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A COURSE OF PURE MATHEMATICS

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CHAPTER V

LIMITS OF FUNCTIONS OF A CONTINUOUS VARIABLE. CONTINUOUS AND DISCONTINUOUS FUNCTIONS

89. Limits as x tends to ∞. We shall now return to functions of a continuous real variable. We shall confine ourselves entirely to *one-valued* functions*, and we shall denote such a function by $\phi(x)$. We suppose x to assume successively all values corresponding to points on our fundamental straight line Λ, starting from some definite point on the line and progressing always to the right. In these circumstances we say that x *tends to infinity*, or *to* ∞, and write $x \to \infty$. The only difference between the 'tending of n to ∞' discussed in the last chapter, and this 'tending of x to ∞', is that x assumes all values as it tends to ∞, *i.e.* that the point P which corresponds to x coincides in turn with every point of Λ to the right of its initial position, whereas n tended to ∞ by a series of jumps. We can express this distinction by saying that x tends *continuously* to ∞.

As we explained at the beginning of the last chapter, there is a very close correspondence between functions of x and functions of n. Every function of n may be regarded as a selection from the values of a function of x. In the last chapter we discussed the peculiarities which may characterise the behaviour of a function $\phi(n)$ as n tends to ∞. Now we are concerned with the same problem for a function $\phi(x)$; and the definitions and theorems to which we are led are practically repetitions of those of the last chapter. Thus corresponding to Def. 1 of § 58 we have:

* Thus $\sqrt{x}$ stands in this chapter for the one-valued function $+\sqrt{x}$ and not (as in § 26) for the two-valued function whose values are $+\sqrt{x}$ and $-\sqrt{x}$.

DEFINITION 1. *The function $\phi(x)$ is said to tend to the limit l as x tends to ∞ if, when any positive number δ, however small, is assigned, a number $x_0(\delta)$ can be chosen such that, for all values of x equal to or greater than $x_0(\delta)$, $\phi(x)$ differs from l by less than δ, i.e. if*

$$|\phi(x) - l| < \delta$$

when $x \geqq x_0(\delta)$.

When this is the case we may write

$$\lim_{x \to \infty} \phi(x) = l,$$

or, when there is no risk of ambiguity, simply $\lim \phi(x) = l$, or $\phi(x) \to l$. Similarly we have:

DEFINITION 2. *The function $\phi(x)$ is said to tend to ∞ with x if, when any number Δ, however large, is assigned, we can choose a number $x_0(\Delta)$ such that*

$$\phi(x) > \Delta$$

when $x \geqq x_0(\Delta)$.

We then write

$$\phi(x) \to \infty.$$

Similarly we define $\phi(x) \to -\infty$*. Finally we have:

DEFINITION 3. *If the conditions of neither of the two preceding definitions are satisfied, then $\phi(x)$ is said to oscillate as x tends to ∞. If $|\phi(x)|$ is less than some constant K when $x \geqq x_0$†, then $\phi(x)$ is said to oscillate finitely, and otherwise infinitely.*

The reader will remember that in the last chapter we considered very carefully various less formal ways of expressing the facts represented by the formulae $\phi(n) \to l$, $\phi(n) \to \infty$. Similar modes of expression may of course be used in the present case. Thus we may say that $\phi(x)$ is small or nearly equal to l or large when x is large, using the words 'small', 'nearly', 'large' in a sense similar to that in which they were used in Ch. IV.

* We shall sometimes find it convenient to write $+\infty$, $x \to +\infty$, $\phi(x) \to +\infty$ instead of ∞, $x \to \infty$, $\phi(x) \to \infty$.

† In the corresponding definition of § 62, we postulated that $|\phi(n)| < K$ for *all* values of n, and not merely when $n \geqq n_0$. But then the two hypotheses would have been equivalent; for if $|\phi(n)| < K$ when $n \geqq n_0$, then $|\phi(n)| < K'$ for all values of n, where K' is the greatest of $\phi(1)$, $\phi(2)$, ..., $\phi(n_0 - 1)$ and K. Here the matter is not quite so simple, as there are infinitely many values of x less than x_0.

