SPACE AND GEOMETRY FROM THE POINT OF VIEW OF PHYSICAL INQUIRY.[1]

Our notions of space are rooted in our *physiological* organism. Geometric concepts are the product of the idealization of *physical* experiences of space. Systems of geometry, finally, originate in the *logical* classification of the conceptual materials so obtained. All three factors have left their indubitable traces in modern geometry. Epistemological inquiries regarding space and geometry accordingly concern the physiologist, the psychologist, the physicist, the mathematician, the philosopher, and the logician alike, and they can be gradually carried to their definitive solution only by the consideration of the widely disparate points of view which are here offered.

Awakening in early youth to full consciousness, we find ourselves in possession of the notion of a *space* surrounding and encompassing our body, in which space move divers *bodies*, now altering and

[1] I shall endeavor in this essay to define my attitude as a physicist toward the subject of metageometry so called. Detailed geometric developments will have to be sought in the sources. I trust, however, that by the employment of illustrations which are familiar to every one I have made my expositions as popular as the subject permitted.

now retaining their size and shape. It is impossible for us to ascertain how this notion has been begotten. Only the most thoroughgoing analysis of experiments purposefully and methodically performed has enabled us to conjecture that inborn idiosyncracies of the body have coöperated to this end with simple and crude experiences of a purely physical character.

Sensational and Locative Qualties.

An object seen or touched is distinguished not only by a *sensational quality* (as "red," "rough," "cold," etc.), but also by a *locative quality* (as "to the left," "above," "before," etc.). The sensational quality may remain the same, while the locative quality continuously changes; that is, the same sensuous object may move in space. Phenomena of this kind being again and again induced by physico-physiological circumstances, it is found that however varied the accidental sensational qualities may be, the same order of locative qualities invariably occurs, so that the latter appear perforce as a fixed and permanent system or register in which the sensational qualities are entered and classified. Now, although these qualities of sensation and locality can be excited only in conjunction with one another, and can make their appearance only concomitantly, the impression nevertheless easily arises that the more familiar system of locative qualities is given antecedently to the sensational qualities (Kant).

Extended objects of vision and of touch consist of more or less distinguishable sensational qualities, conjoined with adjacent distinguishable, continuously graduated locative qualities. If such objects move, particularly in the domain of our hands, we perceive them to shrink or swell (in whole or in part), or we perceive them to remain the same; in other words, the contrasts characterizing their bounding locative qualities change or remain constant. In the latter case, we call the objects rigid. By the recognition of permanency as coincident with spatial displacement, the various constituents of our intuition of space are rendered *comparable* with one another,—at first in the *physiological* sense. By the comparison of different bodies with one another, by the introduction of *physical* measures, this comparability is rendered quantitative and more exact, and so transcends the limitations of individuality. Thus, in the place of an individual and non-transmittable intuition of space are substituted the universal concepts of geometry, which hold good for all men. Each person has his own individual intuitive space; geometric space is common to all. Between the space of intuition and *metric* space, which contains physical experiences, we must distinguish sharply.

Riemann's Physical Conception of Geometry.

The need of a thoroughgoing epistemological elucidation of the foundations of geometry induced Riemann,[1] about the middle of the century just

[1] *Ueber die Hypothesen, welche der Geometrie zu Grunde liegen.* Göttingen, 1867.

closed, to propound the question of the nature of space; the attention of Gauss, Lobachévski, and Bolyai having before been drawn to the empirically hypothetical character of certain of the fundamental assumptions of geometry. In characterizing space as a special case of a multiply-extended "magnitude," Riemann had doubtless in mind some geometric construct, which may in the same manner be imagined to fill all space,—for example, the system of Cartesian co-ordinates. Riemann further asserts that "the propositions of geometry cannot be deduced from general conceptions of magnitude, but that the peculiar properties by which space is distinguished from other conceivable triply-extended magnitudes can be derived from experience only. . . . These facts, like all facts, are in no wise necessary, but possess empirical certitude only,—they are hypotheses." Like the fundamental assumptions of every natural science, so also, on Riemann's theory, the fundamental assumptions of geometry, to which experience has led us, are merely *idealizations* of experience.

In this physical conception of geometry, Riemann takes his stand on the same ground as his master Gauss, who once expressed the conviction that it was impossible to establish the foundations of geometry entirely *a priori*,[1] and who further asserted that "we must in humility confess that if number is exclusively a product of the mind, space

[1] *Brief von Gauss an Bessel*, 27. Januar 1829.

possesses in addition a reality outside of our mind, of which reality we cannot fully dictate *a priori* the laws."[1]

Analogies of Space with Colors.

Every inquirer knows that the knowledge of an object he is investigating is materially augmented by *comparing* it with related objects. Quite naturally therefore Riemann looks about him for objects which offer some analogy to space. Geometric space is defined by him as a triply-extended continuous manifold, the elements of which are the points determined by every possible three co-ordinate values. He finds that "the places of sensuous objects and colors are probably the only concepts [*sic*] whose modes of determination form a multiply-extended manifold." To this analogy others were added by Riemann's successors and elaborated by them, but not always, I think, felicitously.[2]

[1] *Brief von Gauss an Bessel.* April 9, 1830.—The phrase, "Number is a product or creation of the mind," has since been repeatedly used by mathematicians. Unbiased psychological observation informs us, however, that the formation of the concept of number is just as much initiated by experience as the formation of geometric concepts. We must at least know that virtually *equivalent* objects exist in multiple and unalterable form before concepts of number can originate. Experiments in counting also play an important part in the development of arithmetic.

[2] When acoustic pitch, intensity, and *timbre*, when chromatic tone, saturation, and luminous intensity are proposed as analogues of the three dimensions of space, few persons will be satisfied. *Timbre*, like chromatic tone, is dependent on several variables. Hence, if the analogy has any meaning whatever, several dimensions will be found to correspond to *timbre* and chromatic tone.

Comparing *sensation* of space with *sensation* of color, we discover that to the continuous series "above and below," "right and left," "near and far," correspond the three sensational series of mixed colors, black-white, red-green, blue-yellow. The system of sensed (seen) places is a triple continuous manifold like the system of color-sensations. The objection which is raised against this analogy, viz., that in the first instance the three variations (dimensions) are homogeneous and interchangeable with one another, while in the second instance they are heterogeneous and not interchangeable, does not hold when space-*sensation* is compared with color-*sensation*. For from the psycho-physiological point of view "right and left" as little permit of being interchanged with "above and below" as do red and green with black and white. It is only when we compare *geometric* space with the system of colors that the objection is apparently justified. But there is still a great deal lacking to the establishment of a complete analogy between the space of intuition and the system of color-sensation. Whereas nearly equal distances in sensuous space are immediately recognized as such, a like remark cannot be made of differences of colors, and in this latter province it is not possible to compare physiologically the different portions with one another. And, furthermore, even if there be no difficulty, by resorting to physical experience, in characterizing every color of a system by three numbers, just as the places of geometric space are characterized, and so in creating a metric system similar to the latter, it will nevertheless be difficult to find anything which corresponds to distance or volume and which has an analogous physical significance for the system of colors.

Analogies of Space with Time.

There is always an *arbitrary* element in analogies, for they are concerned with the coincidences to which the attention is directed. But between space and time doubtless the analogy is fully conceded, whether we use the word in its physiological or its physical sense. In both meanings of the term, space is a triple, and time a simple, continuous manifold. A physical event, precisely determined by its conditions, of moderate, not too long or too short duration, seems to us physiologically, *now and at any other time,* as having the same duration. Physical events which at any time are temporarily coincident are likewise temporarily coincident at any other time. Temporal congruence exists, therefore, just as much as does spatial congruence. Unalterable physical temporal objects exist, therefore, as much as unalterable physical spatial objects (rigid bodies). There is not only spatial but there is also temporal substantiality. Galileo employed corporeal phenomena, like the beats of the pulse and breathing, for the determination of time, just as anciently the hands and the feet were employed for the estimation of space.

