G_1, \ldots, G_n)$, and in this event they are equated to 'G_1', ..., 'G_n', so that again $A(F_1, \ldots, F_n)$.)

The shift of system was of course farcical. We merely rewrote the primitive predicates of θ as new letters, keeping the old chemical interpretations, and then pleonastically defined the old predicate letters anew in terms of these so that their chemical interpretations were again preserved (extensionally anyway). Yet the erstwhile chemical axioms of θ became, under this definitional hocus pocus, arithmetically true.

I do not speak of arithmetical demonstrability, for a question there arises of choosing among incomplete systems of number theory. I speak of arithmetical truth.

The doctrine that axioms are implicit definitions thus gains support. If axioms are satisfiable at all, they can be viewed as a shorthand instruction to adopt definitions as above, rendering one's theory true by arithmetic. And, if the axioms were true on a literal reading, the interpretation of their predicates remains undisturbed.

The doctrine of implicit definition has been deplored as a too facile way of making any desired truth analytic: just call it an axiom. Now we see that such claims to analyticity are every bit as firm as can be made for sentences whose truth follows by definition from arithmetic. So much the worse, surely, for the notion of analyticity.

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Ontological Reduction and the World of Numbers

One conspicuous concern of analytical or scientific philosophy has been to reduce some notions to others, preferably to less putative ones. A familiar case of such reduction is Frege's definition of number. Each natural number n became, if I may speak in circles, the class of all n-member classes. As is also well known, Frege's was not the only good way. Another was von Neumann's. Under it, if I may again speak in circles, each natural number n became the class of all numbers less than n.

In my judgment we have satisfactorily reduced one predicate to others, certainly, if in terms of these others we have fashioned an open sentence that is *co-extensive* with the predicate in question as originally interpreted; i.e., that is satisfied by the same values of the variables. But this standard does not suit the Frege and von Neumann reductions of number; for these reductions are both good, yet not co-extensive with each other.

Again consider Carnap's clarification of measure, or impure number, where he construes 'the temperature of x is n°C' in the fashion 'the temperature-in-degrees-Centigrade of x is n' and so dispenses with the impure numbers n°C in favor of the pure numbers n.[1] There had been, we might say, a two-place predicate

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[1] Carnap, *Physikalische Begriffsbildung*.

'H' of temperature such that '$H(x, \alpha)$' meant that the temperature of x was α. We end up with a new two-place predicate 'H_c' of temperature in degrees Centigrade. '$H(x, n°\mathrm{C})$' is explained away as '$H_c(x, n)$'. But 'H' is not co-extensive with 'H_c', nor indeed with any surviving open sentence at all; 'H' had applied to putative things α, impure numbers, which come to be banished from the universe. Their banishment was Carnap's very purpose. Such reduction is in part *ontological,* as we may say, and co-extensiveness here is clearly not the point.

The definitions of numbers by Frege and von Neumann are best seen as ontological reductions too. Carnap, in the last example, showed how to skip the impure numbers and get by with pure ones. Just so, we might say, Frege and von Neumann showed how to skip the natural numbers and get by with what we may for the moment call *Frege classes* and *von Neumann classes.* There is only this difference of detail: Frege classes and von Neumann classes simulate the behavior of the natural numbers to the point where it is convenient to call them natural numbers, instead of saying that we have contrived to dispense with the natural numbers as Carnap dispensed with impure numbers.

Where reduction is in part ontological, we see, co-extensiveness is not the issue. What then is? Consider again Frege's way and von Neumann's of construing natural number. And there is yet a third well-known way, Zermelo's. Why are these all good? What have they in common? Each is a structure-preserving model of the natural numbers. Each preserves arithmetic, and that is enough. It has been urged that we need more: we need also to provide for translating mixed contexts in which the arithmetical idioms occur in company with expressions concerning physical objects and the like. Specifically, we need to be able to say what it means for a class to have n members. But in fact this is no added requirement. We can say what it means for a class to have n members no matter how we construe the numbers, as long as we have them in order. For to say that a class has n members is simply to say that the members of the class can be correlated with the natural numbers up to n, whatever they are.