Examples XXXIV. 1. Consider the behaviour of the following functions as $x \to \infty$: $1/x$, $1+(1/x)$, x^2, x^k, $[x]$, $x-[x]$, $[x]+\sqrt{\{x-[x]\}}$.

The first four functions correspond exactly to functions of n fully discussed in Ch. IV. The graphs of the last three were constructed in Ch. II (Exs. xvi. 1, 2, 4), and the reader will see at once that $[x] \to \infty$, $x-[x]$ oscillates finitely, and $[x]+\sqrt{\{x-[x]\}} \to \infty$.

One simple remark may be inserted here. The function $\phi(x)=x-[x]$ oscillates between 0 and 1, as is obvious from the form of its graph. It is equal to zero whenever x is an integer, so that the function $\phi(n)$ derived from it is always zero and so tends to the limit zero. The same is true if

$$\phi(x)=\sin x\pi, \quad \phi(n)=\sin n\pi=0.$$

It is evident that $\phi(x) \to l$ or $\phi(x) \to \infty$ or $\phi(x) \to -\infty$ involves the corresponding property for $\phi(n)$, but that the converse is by no means always true.

2. Consider in the same way the functions:

$$(\sin x\pi)/x, \quad x \sin x\pi, \quad (x \sin x\pi)^2, \quad \tan x\pi, \quad a\cos^2 x\pi + b\sin^2 x\pi,$$

illustrating your remarks by means of the graphs of the functions.

3. Give a geometrical explanation of Def. 1, analogous to the geometrical explanation of Ch. IV, § 59.

4. If $\phi(x) \to l$, and l is not zero, then $\phi(x)\cos x\pi$ and $\phi(x)\sin x\pi$ oscillate finitely. If $\phi(x) \to \infty$ or $\phi(x) \to -\infty$, then they oscillate infinitely. The graph of either function is a wavy curve oscillating between the curves $y=\phi(x)$ and $y=-\phi(x)$.

5. Discuss the behaviour, as $x \to \infty$, of the function

$$y=f(x)\cos^2 x\pi + F(x)\sin^2 x\pi,$$

where $f(x)$ and $F(x)$ are some pair of simple functions (*e.g.* x and x^2). [The graph of y is a curve oscillating between the curves $y=f(x)$, $y=F(x)$.]

90. Limits as x tends to $-\infty$. The reader will have no difficulty in framing for himself definitions of the meaning of the assertions 'x tends to $-\infty$', or '$x \to -\infty$' and

$$\lim_{x \to -\infty} \phi(x) = l, \qquad \phi(x) \to \infty, \qquad \phi(x) \to -\infty.$$

In fact, if $x=-y$ and $\phi(x)=\phi(-y)=\psi(y)$, then y tends to ∞ as x tends to $-\infty$, and the question of the behaviour of $\phi(x)$ as x tends to $-\infty$ is the same as that of the behaviour of $\psi(y)$ as y tends to ∞.

91. Theorems corresponding to those of Ch. IV, §§ 63–67. The theorems concerning the sums, products, and quotients of functions proved in Ch. IV are all true (with obvious verbal alterations which the reader will have no difficulty in supplying) for functions of the continuous variable x. Not only the enunciations but the proofs remain substantially the same.

92. Steadily increasing or decreasing functions. The definition which corresponds to that of § 69 is as follows: *the function $\phi(x)$ will be said to increase steadily with x if $\phi(x_2) \geqq \phi(x_1)$ whenever $x_2 > x_1$.* In many cases, of course, this condition is only satisfied from a definite value of x onwards, *i.e.* when $x_2 > x_1 \geqq x_0$. The theorem which follows in that section requires no alteration but that of n into x: and the proof is the same, except for obvious verbal changes.

If $\phi(x_2) > \phi(x_1)$, the possibility of equality being excluded, whenever $x_2 > x_1$, then $\phi(x)$ will be said to be *steadily increasing in the stricter sense.* We shall find that the distinction is often important (cf. §§ 108–109).

The reader should consider whether or no the following functions increase steadily with x (or at any rate increase steadily from a certain value of x onwards): x^2-x, $x+\sin x$, $x+2\sin x$, $x^2+2\sin x$, $[x]$, $[x]+\sin x$, $[x]+\sqrt{\{x-[x]\}}$. All these functions tend to ∞ as $x \to \infty$.