The simple manifold of *tonal sensations* is likewise analogous to the triple manifold of space-sensations.[1] The comparability of the different parts of the system of tonal sensations is given by the possibility of directly sensing the musical *interval.* A metric system corresponding to geometric space is most easily obtained by expressing tonal pitch in terms of the logarithm of the rate of vibration. For the constant musical interval we have here the expression,

$$\log \frac{n'}{n} = \log n' - \log n = \log \tau - \log \tau' = \text{const.},$$

where n', n denote the rates, and τ', τ the periods of vibration of the higher and the lower note respectively. The difference between the logarithms here represents the constancy of the length on displacement. The unalterable, substantial physical object which we sense as an interval is for the ear *temporally* determined, whereas the analogous object for the senses of sight and touch is spatially determined. Spatial measure seems to us simpler solely because we have chosen for the fundamental measure of geometry distance *itself,* which remains unalterable for sensation, whereas in the province of tones we have reached our measure only by a long and circuitous physical route.

[1] My attention was drawn to this analogy in 1863 by my study of the organ of hearing, and I have since then further developed the subject. See my *Analysis of the Sensations.*

Differences of the Analogies.

Having dwelt on the coincidences of our analogized constructs, it now remains for us to emphasize their *differences*. Conceiving time and space as sensational manifolds, the objects whose motions are made perceptible by the alteration of temporal and spatial qualities are characterized by other sensational qualities, as colors, tactual sensations, tones, etc. If the system of tonal sensations is regarded as analogous to the optical space of sense, the curious fact results that in the first province the spatial qualities occur *alone*, unaccompanied by sensational qualities corresponding to the objects, just as if one could see a place or motion without seeing the object which occupied this place or executed this motion. Conceiving spatial qualities as organic sensations which can be excited only *concomitantly* with sensational qualities,[1] the analogy in question does not appear particularly attractive. For the manifold-mathematician, essentially the same case is presented whether an object of definite color moves continuously in optical space, or whether an object spatially fixed passes continuously through the manifold of colors. But for the physiologist and psychologist the two cases are widely different, not only because of what was above adduced, but also, and specifically, because of the fact that the system of spatial qualities is very familiar to us, whereas we can represent to ourselves a system of

[1] Compare *supra*, page 14 et seq.

color-sensations only laboriously and artificially, by means of scientific devices. Color appears to us as an excerpted member of a manifold the arrangement of which is in no wise familiar to us.

The Extension of Symbols.

The manifolds here analogized with space are, like the color-system, also threefold, or they represent a *smaller* number of variations. Space contains surfaces as twofold and lines as onefold manifolds, to which the mathematician, generalizing, might also add points as zero-fold manifolds. There is also no difficulty in conceiving analytical mechanics, with Lagrange, as an analytical geometry of four dimensions, time being considered the fourth co-ordinate. In fact, the equations of analytical geometry, in their conformity to the co-ordinates, suggest very clearly to the mathematician the extension of these considerations to an unlimited *larger* number of dimensions. Similarly, physics would be justified in considering an extended material continuum, to each point of which a temperature, a magnetic, electric, and gravitational potential were ascribed, as a portion or section of a multiple manifold. Employment with such symbolic representations must, as the history of science shows us, by no means be regarded as entirely unfruitful. Symbols which initially appear to have no meaning whatever, acquire gradually, after subjection to what might be called intellectual experimenting, a lucid and precise significance. Think only of the negative, fractional, and variable exponents of algebra, or of the cases in which important and vital extensions of ideas have taken place which otherwise would have been totally lost or have made their appearance at a much later date. Think only of the so-called imaginary quantities with which mathematicians long operated, and from which they even obtained important results ere they were in a position to assign to them a perfectly determinate and withal visualizable meaning. But symbolic representation has likewise the disadvantage that the object represented is very easily lost sight of, and that operations are continued with the symbols to which frequently no object whatever corresponds.[1]

[1] As a young student I was always irritated with symbolic deductions of which the meaning was not perfectly clear and palpable. But historical studies are well adapted to eradicating the tendency to mysticism which is so easily fostered and bred by the somnolent employment of these methods, in that they clearly show the heuristic function of them and at the same time elucidate epistemologically the points wherein they furnish their essential assistance. A symbolical representation of a method of calculation has the same significance for a mathematician as a model or a visualisable working hypothesis has for the physicist. The symbol, the model, the hypothesis runs parallel with the thing to be represented. But the parallelism may extend farther, or be extended farther, than was originally intended on the adoption of the symbol. Since the thing represented and the device representing are after all *different*, what would be concealed in the one is apparent in the other. It is scarcely possible to light directly on an operation like $a^{\frac{2}{3}}$. But operating with such symbols leads us to attribute to them an intelligible meaning. Mathematicians worked many years with expressions like $\cos x \times \sqrt{-1} \sin x$ and with exponentials having imaginary exponents before in the struggle for adapting concept and symbol to each other the idea that had been germinating for a century finally found expression in 1806 in Argand, viz., that a relationship could be conceived between magnitude and *direction* by which $\sqrt{-1}$ was represented as a mean direction-proportional between $+1$ and -1.

Another View of Riemann's Manifold.

It is easy to rise to Riemann's conception of an n-fold continuous manifold, and it is even possible to realize and visualize portions of such a manifold. Let $a_1, a_2, a_3, a_4 \ldots . c_{n+1}$ be any elements whatsoever (sensational qualities, substances, etc.). If we conceive these elements intermingled in all their possible relations, then each single composite will be represented by the expression

$$\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3 + \ldots\ldots \alpha_{n+1} a_{n+1} = 1,$$

where the coefficients α satisfy the equation

$$\alpha_1 + \alpha_2 + \alpha_3 + \ldots\ldots \alpha_{n+1} = 1.$$

Inasmuch, therefore, as n of these coefficients α may be selected at pleasure, the totality of the composites of the $n+1$ elements will represent an n-fold continuous manifold.[1] As co-ordinates of a point of this manifold, we may regard expressions of the form

$$\frac{\alpha_m}{\alpha_1}, \text{ or } f\left(\frac{\alpha_m}{\alpha_1}\right), \text{ for example, } \log\left(\frac{\alpha_m}{\alpha_1}\right).$$

But in choosing definition of distance, or that of any other notion analogous to geometrical concepts, we shall have to proceed very arbitrarily unless *experiences* of the manifold in question inform us that certain metric concepts have a real meaning, and are therefore to be preferred, as is the case for geomet-

[1] If the six fundamental color-sensations were totally independent of one another, the system of color-sensations would represent a five-fold manifold. Since they are contrasted in pairs, the system corresponds to a three-fold manifold.

ric space with the definition[1] derived from the volum-inal constancy of bodies for the element of distances $ds^2 = dx^2 + dy^2 + dz^2$, and as is likewise the case for sensations of tone with the logarithmic expression mentioned above. In the majority of cases where such an artificial construction is involved, fixed points of this sort are wanting, and the entire consideration is therefore an ideal one. The analogy with space loses thereby in completeness, fruitfulness, and stimulating power.

Measure of Curvature, and Curvature of Space.

In still another direction Riemann elaborated ideas of Gauss; beginning with the latter's investigations concerning curved surfaces. Gauss's measure of the curvature[2] of a surface at any point is given by the expression $k = \frac{d\sigma}{ds}$ where ds is an element of the surface and $d\sigma$ is the superficial element of the unit-sphere, the limiting radii of which are parallel to the limiting normals of the element ds. This measure of curvature may also be expressed in the form $k = \frac{1}{\rho_1 \rho_2}$, where ρ_1, ρ_2 are the principal radii of curvature of the surface at the point in question. Of special interest are the surfaces whose measure of curvature for all points has the same

[1] Comp. *supra*, p. 73 *et passim*.