The real numbers, like the natural numbers, can be taken in a variety of ways. The Dedekind cut is the central idea, but you can use it to explain real numbers either as certain classes of ratios, or as certain relations of natural numbers, or as certain classes of natural numbers. Under the first method, if I may again speak in circles, each real number x becomes the class of all ratios less than x. Under the second method, x becomes this relation of natural numbers: m bears the relation to n if m stands to n in a ratio less than x. For the third version, we change this relation of natural numbers to a class of natural numbers by mapping the ordered pairs of natural numbers into the natural numbers.

All three alternatives are admissible, and what all three conspicuously have in common is, again, just the relevant structure: each is a structure-preserving model of the real numbers. Again it seems that no more is needed to assure satisfactory translation also of any mixed contexts. When real numbers are applied to magnitudes in the physical world, any model of the real numbers could be applied as well.

The same proves true when we come to the imaginary numbers and the infinite numbers, cardinal and ordinal: the problem of construing comes to no more, again, than modeling. Once we find a model that reproduces the formal structure, there seems to be no difficulty in translating any mixed contexts as well.

These cases suggest that what justifies the reduction of one system of objects to another is preservation of relevant structure. Since, according to the Löwenheim–Skolem theorem, any theory that admits of a true interpretation at all admits of a model in the natural numbers, G. D. W. Berry concluded that only common sense stands in the way of adopting an all-purpose Pythagorean ontology: natural numbers exclusively.

There is an interesting reversal here. Our first examples of ontological reduction were Frege's and von Neumann's reductions of natural number to set theory. These and other examples encouraged the thought that what matters in such reduction is the discovery of a model. And so we end up saying, in view of the Löwenheim–Skolem theorem, that theories about objects of any sort can, when true, be reduced to theories of natural numbers. Instead of reducing talk of numbers to talk of sets, we may reduce talk of sets—and of all else—to talk of natural numbers. And here there is an evident gain, since the natural numbers are relatively clear and, as infinite sets go, economical.

But is it true that all that matters is a model? Any interpretable theory can, in view of the Löwenheim–Skolem theorem, be *modeled* in the natural numbers, yes; but does this entitle us to say that it is once and for all *reducible* to that domain, in a sense that would allow us thenceforward to repudiate the old objects for all purposes and recognize just the new ones, the natural numbers? Examples encouraged in us the impression that modeling assured such reducibility, but we should be able to confirm or remove the impression with a little analysis.

What do we require of a reduction of one theory to another? Here is a complaisant answer: any effective mapping of closed sentences on closed sentences will serve if it preserves truth. If we settle for this, then what of the thesis that every true theory θ can be reduced to a theory about natural numbers? It can be proved, even without the Löwenheim–Skolem theorem. For we can translate each closed sentence S of θ as 'Tx' with x as the Gödel number of S and with 'T' as the *truth predicate* for θ, a predicate satisfied by all and only the Gödel numbers of true sentences of θ.

Of this trivial way of reducing an ontology to natural numbers, it must be said that whatever it saves in ontology it pays for in *ideology:* we have to strengthen the primitive predicates. For we know from Gödel and Tarski that the truth predicate of θ is expressible only in terms that are stronger in essential ways than any originally available in θ itself.[2]

Nor is this a price that can in general be saved by invoking the Löwenheim–Skolem theorem. I shall explain why not. When, in conformity with the proof of the Löwenheim–Skolem theorem, we reinterpret the primitive predicates of a theory θ so as to make them predicates of natural numbers, we do not in general make them arithmetical predicates. That is, they do not in general go over into predicates that can be expressed in terms of sum, product, equality, and logic. If we are modeling merely the *theorems* of a deductive system—the implicates of an effective if not finite set of axioms—then certainly we can get arithmetical reinterpretations of the predicates.[3] But that is not what we are about. We are concerned rather to accommodate all the *truths* of θ—all the sentences, regardless of axiomatizability, that were true under the original interpretation of the predicates of θ. There

[2] See Tarski, *Logic, Semantics, Metamathematics,* p. 273. There are exceptions where θ is especially weak; see Myhill, p. 194.