93. Limits as x tends to 0. Let $\phi(x)$ be such a function of x that $\lim_{x \to \infty} \phi(x) = l$, and let $y=1/x$. Then

$$\phi(x) = \phi(1/y) = \psi(y),$$

say. As x tends to ∞, y tends to the limit 0, and $\psi(y)$ tends to the limit l.

Let us now dismiss x and consider $\psi(y)$ simply as a function of y. We are for the moment concerned only with those values of y which correspond to large positive values of x, that is to say with small positive values of y. And $\psi(y)$ has the property that by making y sufficiently small we can make $\psi(y)$ differ by as little as we please from l. To put the matter more precisely, the statement expressed by $\lim \phi(x)=l$ means that, when any positive number δ, however small, is assigned, we can choose x_0 so that $|\phi(x)-l|<\delta$ for all values of x greater than or equal to x_0. But this is the same thing as saying that we can choose $y_0=1/x_0$ so that $|\psi(y)-l|<\delta$ for all positive values of y less than or equal to y_0.

We are thus led to the following definitions:

A. *If, when any positive number δ, however small, is assigned, we can choose $y_0(\delta)$ so that*

$$|\phi(y) - l| < \delta$$

when $0 < y \leqq y_0(\delta)$, then we say that $\phi(y)$ tends to the limit l as y tends to 0 by positive values, and we write

$$\lim_{y \to +0} \phi(y) = l.$$

B. *If, when any number Δ, however large, is assigned, we can choose $y_0(\Delta)$ so that*

$$\phi(y) > \Delta$$

when $0 < y \leqq y_0(\Delta)$, then we say that $\phi(y)$ tends to ∞ as y tends to 0 by positive values, and we write

$$\phi(y) \to \infty.$$

We define in a similar way the meaning of '$\phi(y)$ tends to the limit l as y tends to 0 by negative values', or '$\lim \phi(y) = l$ when $y \to -0$'. We have in fact only to alter $0 < y \leqq y_0(\delta)$ to $-y_0(\delta) \leqq y < 0$ in definition A. There is of course a corresponding analogue of definition B, and similar definitions in which

$$\phi(y) \to -\infty$$

as $y \to +0$ or $y \to -0$.

If $\lim_{y \to +0} \phi(y) = l$ *and* $\lim_{y \to -0} \phi(y) = l$, we write simply

$$\lim_{y \to 0} \phi(y) = l.$$

This case is so important that it is worth while to give a formal definition.

If, when any positive number δ, however small, is assigned, we can choose $y_0(\delta)$ so that, for all values of y different from zero but numerically less than or equal to $y_0(\delta)$, $\phi(y)$ differs from l by less than δ, then we say that $\phi(y)$ tends to the limit l as y tends to 0, and write

$$\lim_{y \to 0} \phi(y) = l.$$

So also, if $\phi(y) \to \infty$ as $y \to +0$ and also as $y \to -0$, we say that $\phi(y) \to \infty$ as $y \to 0$. We define in a similar manner the statement that $\phi(y) \to -\infty$ as $y \to 0$.

Finally, if $\phi(y)$ does not tend to a limit, or to ∞, or to $-\infty$, as $y \to +0$, we say that $\phi(y)$ oscillates as $y \to +0$, finitely or infinitely as the case may be; and we define oscillation as $y \to -0$ in a similar manner.

The preceding definitions have been stated in terms of a variable denoted by y: what letter is used is of course immaterial, and we may suppose x written instead of y throughout them.