[2] *Disquisitiones generales circa superficies curvas*, 1827.

value,—the surfaces of *constant* curvature. Conceiving the surfaces as infinitely thin, non-distensible, but flexible bodies, it will be found that surfaces of like curvature may be made to coincide by bending,—as for example a plane sheet of paper wrapped round a cylinder or cone,—but cannot be made to coincide with the surface of a sphere. During such deformation, nay, even on crumpling, the proportional parts of figures drawn *in the surface* remain invariable as to lengths and angles, provided we do not go out of the two dimensions of the surface in our measurements. Conversely, likewise, the curvature of the surface does not depend on its conformation in the third dimension of space, but solely upon its *interior proportionalities*. Riemann, now, conceived the idea of generalizing the notion of measure of curvature and applying it to spaces of three or more dimensions. Conformably thereto, he assumes that finite unbounded spaces of constant positive curvature are possible, corresponding to the unbounded but finite two-dimensional surface of the sphere, while what we commonly take to be infinite space would correspond to the unlimited plane of curvature zero, and similarly a third species of space would correspond to surfaces of negative curvature. Just as the figures drawn upon a surface of determinate constant curvature can be displaced without distortion upon this surface only (for example, a spherical figure on the surface of its sphere only, or a plane figure in its plane only), so should analogous conditions necessarily hold for spatial figures and rigid bodies. The latter are capable of free motion only in spaces of constant curvature, as Helmholtz[1] has shown at length. Just as the shortest lines of a plane are infinite, but on the surface of a sphere occur as great circles of definite finite length, closed and reverting into themselves, so Riemann conceived in the three-dimensional space of positive curvature analogues of the straight line and the plane as finite but unbounded. But there is a difficulty here. If we possessed the notion of a measure of curvature for a four-dimensional space, the transition to the special case of three-dimensional space could be easily and rationally executed; but the passage from the special to the more general case involves a certain arbitrariness, and, as is natural, different inquirers have adopted here different courses[2] (Riemann and Kronecker). The very fact that for a one-dimensional space (a curved line of any sort) a measure of curvature does not exist having the significance of an interior measure, and that such a measure first occurs in connection with two-dimensional figures, forces upon us the question whether and to what extent something analogous has any meaning for three-dimensional figures. Are we not subject here to an illusion, in that we operate with symbols to which perhaps nothing real corresponds, or at least nothing representable to the senses, by means of which we can verify and rectify our ideas?

[1] "Ueber die Thatsachen, welche der Geometrie zu Grunde liegen." *Göttinger Nachrichten*, 1868, June 3.

[2] Compare, for example, Kronecker, "Ueber Systeme von Functionen mehrerer Variablen." *Ber. d. Berliner Akademie*, 1869.

Thus were reached the highest and most universal notions regarding space and its relations to analogous manifolds which resulted from the conviction of Gauss concerning the empirical foundations of geometry. But the genesis of this conviction has a preliminary history of two thousand years, the chief phenomena of which we can perhaps better survey from the height which we have now gained.

The Early Discoveries in Geometry.

The unsophisticated men, who, rule in hand, acquired our first geometric knowledge, held to the simplest bodily objects (figures): the straight line, the plane, the circle, etc., and investigated, by means of forms which could be conceived as combinations of these simple figures, the connection of their measurements. It could not have escaped them that the mobility of a body is restricted when one and then two of its points are fixed, and that finally it is altogether checked by fixing three of its points. Granting that rotation about an axis (two points), or rotation about a point in a plane, as likewise displacement with constant contact of two points with a straight line and of a third point with a fixed plane laid through that straight line,—granting that these facts were *separately observed*, it would be known how to distinguish between *pure* rotation, *pure* displacement, and the motion compounded of these two independent motions. The first geometry was of course not based on purely metric notions, but made many considerable concessions to the physiological factors of sense.[1] Thus is the appearance explained of two different fundamental measures: the (straight) length and the angle (circular measure). The straight line was conceived as a rigid mobile body (measuring-rod), and the angle as the

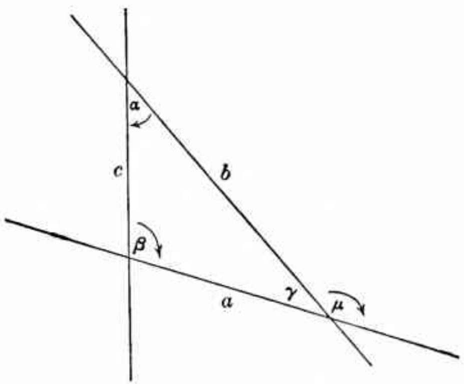

Fig. 14.

rotation of a straight line with respect to another (measured by the arc so described). Doubtless no one ever demanded special proof for the equality of angles at the origin described by the same rotation. Additional propositions concerning angles resulted quite easily. Turning the line b about its intersection with c so as to describe the angle α (Fig. 14), and after coincidence with c turning it again about

[1] Comp. *supra*, p. 83.

its intersection with a till it coincides with a and so describes the angle β, we shall have rotated b from its initial to its final position a through the angle μ in the same sense.[1] Therefore the exterior angle $\mu = \alpha + \beta$, and since $\mu + \gamma = 2R$, also $\alpha + \beta + \gamma = 2R$. Displacing (Fig. 15) the rigid system of lines a, b, c, which intersect at 1, within their plane to the position 2, the line a always remaining within itself, no alteration of angles will be caused by the mere

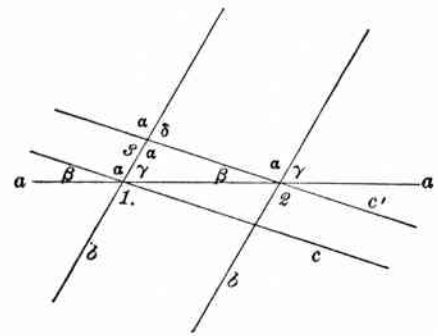

Fig. 15.

motion. The sum of the interior angles of the triangle 1 2 3 so produced is evidently $2R$. The same consideration also throws into relief the properties

[1] C. R. Kosack, *Beiträge zu einer systematischen Entwickelung der Geometrie aus der Anschauung*, Nordhausen, 1852. I was able to see this programme through the kindness of Prof. F. Pietzker of Nordhausen. Similar simple deductions are found in Bernhard Becker's *Leitfaden für den ersten Unterricht in der Geometrie*, Frankfort on the Main, 1845, and in the same author's treatise *Ueber die Methoden des geometrischen Unterrichts*, Frankfort, 1845. I gained access to the first-named book through the kindness of Dr. M. Schuster of Oldenburg.

of parallel lines. Doubts as to whether successive rotation about several points is equivalent to rotation about *one* point, whether *pure* displacement is at all possible,—which are justified when a surface of curvature differing from zero is substituted for the Euclidean plane,—could never have arisen in the mind of the ingenuous and delighted *discoverer* of these relations, at the period we are considering. The study of the movement of rigid bodies, which Euclid studiously avoids and only covertly introduces in his principle of congruence, is to this day the device best adapted to elementary instruction in geometry. An idea is best made the possession of the learner by the method by which it has been found.

Deductive Geometry.