[3] See Wang; also Kleene, pp. 389–398 and more particularly p. 431. For exposition see also my "Interpretations of sets of conditions."

is, under the Löwenheim–Skolem theorem, a reinterpretation that carries all these truths into truths about natural numbers; but there may be no such interpretation in arithmetical terms. There will be if θ admits of complete axiomatization, of course, and there will be under some other circumstances, but not under all. In the general case the most that can be said is, again, that the numerical reinterpretations are expressible in the notation of arithmetic plus the truth predicate for θ.[4]

So on the whole the reduction to a Pythagorean ontology exacts a price in ideology whether we invoke the truth predicate directly or let ourselves be guided by the argument of the Löwenheim–Skolem theorem. Still there is a reason for preferring the latter, longer line. When I suggested simply translating S as 'Tx' with x as Gödel number of S, I was taking advantage of the liberal standard: reduction was just any effective and truth-preserving mapping of closed sentences on closed sentences. Now the virtue of the longer line is that it works also for a less liberal standard of reduction. Instead of accepting just any and every mapping of closed sentences on closed sentences so long as it is effective and truth-preserving, we can insist rather that it preserve predicate structure. That is, instead of mapping just whole sentences of θ on sentences, we can require that each of the erstwhile primitive predicates of θ carry over into a predicate or open sentence about the new objects (the natural numbers).

Whatever its proof and whatever its semantics, a doctrine of blanket reducibility of ontologies to natural numbers surely trivializes most further ontological endeavor. If the universe of discourse of every theory can as a matter of course be standardized as the Pythagorean universe, then apparently the only special ontological reduction to aspire to in any particular theory is reduction to a finite universe. Once the size is both finite and specified, of course, ontological considerations lose all force; for we can then reduce all quantifications to conjunctions and alternations and so retain no recognizably referential apparatus.

Some further scope for ontological endeavor does still remain, I suppose, in the relativity to ideology. One can try to reduce a given theory to the Pythagorean ontology without stepping up its

[4] This can be seen by examining the general construction in §1 of "Interpretations of sets of conditions."

ideology. This endeavor has little bearing on completely axiomatized theories, however, since they reduce to pure arithmetic, or elementary number theory.[5]

Anyway we seem to have trivialized most ontological contrasts. Perhaps the trouble is that our standard of ontological reduction is still too liberal. We narrowed it appreciably when we required that the predicates be construed severally. But we still did not make it very narrow. We continued to allow the several predicates of a theory θ to go over into any predicates or open sentences concerning natural numbers, so long merely as the truth values of closed sentences were preserved.

Let us return to the Carnap case of impure number for a closer look. We are initially confronted with a theory whose objects include place-times x and impure numbers α and whose primitive predicates include 'H'. We reduce the theory to a new one whose objects include place-times and pure numbers, and whose predicates include 'H_c'. The crucial step consists of explaining '$H(x, n°\text{C})$' as '$H_c(x, n)$'.

Now this is successful, if it is, because three conditions are met. One is, of course, that '$H_c(x, n)$' under the intended interpretation agrees in truth value with '$H(x, n°\text{C})$', under its originally intended interpretation, for all values of x and n. A second condition is that, in the original theory, all mention of impure numbers α was confined or confinable to the specific form of context '$H(x, \alpha)$'. Otherwise the switch to '$H_c(x, n)$' would not eliminate such mention. But if this condition were to fail, through there being further predicates (say a predicate of length or of density) and further units (say meters) along with 'H' and degrees, we could still win through by just treating them similarly. A third condition, finally, is that an impure number α can always be referred to in terms of a pure number and a unit: thus n°C, n meters. Otherwise explaining '$H(x, n°\text{C})$' as '$H_c(x, n)$' would not take care of '$H(x, \alpha)$'.

This third condition is that we be able to specify what I shall call a *proxy function:* a function which assigns one of the new things, in this example a pure number, to each of the old things—each of the impure numbers of temperature. In this example the

[5] Thus far in this paper I have been recording things that I said in the Shearman Lectures at University College, London, February 1954. Not so from here on.

proxy function is the function "how many degrees centigrade"—the function f such that $f(n°\text{C}) = n$. It is not required that such a function be expressible in the original theory θ to which 'H' belonged, much less that it be available in the final theory θ' to which 'H_c' belongs. It is required rather of *us*, out in the metatheory where we are explaining and justifying the discontinuance of θ in favor of θ', that we have some means of expressing a proxy function. Only upon us, who explain '$H(x, \alpha)$' away by '$H_c(x, n)$', does it devolve to show how every α that was intended in the old θ determines an n of the new θ'.