94. Limits as x tends to a. Suppose that $\phi(y) \to l$ as $y \to 0$, and write

$$y = x - a, \quad \phi(y) = \phi(x - a) = \psi(x).$$

If $y \to 0$ then $x \to a$ and $\psi(x) \to l$, and we are naturally led to write

$$\lim_{x \to a} \psi(x) = l,$$

or simply $\lim \psi(x) = l$ or $\psi(x) \to l$, and to say that $\psi(x)$ *tends to the limit l as x tends to a*. The meaning of this equation may be formally and directly defined as follows: *if, given δ, we can always determine $\epsilon(\delta)$ so that*

$$|\phi(x) - l| < \delta$$

when $0 < |x - a| \leqq \epsilon(\delta)$, then

$$\lim_{x \to a} \phi(x) = l.$$

By restricting ourselves to values of x greater than a, *i.e.* by replacing $0 < |x - a| \leqq \epsilon(\delta)$ by $a < x \leqq a + \epsilon(\delta)$, we define '$\phi(x)$ tends to l when x approaches a from the right', which we may write as

$$\lim_{x \to a+0} \phi(x) = l.$$

In the same way we can define the meaning of

$$\lim_{x \to a-0} \phi(x) = l.$$

Thus $\lim_{x \to a} \phi(x) = l$ is equivalent to the two assertions

$$\lim_{x \to a+0} \phi(x) = l, \quad \lim_{x \to a-0} \phi(x) = l.$$

We can give similar definitions referring to the cases in which $\phi(x) \to \infty$ or $\phi(x) \to -\infty$ as $x \to a$ through values greater or less than a; but it is probably unnecessary to dwell further on these definitions, since they are exactly similar to those stated above in the special case when $a=0$, and we can always discuss the behaviour of $\phi(x)$ as $x \to a$ by putting $x-a=y$ and supposing that $y \to 0$.

95. Steadily increasing or decreasing functions. If there is a number ϵ such that $\phi(x') \leqq \phi(x'')$ whenever $a-\epsilon<x'<x''<a+\epsilon$, then $\phi(x)$ will be said to *increase steadily in the neighbourhood of* $x=a$.

Suppose first that $x<a$, and put $y=1/(a-x)$. Then $y \to \infty$ as $x \to a-0$, and $\phi(x)=\psi(y)$ is a steadily increasing function of y, never greater than $\phi(a)$. It follows from § 92 that $\phi(x)$ tends to a limit not greater than $\phi(a)$. We shall write

$$\lim_{x \to a+0} \phi(x)=\phi(a+0).*$$

We can define $\phi(a-0)$ in a similar manner; and it is clear that

$$\phi(a-0) \leqq \phi(a) \leqq \phi(a+0).$$

It is obvious that similar considerations may be applied to *decreasing* functions.

If $\phi(x')<\phi(x'')$, the possibility of equality being excluded, whenever $a-\epsilon<x'<x''<a+\epsilon$, then $\phi(x)$ will be said to be *steadily increasing in the stricter sense.*

96. Limits of indetermination and the principle of convergence. All of the argument of §§ 80—84 may be applied to functions of a continuous variable x which tends to a limit a. In particular, if $\phi(x)$ is *bounded* in an interval including a (*i.e.* if we can find ϵ, H, and K so that $H<\phi(x)<K$ when $a-\epsilon \leqq x \leqq a+\epsilon$)†, then we can define λ and Λ, the lower and upper limits of indetermination of $\phi(x)$ as $x \to a$, and prove that the necessary and sufficient condition that $\phi(x) \to l$ as $x \to a$ is that $\lambda=\Lambda=l$. We can also establish the analogue of the principle of convergence, *i.e.* prove that *the necessary and sufficient condition that* $\phi(x)$ *should tend to a limit as* $x \to a$ *is that, when* δ *is given, we can choose* $\epsilon(\delta)$ *so that* $|\phi(x_2)-\phi(x_1)|<\delta$ *when* $0<|x_2-a|<|x_1-a| \leqq \epsilon(\delta)$.

Examples XXXV. 1. If $\phi(x) \to l$, $\psi(x) \to l'$, as $x \to a$, then

$$\phi(x)+\psi(x) \to l+l', \quad \phi(x)\psi(x) \to ll', \quad \phi(x)/\psi(x) \to l/l',$$

unless in the last case $l'=0$.

[We saw in § 91 that the theorems of Ch. IV, §§ 63 *et seq.* hold also for functions of x when $x \to \infty$ or $x \to -\infty$. By putting $x=1/y$ we may extend them to functions of y, when $y \to 0$, and by putting $y=z-a$ to functions of z, when $z \to a$.

* It will of course be understood that $\phi(a+0)$ has no meaning other than that of a conventional abbreviation for the limit on the left hand side.