This sound and naïve conception of things vanished and the treatment of geometry underwent essential modifications when it became the subject of *professional* and *scholarly* contemplation. The object now was to systematize the knowledge of this province for purposes of individual survey, to separate what was directly cognizable from what was deducible and deduced, and to throw into distinct relief the thread of deduction. For the purpose of instruction the simplest principles, those most easily gained and apparently free from doubt and contradiction, are placed at the beginning, and the remainder based upon them. Efforts were made to reduce

these initial principles to a minimum, as is observable in the system of Euclid. Through this endeavor to support every notion by another, and to leave to direct knowledge the least possible scope, geometry was gradually detached from the empirical soil out of which it had sprung. People accustomed themselves to regard the derived truths as of higher dignity than the directly perceived truths, and ultimately came to demand proofs for propositions which no one ever seriously doubted. Thus arose,—as tradition would have it, to check the onslaughts of the Sophists,—the system of Euclid with its logical perfection and finish. Yet not only were the ways of research designedly concealed by this artificial method of stringing propositions on an arbitrarily chosen thread of deduction, but the varied organic connection between the principles of geometry was quite lost sight of.[1] This system was more fitted to produce narrow-minded and sterile pedants than fruitful, productive investigators.

[1] Euclid's system fascinated thinkers by its logical excellences, and its drawbacks were overlooked amid this admiration. Great inquirers, even in recent times, have been misled into following Euclid's example in the presentation of the results of their inquiries, and so into actually concealing their methods of investigation, to the great detriment of science. But science is not a feat of legal casuistry. Scientific presentation aims so to expound all the grounds of an idea so that it can at any time be thoroughly examined as to its tenability and power. The learner is not to be led half-blindfolded. There therefore arose in Germany among philosophers and educationists a healthy reaction, which proceeded mainly from Herbart, Schopenhauer, and Trendelenburg. The effort was made to introduce greater perspicuity, more genetic methods, and logically more lucid demonstrations into geometry.

And these conditions were not improved when scholasticism, with its preference for slavish comment on the intellectual products of others, cultivated in thinkers scarcely any sensitiveness for the rationality of their fundamental assumptions and by way of compensation fostered in them an exaggerated respect for the logical form of their deductions. The entire period from Euclid to Gauss suffered more or less from this affection of mind.

Euclid's Fifth Postulate.

Among the propositions on which Euclid based his system is found the so-called Fifth Postulate (also called the Eleventh Axiom and by some the Twelfth): "If a straight line meet two straight lines, so as to make the two interior angles on the same side of it taken together less than two right angles, these straight lines being continually produced, shall at length meet upon that side on which are the angles which are less than two right angles." Euclid easily proves that if a straight line falling on two other straight lines makes the alternate angles equal to each other, the two straight lines will *not* meet but are *parallel*. But for the proof of the converse, that parallels make equal alternate angles with *every* straight line falling on them, he is obliged to resort to the Fifth Postulate. This converse is equivalent to the proposition that *only one* parallel to a straight line can be drawn through a point. Further, by the fact that with the

aid of this converse it can be proved that the sum of the angles of a triangle is equal to two right angles and that from this last theorem again the first follows, the relationship between the propositions in question is rendered distinct and the fundamental significance of the Fifth Postulate for Euclidean geometry is made plain.

The intersection of slowly converging lines lies without the province of construction and observation. It is therefore intelligible that in view of the great importance of the assertion contained in the Fifth Postulate the successors of Euclid, habituated by him to rigor, should, even in ancient times, have strained every nerve to demonstrate this postulate, or to replace it by some immediately obvious proposition. Numberless futile efforts were made from Euclid to Gauss, to deduce this Fifth Postulate from the other Euclidean assumptions. It is a sublime spectacle which these men offer: laboring for centuries, from a sheer thirst for scientific elucidation, in quest of the hidden sources of a truth which no person of theory or of practice ever really doubted! With eager curiosity we follow the pertinacious utterances of the ethical power resident in this human search for knowledge, and with gratification we note how the inquirers gradually are led by their failures to the perception that the true basis of geometry is experience. We shall content ourselves with a few examples.

SACCHERI'S THEORY OF PARALLELS.

Among the inquirers notable for their contributions to the theory of parallels are the Italian Saccheri and the German mathematician Lambert. In order to render their mode of attack intelligible, we will remark first that the existence of rectangles and squares, which we fancy we constantly observe, cannot be demonstrated without the aid of the Fifth Postulate. Let us consider, for example, two congruent isosceles triangles *ABC, DBC,* having right angles at *A* and *D* (Fig. 16), and let them be laid together at their hypothenuses *BC* so as to form the

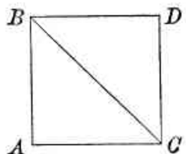

Fig. 16.

equilateral quadrilateral *ABCD*; the first twenty-seven propositions of Euclid do not suffice to determine the character and magnitude of the two equal (right) angles at *B* and *C*. For measure of length and measure of angle are fundamentally different and directly not comparable; hence the first propositions regarding the connection of sides and angles are *qualitative* only, and hence the imperative necessity of a *quantitative* theorem regarding angles, like that of the angle-sum. Be it further remarked that theorems analogous to the twenty-seven planimetric propositions of Euclid may be set up for the surface

of a sphere and for surfaces of constant negative curvature, and that in these cases the analogous construction gives respectively obtuse and acute angles at *B* and *C*.

Saccheri's cardinal achievement was his form of stating the problem.[1] If the Fifth Postulate is involved in the remaining assumptions of Euclid, then it will be possible to prove without its aid that in the quadrilateral *ABCD* (Fig. 17) having right angles at *A* and *B* and $AC = BD$, the angles at *C* and *D* likewise are right angles. And, on the other hand, in this event, the assumption that *C* and *D*

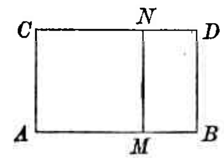

Fig. 17.

are either obtuse or acute will lead to contradictions. Saccheri, in other words, seeks to draw conclusions from the hypothesis of the right, the obtuse, or the acute angle. He shows that each of these hypotheses will hold in all cases if it be proved to hold in one. It is needful to have only one triangle with its angles $\lesseqgtr 2R$ in order to demonstrate the universal validity of the hypothesis of the acute, the right, or the obtuse angle. Notable is the fact that Saccheri also adverts to *physico-geometrical* experi-

[1] *Euclides ab omni naevo vindicatus.* Milan, 1733. German translation in Engel and Staeckel's *Die Theorie der Parallellinien.* Leipsic, 1895.

ments which support the hypothesis of the right angle. If a line CD (Fig. 17) join the two extremities of the equal perpendiculars erected on a straight line AB, and the perpendicular dropped on AB from any point N of the first line, viz., NM, be equal to $CA = DB$, then is the hypothesis of the right angle demonstrated to be correct. Saccheri rightly does not regard it as self-evident that the line which is equidistant from another straight line is itself a straight line. Think only of a circle parallel to a great circle on a sphere which does not represent a

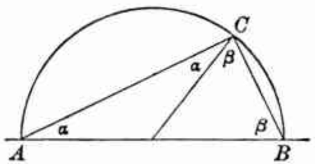

Fig. 18.

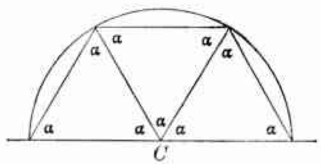

Fig. 19.

shortest line on a sphere and the two faces of which cannot be made congruent.

Other experimental proofs of the correctness of the hypothesis of the right angle are the following. If the angle in a semicircle (Fig. 18) is shown to be a right angle, $\alpha + \beta = R$, then is $2\alpha + 2\beta = 2R$, the sum of the angles of the triangle ABC. If the radius be subtended thrice in a semicircle and the line joining the first and the fourth extremity pass through the center, we shall have at C (Fig. 19) $3\alpha = 2R$, and consequently each of the three triangles will have the angle-sum $2R$. The existence of equiangular triangles of different sizes (similar

triangles) is likewise subject to experimental proof. For (Fig. 20) if the angles at B and C give $\beta + \delta + \gamma + \epsilon = 4R$, so also is $4R$ the angle-sum of the quadrilateral $BCB'C'$. Even Wallis[1] (1663) based his proof of the Fifth Postulate on the assumption of the existence of similar triangles, and a modern geometer, Delbœuf, deduced from the assumption of similitude the entire Euclidean geometry.