In these three conditions we have a further narrowing of what had been too liberal a standard of what to count as a reduction of one theory or ontology to another. We have in fact narrowed it to where, as it seems to me, the things we should like to count as reduction do so count and the rest do not. Carnap's elimination of impure number so counts; likewise Frege's and von Neumann's reduction of natural arithmetic to set theory; likewise the various essentially Dedekindian reductions of the theory of real numbers. Yet the general trivialization of ontology fails; there ceases to be any evident way of arguing, from the Löwenheim–Skolem theorem, that ontologies are generally reducible to the natural numbers.

The three conditions came to us in an example. If we restate them more generally, they lose their tripartite character. The standard of reduction of a theory θ to a theory θ' can now be put as follows. We specify a function, not necessarily in the notation of θ or θ', which admits as arguments all objects in the universe of θ and takes values in the universe of θ'. This is the proxy function. Then to each n-place primitive predicate of θ, for each n, we effectively associate an open sentence of θ' in n free variables, in such a way that the predicate is fulfilled by an n-tuple of arguments of the proxy function always and only when the open sentence is fulfilled by the corresponding n-tuple of values.

For brevity I am supposing that θ has only predicates, variables, quantifiers, and truth functions. The exclusion of singular terms, function signs, abstraction operators, and the like is no real restriction, for these accessories are reducible to the narrower basis in familiar ways.

Let us try applying the above standard of reduction to the

Frege case: Frege's reduction of number to set theory. Here the proxy function f is the function which, applied, e.g., to the "genuine" number 5, gives as value the class of all five-member classes (Frege's so-called 5). In general fx is describable as the class of all x-member classes.

When the real numbers are reduced (by what I called the first method) to classes of ratios, fx is the class of all ratios less than the "genuine" real number x.

I must admit that my formulation suffers from a conspicuous element of make-believe. Thus, in the Carnap case I had to talk as if there *were* such things as x°C, much though I applaud Carnap's repudiation of them. In the Frege case I had to talk as if the "genuine" number 5 were really something over and above Frege's, much though I applaud his reduction. My formulation belongs, by its nature, in an inclusive theory that admits the objects of θ, as unreduced, and the objects of θ' on an equal footing.

But the formulation seems, if we overlook this imperfection, to mark the boundary we want. Ontological reductions that were felt to be serious do conform. Another that conforms, besides those thus far mentioned, is the reduction of an ontology of place-times to an ontology of number quadruples by means of Cartesian co-ordinates. And at the same time any sweeping Pythagoreanization on the strength of the Löwenheim–Skolem theorem is obstructed. The proof of the Löwenheim–Skolem theorem is such as to enable us to give the predicates of the numerical model; but the standard of ontological reduction that we have now reached requires more than that. Reduction of a theory θ to natural numbers—true reduction by our new standard, and not mere modeling—means determining a proxy function that actually assigns numbers to all the objects of θ and maps the predicates of θ into open sentences of the numerical model. Where this can be done, with preservation of truth values of closed sentences, we may well speak of reduction to natural numbers. But the Löwenheim–Skolem argument determines, in the general case, no proxy function. It does not determine which numbers are to go proxy for the respective objects of θ. Therein it falls short of our standard of ontological reduction.

It emerged early in this paper that what justifies an ontological reduction is, vaguely speaking, preservation of relevant structure.

What we now perceive is that this relevant structure runs deep; the objects of the one system must be assigned severally to objects of the other.

Goodman argued along other lines to this conclusion and more;[6] he called for isomorphism, thereby requiring one-to-one correspondence between the old objects and their proxies. I prefer to let different things have the same proxy. For instance n is wanted as proxy for both n°C and n meters. Or again consider hidden inflation, as described in the preceding essay. Relieving such inflation is a respectable brand of ontological reduction, and it consists precisely in taking one thing as proxy for all the things that were indiscriminable from it.[7]

[6] Pp. 5–19.

[7] I am indebted for this observation to Paul Benacerraf. On such deflation see further my discussion of identification of indiscernibles in *Word and Object*, p. 230; in *From a Logical Point of View*, pp. 71f; and in "Reply to Professor Marcus," which is Essay 14 above.