† See § 102.

The reader should however try to prove them directly from the formal definition given above. Thus, in order to obtain a strict direct proof of the first result he need only take the proof of Theorem I of § 63 and write throughout x for n, a for ∞ and $0<|x-a| \leqq \epsilon$ for $n \geqq n_0$.]

2. If m is a positive integer then $x^m \to 0$ as $x \to 0$.

3. If m is a negative integer then $x^m \to +\infty$ as $x \to +0$, while $x^m \to -\infty$ or $x^m \to +\infty$ as $x \to -0$, according as m is odd or even. If $m=0$ then $x^m=1$ and $x^m \to 1$.

4. $\lim\limits_{x \to 0}(a+bx+cx^2+\dots+kx^m)=a$.

5. $\lim\limits_{x \to 0}\{(a+bx+\dots+kx^m)/(\alpha+\beta x+\dots+\kappa x^\mu)\}=a/\alpha$, unless $\alpha=0$. If $\alpha=0$ and $a \neq 0$, $\beta \neq 0$, then the function tends to $+\infty$ or $-\infty$, as $x \to +0$, according as a and β have like or unlike signs; the case is reversed if $x \to -0$. The case in which both a and α vanish is considered in Ex. XXXVI. 5. Discuss the cases which arise when $a \neq 0$ and more than one of the first coefficients in the denominator vanish.

6. $\lim\limits_{x \to a} x^m=a^m$, if m is any positive or negative integer, except when $a=0$ and m is negative. [If $m>0$, put $x=y+a$ and apply Ex. 4. When $m<0$, the result follows from Ex. 1 above. It follows at once that $\lim P(x)=P(a)$, if $P(x)$ is any polynomial.]

7. $\lim\limits_{x \to a} R(x)=R(a)$, if R denotes any rational function and a is not one of the roots of its denominator.

8. Show that $\lim\limits_{x \to a} x^m=a^m$ for all rational values of m, except when $a=0$ and m is negative. [This follows at once, when a is positive, from the inequalities (9) or (10) of § 74. For $|x^m-a^m|<H|x-a|$, where H is the greater of the absolute values of mx^{m-1} and ma^{m-1} (cf. Ex. XXVIII. 4). If a is negative we write $x=-y$ and $a=-b$. Then

$$\lim x^m=\lim(-1)^m y^m=(-1)^m b^m=a^m.]$$

97. The reader will probably fail to see at first that any proof of such results as those of Exs. 4, 5, 6, 7, 8 above is necessary. He may ask 'why not simply put $x=0$, or $x=a$? Of course we then get $a, a/\alpha, a^m, P(a), R(a)$'. It is very important that he should see exactly where he is wrong. We shall therefore consider this point carefully before passing on to any further examples.

The statement $\quad \lim\limits_{x \to 0} \phi(x)=l$

is a statement about the values of $\phi(x)$ when x has any value *distinct from but differing by little from zero**. It is *not* a statement about the *value of* $\phi(x)$ *when* $x=0$. When we make the statement we assert that, when x is *nearly* equal to zero, $\phi(x)$ is nearly equal to l. We assert nothing whatever about what happens when x is *actually* equal to 0. So far as we know, $\phi(x)$ may not be defined at all for $x=0$; or it may have some value other than l. For example, consider the function defined for all values of x by the equation $\phi(x)=0$. It is obvious that

$$\lim \phi(x)=0 \quad \text{.......................(1).}$$

Now consider the function $\psi(x)$ which differs from $\phi(x)$ only in that $\psi(x)=1$ when $x=0$. Then

$$\lim \psi(x)=0 \quad \text{.......................(2),}$$

for, when x is nearly equal to zero, $\psi(x)$ is not only nearly but exactly equal to zero. But $\psi(0)=1$. The graph of this function consists of the axis of x, with the point $x=0$ left out, and one isolated point, viz. the point (0, 1). The equation (2) expresses the fact that if we move along the graph towards the axis of y, from either side, then the ordinate of the curve, being always equal to zero, tends to the limit zero. This fact is in no way affected by the position of the isolated point (0, 1).