The hypothesis of the obtuse angle, Saccheri fan-

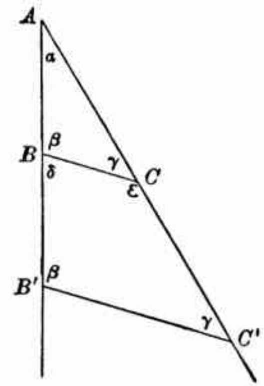

Fig. 20.

cied he could easily refute. But the hypothesis of the acute angle presented to him difficulties, and in his quest for the expected contradictions he was carried to the most far-reaching conclusions, which Lobachévski and Bolyai subsequently rediscovered by methods of their own. Ultimately he felt compelled to reject the last-named hypothesis as incompatible with the nature of the straight line; for it

[1] **Engel and Staeckel,** ***loc. cit.,*** **p. 21 et seq.**

led to the assumption of different kinds of straight lines, which met at infinity, that is, had there a common perpendicular. Saccheri did much in anticipation and promotion of the labors that were subsequently to elucidate these matters, but exhibited withal toward the traditional views a certain bias.

Lambert's Investigations.

Lambert's treatise[1] is allied in method to that of Saccheri, but it proceeds farther in its conclusions, and gives evidence of a less constrained vision. Lambert starts from the consideration of a quadrilateral with three right angles, and examines the consequences that would follow from the assumption that the fourth angle was right, obtuse, or acute. The similarity of figures he finds to be incompatible with the second and third assumptions. The case of the obtuse angle, which requires the sum of the angles of a triangle to exceed $2R$, he discovers to be realized in the *geometry of spherical surfaces,* in which the difficulty of parallel lines entirely vanishes. This leads him to the conjecture that the case of the acute angle, where the sum of the angles of a triangle is less than $2R$, might be realized on the surface of a sphere of imaginary radius. The amount of the departure of the angle-sum from $2R$ is in both cases proportional to the area of the triangle, as may be demonstrated by appropriately dividing large triangles into small triangles, which on diminution may be made to approach as near as we please to the angle-sum $2R$. Lambert advanced very closely in this conception to the point of view of modern geometers. Admittedly a sphere of imaginary radius, $r\sqrt{-1}$ is not a visualizable geometric construct, but analytically it is a surface having a negative constant Gaussian measure of curvature. It is evident again from this example how experimenting with *symbols* also may direct inquiry to the right path, in periods where other points of support are entirely lacking and where every helpful device must be esteemed at its worth.[2] Even Gauss appears to have thought of a sphere of imaginary radius, as is obvious from his formula for the circumference of a circle (*Letter to Schumacher,* July 12, 1831). Yet in spite of all, Lambert actually fancied he had approached so near to the proof of the Fifth Postulate that what was lacking could be easily supplied.

[1]Published in 1766. Engel and Staeckel, *loc cit.*, p. 152 et seq.

View of Gauss.

We may turn now to the investigators whose views possess a most radical significance for our conception of geometry, but who announced their opinion only briefly, by word of mouth or letter. "Gauss regarded geometry merely as a logically consistent system of constructs, with the theory of parallels placed at the pinnacle as an axiom; yet he had

[2] See note, p. 104.

reached the conviction that this proposition could not be proved, though it was known from *experience*,—for example, from the angles of the triangle joining the Brocken, Hohenhagen, and Inselsberg, —that it was approximately correct. But if this axiom be not conceded, then, he contends, there results from its non-acceptance a different and entirely independent geometry, which he had once investigated and called by the name of the Anti-Euclidean

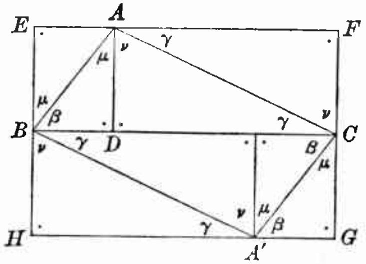

Fig. 21.

geometry." Such, according to Sartorius von Waltershausen, was the view of Gauss.[1]

Researches of Stolz.

Starting at this point, O. Stolz, in a small but very instructive pamphlet,[2] sought to deduce the principal propositions of the Euclidean geometry from the purely observable facts of experience. We shall reproduce here the most important point of Stolz's brochure. Let there be given (Fig. 21) *one*

[1] *Gauss zum Gedächtniss*, Leipsic, 1856.

[2] "Das letzte Axiom der Geometrie," *Berichte des naturw.-medicin. Vereins zu Innsbruck*, 1886, pp. 25-34.

large triangle ABC having the angle-sum $2R$. We draw the perpendicular AD on BC, complete the figure by $BAE \cong ABD$ and $CAF \cong ACD$, and add to the figure $BCFAE$ the congruent figure $CBHA'G$. We obtain thus a *single* rectangle, for the angles E, F, G, H are right angles and those at A, C, A', B are straight angles (equal to $2R$), the boundary lines therefore straight lines and the opposite sides equal. A rectangle can be divided into two congruent rectangles by a perpendicular erected at the middle point of one of its sides, and by continuing this procedure the line of division may be brought

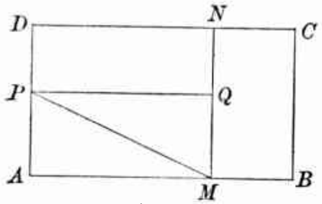

Fig. 22.

to any point we please in the divided side. And the same holds true of the other two opposite sides. It is possible, therefore, from a given rectangle $ABCD$ (Fig. 22) to cut out a smaller $AMPQ$ having sides bearing any proportion to one another. The diagonal of this last divides it into two congruent *right-angled* triangles, of which each, independently of the ratio of the sides, has the angle-sum $2R$. Every oblique-angled triangle can by the drawing of a perpendicular be decomposed into right-angled triangles, each of which can again be decomposed into right-angled triangles having smaller sides,—so that $2R$, therefore, results for the angle-sum of *every* triangle if it holds true exactly of *one*. By the aid of these propositions which repose on observation we *conclude* easily that the two opposite sides of a rectangle (or of any so-called parallelogram) are everywhere, no matter how far prolonged, the same distance apart, that is, never intersect. They have the properties of the Euclidean *parallels*, and may be called and *defined* as such. It likewise *follows*, now, from the properties of triangles and rectangles, that two straight lines which are cut by a third straight line so as to make the sum of the interior angles on the same side of them less than two right angles will meet on that side, but in either direction from their point of intersection will move indefinitely far away from each other. The straight line therefore is *infinite*. What was a *groundless* assertion stated as an axiom or an initial principle may as *inference* have a sound meaning.

Geometry and Physics Compared.

Geometry, accordingly, consists of the application of mathematics to experiences concerning space. Like mathematical physics, it can become an exact deductive science only on the condition of its representing the objects of experience by means of schematizing and idealizing concepts. Just as mechanics can assert the constancy of masses or reduce the interactions between bodies to *simple* accelerations *only within the limits of errors of observation*,

so likewise the existence of straight lines, planes, the amount of the angle-sum, etc., can be maintained only on a similar restriction. But just as physics sometimes finds itself constrained to replace its ideal assumptions by other more general ones, viz., to put in the place of a constant acceleration of falling bodies one dependent on the distance, instead of a constant quantity of heat a variable quantity,—so a similar procedure is permissible in geometry, when it is demanded by the facts or is necessary temporarily for scientific elucidation. And now the endeavors of Legendre, Lobachévski, and the two Bolyais, the younger of whom was probably indirectly inspired by Gauss, will appear in their right light.