The reader may object to this example on the score of artificiality: but it is easy to write down simple formulae representing functions which behave precisely like this near $x=0$. One is

$$\psi(x)=[1-x^2],$$

where $[1-x^2]$ denotes as usual the greatest integer not greater than $1-x^2$. For if $x=0$ then $\psi(x)=[1]=1$; while if $0<x<1$, or $-1<x<0$, then $0<1-x^2<1$ and so $\psi(x)=[1-x^2]=0$.

Or again, let us consider the function

$$y=x/x$$

already discussed in Ch. II, § 24, (2). This function is equal to 1 for all values of x save $x=0$. It is *not* equal to 1 when $x=0$: it is in fact not defined at all for $x=0$. For when we say

* Thus in Def. A of § 93 we make a statement about values of y such that $0<y\leqq y_0$, the first of these inequalities being inserted expressly in order to exclude the value $y=0$.

that $\phi(x)$ is defined for $x=0$ we mean (as we explained in Ch. II, *l.c.*) that we can calculate its value for $x=0$ by putting $x=0$ in the actual expression of $\phi(x)$. In this case we cannot. When we put $x=0$ in $\phi(x)$ we obtain 0/0, which is a meaningless expression. The reader may object 'divide numerator and denominator by x'. But he must admit that when $x=0$ this is impossible. Thus $y=x/x$ is a function which differs from $y=1$ solely in that it is not defined for $x=0$. None the less

$$\lim (x/x)=1,$$

for x/x is equal to 1 so long as x differs from zero, however small the difference may be.

Similarly $\phi(x)=\{(x+1)^2-1\}/x=x+2$ so long as x is not equal to zero, but is undefined when $x=0$. None the less $\lim \phi(x)=2$.

On the other hand there is of course nothing to prevent the limit of $\phi(x)$ as x tends to zero from being equal to $\phi(0)$, the value of $\phi(x)$ for $x=0$. Thus if $\phi(x)=x$ then $\phi(0)=0$ and $\lim \phi(x)=0$. This is in fact, from a practical point of view, *i.e.* from the point of view of what most frequently occurs in applications, the ordinary case.

Examples XXXVI. 1. $\lim\limits_{x\to a} (x^2-a^2)/(x-a)=2a$.

2. $\lim\limits_{x\to a} (x^m-a^m)/(x-a)=ma^{m-1}$, if m is any integer (zero included).

3. Show that the result of Ex. 2 remains true for all rational values of m, provided a is positive. [This follows at once from the inequalities (9) and (10) of § 74.]

4. $\lim\limits_{x\to 1} (x^7-2x^5+1)/(x^3-3x^2+2)=1$. [Observe that $x-1$ is a factor of both numerator and denominator.]

5. Discuss the behaviour of

$$\phi(x)=(a_0x^m+a_1x^{m+1}+\dots+a_kx^{m+k})/(b_0x^n+b_1x^{n+1}+\dots+b_lx^{n+l})$$

as x tends to 0 by positive or negative values.

[If $m>n$, $\lim \phi(x)=0$. If $m=n$, $\lim \phi(x)=a_0/b_0$. If $m<n$ and $n-m$ is even, $\phi(x)\to+\infty$ or $\phi(x)\to-\infty$ according as $a_0/b_0>0$ or $a_0/b_0<0$. If $m<n$ and $n-m$ is odd, $\phi(x)\to+\infty$ as $x\to+0$ and $\phi(x)\to-\infty$ as $x\to-0$, or $\phi(x)\to-\infty$ as $x\to+0$ and $\phi(x)\to+\infty$ as $x\to-0$, according as $a_0/b_0>0$ or $a_0/b_0<0$.]

6. Orders of smallness. When x is small x^2 is very much smaller, x^3 much smaller still, and so on: in other words

$$\lim_{x\to 0}(x^2/x)=0,\quad \lim_{x\to 0}(x^3/x^2)=0,\quad \dots .$$

Another way of stating the matter is to say that, when x tends to 0, x^2, x^3, ... all also tend to 0, but x^2 tends to 0 more rapidly than x, x^3 than x^2, and so on. It is convenient to have some scale by which to measure the rapidity with which a function, whose limit, as x tends to 0, is 0, diminishes with x, and it is natural to take the simple functions x, x^2, x^3, ... as the measures of our scale.