The Contributions of Lobachevski and Bolyai.

Of the labors of Schweickart and Taurinus, also contemporaries of Gauss, we will not speak. Lobachévski's works were the first to become known to the thinking world and so productive of results (1829). Very soon afterward the publication of the younger Bolyai appeared (1833), which agreed in all essential points with Lobachévski's, departing from it only in the form of its developments. According to the originals which have been made almost completely accessible to us in the beautiful editions of Engel and Staeckel,[1] it is permissible to

[1] *Urkunden zur Geschichte der nichteuklidischen Geometrie.* I. N. I. Lobatschefskij. Leipzig, 1899.

assume that Lobachévski also undertook his investigations in the hope of becoming involved in contradictions by the rejection of the Euclidean axiom. But after he found himself mistaken in this expectation, he had the ***intellectual courage*** to draw all the consequences from this fact. Lobachévski gives his conclusions in synthetic form. But we can fairly well imagine the general analyzing considerations that paved the way for the construction of his geometry.

From a point lying outside a straight line *g* (Fig. 23) a perpendicular *p* is dropped and through the

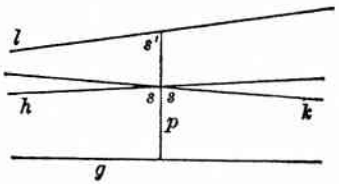

Fig. 23.

same point in the plane *pg* a straight line *h* is drawn, making with the perpendicular an acute angle *s*. Making tentatively the assumption that *g* and *h* do not meet but that on the slightest diminution of the angle *s* they would meet, we are at once forced by the homogeneity of space to the conclusion that a *second* line *k* having the same angle *s* similarly deports itself on the other side of the perpendicular. Hence all non-intersecting lines drawn through the same point are situate between *h* and *k*. The latter form the *boundaries* between the intersecting and

non-intersecting lines and are called by Lobachévski *parallels*.

In the Introduction to his *New Elements of Geometry* (1835) Lobachévski proves himself a thorough natural inquirer. No one would think of attributing even to an ordinary man of sense the crude view that the "parallel-angle" was very much less than a right angle, when on slight prolongation it could be distinctly seen that they would intersect. The relations here considered admit of representation only in drawings that distort the true proportions, and we have rather to picture to ourselves that in the dimensions of the illustration the variation of *s* from a right angle is so small that *h* and *k* are to the eye undistinguishably coincident. Prolonging, now, the perpendicular *p* to a point beyond its intersection with *h*, and drawing through its extremity a new line *l* parallel to *h* and therefore parallel also to *g*, it follows that the parallel-angle *s'* must necessarily be less than *s*, if *h* and *l* are not again to fulfill the conditions of the Euclidean case. Continuing in the same manner, the prolongation of the perpendicular and the drawing of parallels, we obtain a parallel-angle that constantly decreases. Considering, now, parallels which are more remote and consequently converge more rapidly on the side of convergence, we shall logically be compelled to assume, not to be at variance with the preceding supposition, that on approach or on the decrease of the length of the perpendicular the parallel-angle will again increase. The angle of parallelism, therefore, is an inverse function of the perpendicular p, and has been designated by Lobachévski by $\Pi(p)$. A group of parallels in a plane has the arrangement shown schematically in Figure 24. They all approach one another asymptotically toward the side of their convergence. The homogeneity of space requires that every "strip" between two parallels can be made to coincide with every other strip provided it be displaced the requisite distance in a longitudinal direction.

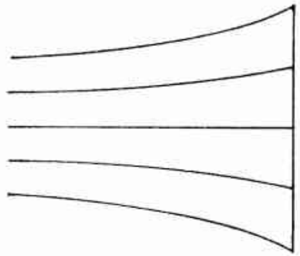

Fig. 24.

If a circle be imagined to increase indefinitely, its radii will cease to intersect the moment the increasing arcs reach the point where the convergence of the radii corresponds to parallelism. The circle then passes over into the so-called *"boundary-line."* Similarly the surface of a sphere, if it indefinitely increase, will pass into what Lobachévski calls a *"boundary-surface."* The boundary-lines bear a relation to the boundary-surface analogous to that which a great circle bears to the surface of a sphere. The geometry of the surface of a sphere is inde-

pendent of the axiom of parallels. But since it can be demonstrated that triangles formed from boundary-lines on a boundary-surface no more exhibit an excess of angle-sum than do finite triangles on a sphere of infinite radius, consequently the rules of the Euclidean geometry likewise hold good for these boundary-triangles. To find points of the boundary-line, we determine (Fig. 25) in a bundle of parallels, $a\alpha$, $b\beta$, $c\gamma$, $d\delta$..... lying in a plane points a, b, c, d in each of these parallels so situated with respect to the point a in $a\alpha$

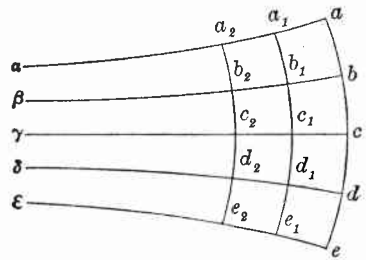

Fig. 25.

that $\angle \alpha ab = \angle \beta ba$, $\angle \alpha ac = \angle \gamma ca$, $\angle \alpha ad = \angle \delta da$.....Owing to the sameness of the entire construction, each of the parallels may be regarded as the *"axis"* of the boundary line, which will generate, when revolved about this axis, the boundary-surface. Likewise each of the parallels may be regarded as the axis of the boundary-surface. For the same reason all boundary-lines and all boundary-surfaces are *congruent*. The intersection of every plane with the boundary-surface is a *circle*; it is a boundary-line only when the cutting plane contains the axis. In the Euclidean geometry there is no boundary-line, nor boundary-surface. The analogues of them are here the straight line and the plane. If no boundary-line exists, then necessarily must any three points not in a straight line lie on a circle. Hence the younger Bolyai was able to replace the Euclidean axiom by this last postulate.

Let $a\alpha$, $b\beta$, $c\gamma$ be a system of parallels, and ae, a_1e_1, a_2e_2..a system of boundary-lines, each of which systems divides the other into equal parts (Fig. 25). The ratio to each other of any two boundary-arcs between the same parallels, e. g., the arcs $ae=u$ and $a_2e_2=u'$, is dependent therefore solely on their distance apart $aa_2=x$. We may put generally $\frac{u}{u'}=e^{\frac{x}{k}}$, where k is so chosen that e shall be the base of the Naperian system of logarithms. In this manner exponentials and by means of these hyperbolic functions are introduced. For the angle of parallelism we obtain $s= cot\frac{1}{2}\Pi(p)=e^{\frac{p}{k}}$. If $p=o$, $s=\frac{\pi}{2}$; if $p=\infty$, $s=0$.

An example will illustrate the relation of the Lobachévskian to the Euclidean and spherical geometries. For a rectilinear Lobachévskian triangle having the sides a, b, c, and the angles A, B, C, we obtain, when C is a right angle,

$$\sinh\frac{a}{k}=\sinh\frac{c}{k}\sin A.$$

Here *sinh* stands for the hyperbolic sine,

$$\sinh x=\frac{e^{x}-e^{-x}}{2}$$

whereas $$\sin x=\frac{e^{xi}-e^{-xi}}{2i},$$

or, $$\sinh x=\frac{x}{1\,!}+\frac{x^3}{3\,!}+\frac{x^5}{5\,!}+\frac{x^7}{7\,!}+\ldots\ldots,$$

and $$\sin x=\frac{x}{1\,!}-\frac{x^3}{3\,!}+\frac{x^5}{5\,!}-\frac{x^7}{7\,!}+\ldots\ldots$$

Considering the relations $\sin(xi)=i\,(\sinh x)$, or $\sinh(xi)=i\sin x$, involved in the foregoing formulæ, it will be seen that the above-given formula for the Lobachévskian triangle passes over into the formula holding for the *spherical* triangle, viz., $\sin\frac{a}{k}=\sin\frac{c}{k}\sin A$, when ki is put in the place of k in the former and k is considered as the radius of the sphere, which in the usual formulæ assumes the value unity. The re-transformation of the spherical formula into the Lobachévskian by the same method is obvious. If k be very great in comparison with a and c, we may restrict ourselves to the first member of the series for sinh or sin, obtaining in both cases, $\frac{a}{k}=\frac{c}{k}\sin A$ or $a=c\sin A$, the formula of *plane Euclidean* geometry, which we may regard as a limiting case of both the Lobachévskian and spherical geometries for very large values of k, or for $k=\infty$. It is likewise permissible to say that all three geometries coincide in the domain of the infinitely small.

[1] F. Engel, *N. I. Lobatschefskij, Zwei geometrische Abhandlungen*, Leipsic, 1899.

The Different Systems of Geometry.

As we see, it is possible to construct a self-consistent, non-contradictory system of geometry solely on the assumption of the convergence of parallel lines. True, there is not a single observation of the geometrical facts accessible to us that speaks in favor of this assumption, and admittedly the hypothesis is at so great variance with our geometrical instinct as easily to explain the attitude toward it of the earlier inquirers like Saccheri and Lambert. Our imagination, dominated as it is by our modes of visualizing and by the familiar Euclidean concepts, is competent to grasp only piecemeal and gradually Lobachévski's views. We must suffer ourselves to be led here rather by mathematical *concepts* than by *sensuous images* derived from a single narrow portion of space. But we must grant, nevertheless, that the quantitative mathematical concepts by which we through our own initiative and within a certain arbitrary scope represent the facts of geometrical experience, do not reproduce the latter with absolute exactitude. Different ideas can express the facts with the same exactness in the domain accessible to observation. The *facts* must hence be carefully distinguished from the *intellectual* constructs the formation of which they suggested. The latter—concepts—must be *consistent* with observation, and must in addition be *logically* in accord with one another. Now these two requirements can be fulfilled

in *more than one* manner, and hence the different systems of geometry.

Manifestly the labors of Lobachévski were the outcome of intense and protracted mental effort, and it may be surmised that he first gained a clear conception of his system from general considerations and by analytic (algebraic) methods before he was able to present it synthetically. Expositions in this cumbersome Euclidean form are by no means alluring, and it is possibly due mainly to this fact that the significance of Lobachévski's and Bolyai's labors received such tardy recognition.

Lobachévski developed only the consequences of the modification of Euclid's Fifth Postulate. But if we abandon the Euclidean assertion that "two straight lines cannot enclose a space," we shall obtain a companion-piece to the Lobachévskian geometry. Restricted to a surface, it is the geometry of the surface of a sphere. In place of the Euclidean straight lines we have great circles, all of which intersect twice and of which each pair encloses two spherical lunes. There are therefore no parallels. Riemann first intimated the possibility of an analogous geometry for three-dimensional space (of positive curvature),—a conception that does not appear to have occurred even to Gauss, possibly owing to his predilection for infinity. And Helmholtz,[1] who continued the researches of Riemann physically, neglected in his turn, in his first publication, the de-

[1] "Ueber die thatsächlichen Grundlagen der Geometrie," *Wissensch. Abhandl.*, 1866. II., p. 610 et seq.

velopment of the Lobachévskian case of a space of negative curvature (with an imaginary parameter *k*). The consideration of this case is in point of fact more obvious to the mathematician than it is to the physicist. Helmholtz treats in the publication mentioned only the Euclidean case of the curvature zero and Riemann's space of positive curvature.

Applicability of the Different Systems to Reality.

We are able, accordingly, to represent the facts of spatial observation with all possible precision by both the Euclidean geometry and the geometries of Lobachévski and Riemann, provided in the two latter cases we take the parameter *k* large enough. Physicists have as yet found no reason for departing from the assumption $k = \infty$ of the Euclidean geometry. It has been their practice, the result of long and tried experience, to adhere steadfastly to the *simplest* assumptions until the facts forced their complication or modification. This accords likewise with the attitude of all great mathematicians toward *applied* geometry. The deportment of physicists and mathematicians toward these questions is in the main different, but this is explained by the circumstance that for the former class of inquirers the physical facts are of most significance, geometry being for them merely a convenient implement of investigation, while for the latter class these very questions are the main

material of research, and of greatest technical and particularly epistemological interest. Supposing a mathematician to have modified tentatively the simplest and most immediate assumptions of our geometrical experience, and supposing his attempt to have been productive of fresh insight, certainly nothing is more natural than that these researches should be prosecuted farther from a purely mathematical interest. Analogues of the geometry we are familiar with, are constructed on broader and more general assumptions for any number of dimensions, with no pretension of being regarded as more than intellectual scientific experiments and with no idea of being applied to reality. In support of my remark it will be sufficient to advert to the advances made in mathematics by Clifford, Klein, Lie, and others. Seldom have thinkers become so absorbed in revery, or so far estranged from reality, as to imagine for our space a number of dimensions *exceeding the three of the given space of sense*, or to conceive of representing that space by any geometry that departs appreciably from the Euclidean. Gauss, Lobachévski, Bolyai, and Riemann were perfectly clear on this point, and cannot certainly be held responsible for the grotesque fictions which were subsequently constructed in this domain.

It little accords with the principles of a physicist to make suppositions regarding the deportment of geometrical constructs in infinity and in non-accessible places, then subsequently to compare them with our immediate experience and adapt them to it. He prefers, like Stolz, to regard what is directly given as the source of his ideas, which he likewise considers applicable to what is inaccessible until obliged to change them. But he too may be extremely grateful for the discovery that there exist *several* sufficing geometries, that we can make shift also with a *finite* space, etc.,—grateful in short, for the abolition of certain *conventional barriers* of thought.

If we lived on the surface of a planet with a turbid, opaque atmosphere and if, on the supposition that the surface of the earth was a plane and our only instruments were square and chain, we were to undertake geodetic measurements; then the increase in the excess of the angle-sum of large triangles would soon compel us to substitute a spherometry for our planimetry. The *possibility* of analogous experiences in three-dimensional space the physicist cannot as a matter of *principle* reject, although the phenomena that would compel the acceptance of a Lobachévskian or a Riemannian geometry would present so odd a contrast with those to which we have been hitherto accustomed, that no one will regard their actual occurrence as *probable*.

The question whether a given *physical* object is a straight line or the arc of a circle is not properly formulated. A stretched chord or a ray of light is certainly neither the one nor the other. The question is simply whether the object so spatially reacts that it conforms better to the one concept than to the other, and whether with the exactitude which is sufficient for us and obtainable by us it conforms at all to any geometric concept. Excluding the latter case, the question arises, whether we can in practice remove, or at least in thought determine and make allowance for, the *deviation* from the straight line or circle, in other words, *correct* the result of the measurement. But we are dependent always, in practical measurements, on the comparison of *physical* objects. If on direct investigation these coincided with the geometric concepts to the highest attainable point of accuracy, but the indirect results of the measurement deviated more from the theory than the consideration of all possible errors permitted, then certainly we should be obliged to *change* our physico-metric notions. The physicist will do well to await the occurrence of such a situation, while in the meantime the mathematician may be allowed full and free scope for his speculations.

The Concepts of Mathematics and Physics.

Of all the concepts which the natural inquirer employs, the *simplest* are the concepts of space and time. Spatial and temporal objects conforming to his conceptual constructs can be framed with great *exactness*. Nearly every observable *deviation* can be eliminated. We can imagine any spatial or temporal construct realized without doing violence to any fact. The other physical properties of bodies are so intimately interconnected that in their case arbitrary fictions are subjected to narrow restrictions by the facts. A perfect gas, a perfect fluid, a perfectly elastic body does not exist; the physicist knows that his fictions conform only approximately and by arbitrary simplifications to the facts; he is perfectly aware of the deviation, which cannot be removed. We can conceive a sphere, a plane, etc., constructed *with unlimited exactness*, without running counter to any fact. Hence, when any new physical fact occurs which renders a modification of our concepts necessary, the physicist always prefers to sacrifice the less perfect concepts of physics rather than the simpler, more perfect, and more lasting concepts of geometry, which form the solidest foundation of all his theories.

But the physicist can derive in another direction substantial assistance from the labors of geometers. Our geometry refers always to objects of sensuous experience. But the moment we begin to operate with mere things of thought like atoms and molecules, which from their very nature *can never be made the objects of sensuous contemplation*, we are under no obligation whatever to think of them as standing in spatial relationships which are peculiar to the Euclidean three-dimensional space of our sensuous experience. This may be recommended to the special attention of thinkers who deem atomistic speculations indispensable.[1]

[1] While still an upholder of the atomic theory, I sought to explain the line-spectra of gases by the vibrations of the atomic constituents of a gas-molecule with respect to another. The difficulties which I here encountered suggested to me (1863) the idea that non-sensuous things did not necessarily have to be pictured in our sensuous space of three dimensions. In this way I also lighted upon analogues of spaces of different numbers of dimensions. The collateral study of various physiological manifolds (see footnote on page 98 of this book) led me to the problems discussed in the conclusion of this paper. The notion of finite spaces, converging parallels, etc., which can come only from a historical study of geometry, was at that time remote from me. I believe that my critics would have done well had they not overlooked the italicised paragraph. For details see the notes to my *Erhaltung der Arbeit*, Prague, 1872.

The Relativity of All Spatial Relations.

Let us go back in thought to the origin of geometry in the practical needs of life. The recognition of the spatial substantiality and spatial invariability of spatial objects in spite of their movements is a biological necessity for human beings, for spatial quantity is related directly to the quantitative satisfaction of our needs. When knowledge of this sort is not sufficiently provided for by our physiological organization, we employ our hands and feet for comparing the spatial objects. When we begin to compare *bodies* with one another, we enter the domain of physics, whether we employ our hands or an artificial measure. All *physical* determinations are *relative*. Consequently, likewise all *geometrical* determinations possess validity only *relatively* to the measure. The concept of measurement is a concept of relation, which contains nothing not contained in the measure. In geometry we simply assume that the measure will always and everywhere coincide with that with which it has at some other time and in some other place coincided. But this assumption is determinative of nothing concerning the measure. In place of spatial *physiological* equality is substituted an altogether differently defined *physical* equality, which must not be confounded with the former, no more than the indications of a thermometer are to be identified with the sensation of heat. The practical geometer, it is true, determines the dilatation of a heated measure, by means of a measure kept at a constant temperature, and takes account of the fact that the relation of congruence in question is disturbed by this non-spatial physical circumstance. But to the pure theory of space all assumptions regarding the measure are foreign. Simply the physiologically created habit of regarding the measure as invariable is tacitly but unjustifiably retained. It would be quite superfluous and meaningless to assume that the measure, and therefore bodies generally, suffered alterations on displacement in space, or that they remained unchanged on such displacement,—a fact which in its turn could only be determined by the use of a new measure. The *relativity* of all spatial relations is made manifest by these considerations.

Introduction of the Notion of Number.

If the criterion of spatial equality is substantially modified by the introduction of measure, it is subjected to a still further modification, or intensification, by the introduction of the notion of *number* into geometry. There is nicety of distinction gained by this introduction which the idea of congruence

alone could never have attained. The application of arithmetic to geometry leads to the notion of *incommensurability* and *irrationality*. Our geometric concepts therefore contain adscititious elements not intrinsic to space; they represent space with a certain latitude, and, arbitrarily also, with greater precision than spatial observation alone could possibly ever realize. This imperfect contact between fact and concept explains the possibility of different systems of geometry.[1]

Significance of the Metageometric Movement.

The entire movement which led to the transformation of our ideas of geometry must be characterized as a sound and healthful one. This movement, which began centuries ago but is now greatly intensified, is not to be looked upon as having terminated. On the contrary, we are quite justified in the expectation that it will long continue, and redound not only to the great advancement of mathematics and geometry, especially in an epistemological regard, but also to that of the other sciences. This movement was, it is true, powerfully stimulated by a few eminent men, but it sprang, nevertheless, not from an individual, but from a general need. This will be seen from the difference in the pro-

[1] It would be too much to expect of matter that it should realize all the atomistic fantasies of the physicist. So, too, space, as an object of experience, can hardly be expected to satisfy all the ideas of the mathematician, though there be no doubt whatever as to the general value of their investigations.

fessions of the men who have taken part in it. Not only the mathematician, but also the philosopher and the educationist, have made considerable contributions to it. So, too, the methods pursued by the different inquirers are not unrelated. Ideas which Leibnitz[1] uttered recur in slightly altered form in Fourier,[2] Lobachévski, Bolyai, and H. Erb.[3] The philosopher Ueberweg,[4] closely approaching in his opposition to Kant the views of the psychologist Beneke,[5] and in his geometrical ideas starting from Erb (which later writer mentions K. A. Erb[6] as his predecessor) anticipates a goodly portion of Helmholtz's labors.

SUMMARY.

The results to which the preceding discussion has led, may be summarized as follows:

1. The source of our geometric concepts has been found to be experience.

2. The character of the concepts satisfying the same geometrical facts has been shown to be many and varied.

3. By the comparison of space with other manifolds, more general concepts have been reached, of which the geometric represents a special case. Geometric thought has thus been freed from conventional limitations, heretofore imagined insuperable.

4. By the demonstration of the existence of manifolds allied to but different from space, entirely new questions have been suggested. What is space physiologically, physically, geometrically? To what are its specific properties to be attributed, since others are also conceivable? Why is space three-dimensional, etc.?

With questions such as these, though we must not expect the answer to-day or to-morrow, we stand before the entire profundity of the domain to be investigated. We shall say nothing of the inept strictures of the Bœotians, whose coming Gauss predicted, and whose attitude determined him to reserve. But what shall we say to the acrid and captious criticisms to which Gauss, Riemann and their associates have been subjected by men of highest standing in the scientific world? Have these men never experienced in their own persons the truth that inquirers on the outermost boundaries of knowledge frequently discover many things that will not slip smoothly into all heads, but which are not on that account arrant nonsense? True, such inquirers are liable to error, but even the errors of some men are often more fruitful in their consequences than the discoveries of others.

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[1] See above pp. 66-67.

[2] *Séances de l'Ecole Normale. Débats.* Vol. I., 1800, p. 28.

[3] H. Erb, Grossherzoglich Badischer Finanzrath, *Die Probleme der geraden Linie, des Winkels und der ebenen Fläche,* Heidelberg, 1846.

[4] "Die Principien der Geometrie wissenschaftlich dargestellt." *Archiv für Philologie und Pädagogik.* 1851. Reprinted in Brasch's *Welt- und Lebensanschauung F. Ueberwegs,* Leipzig, 1889, pp. 263-317.

[5] *Logik als Kunstlehre des Denkens,* Berlin, 1842, Vol. II., pp. 51-55.

[6] *Zur Mathematik und Logik,* Heidelberg, 1821. I was unable to examine this work.

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