We say, therefore, that $\phi(x)$ *is of the first order of smallness* if $\phi(x)/x$ tends to a limit other than 0 as x tends to 0. Thus $2x+3x^2+x^7$ is of the first order of smallness, since $\lim(2x+3x^2+x^7)/x=2$.

Similarly we define the second, third, fourth, ... orders of smallness. It must not be imagined that this scale of orders of smallness is in any way complete. If it were complete, then every function $\phi(x)$ which tends to zero with x would be of either the first or second or some higher order of smallness. This is obviously not the case. For example $\phi(x)=x^{7/5}$ tends to zero more rapidly than x and less rapidly than x^2.

The reader may not unnaturally think that our scale might be made complete by including in it *fractional* orders of smallness. Thus we might say that $x^{7/5}$ was of the $\frac{7}{5}$th order of smallness. We shall however see later on that such a scale of orders would still be altogether incomplete. And as a matter of fact the *integral* orders of smallness defined above are so much more important in applications than any others that it is hardly necessary to attempt to make our definitions more precise.

Orders of greatness. Similar definitions are at once suggested to meet the case in which $\phi(x)$ is large (positively or negatively) when x is small. We shall say that $\phi(x)$ is of the kth order of greatness when x is small if $\phi(x)/x^{-k}=x^k\phi(x)$ tends to a limit different from 0 as x tends to 0.

These definitions have reference to the case in which $x\to 0$. There are of course corresponding definitions relating to the cases in which $x\to\infty$ or $x\to a$. Thus if $x^k\phi(x)$ tends to a limit other than zero, as $x\to\infty$, then we say that $\phi(x)$ is of the kth order of smallness when x is large: while if $(x-a)^k\phi(x)$ tends to a limit other than zero, as $x\to a$, then we say that $\phi(x)$ is of the kth order of greatness when x is nearly equal to a.

*7. $\lim\sqrt{(1+x)}=\lim\sqrt{(1-x)}=1$. [Put $1+x=y$ or $1-x=y$, and use Ex. XXXV. 8.]

8. $\lim\{\sqrt{(1+x)}-\sqrt{(1-x)}\}/x=1$. [Multiply numerator and denominator by $\sqrt{(1+x)}+\sqrt{(1-x)}$.]

* In the examples which follow it is to be assumed that limits as $x\to 0$ are required, unless (as in Exs. 19, 22) the contrary is explicitly stated.

9. Consider the behaviour of $\{\sqrt{(1+x^m)}-\sqrt{(1-x^m)}\}/x^n$ as $x\to 0$, m and n being positive integers.

10. $\lim\{\sqrt{(1+x+x^2)}-1\}/x=\frac{1}{2}$.

11. $\lim\dfrac{\sqrt{(1+x)}-\sqrt{(1+x^2)}}{\sqrt{(1-x^2)}-\sqrt{(1-x)}}=1$.

12. Draw a graph of the function

$$y=\left\{\frac{1}{x-1}+\frac{1}{x-\frac12}+\frac{1}{x-\frac13}+\frac{1}{x-\frac14}\right\}\Big/\left\{\frac{1}{x-1}+\frac{1}{x-\frac12}+\frac{1}{x-\frac13}+\frac{1}{x-\frac14}\right\}.$$

Has it a limit as $x\to 0$? [Here $y=1$ except for $x=1,\frac12,\frac13,\frac14$, when y is not defined, and $y\to 1$ as $x\to 0$.]

13. $\lim\dfrac{\sin x}{x}=1$.

[It may be deduced from the definitions of the trigonometrical ratios* that if x is positive and less than $\frac12\pi$ then

$$\sin x<x<\tan x$$

or

$$\cos x<\frac{\sin x}{x}<1$$

or

$$0<1-\frac{\sin x}{x}<1-\cos x=2\sin^2\tfrac12 x.$$

But $2\sin^2\frac12 x<2(\frac12 x)^2<\frac12 x^2$ Hence $\lim\limits_{x\to+0}\left(1-\dfrac{\sin x}{x}\right)=0$, and $\lim\limits_{x\to+0}\dfrac{\sin x}{x}=1$.

As $\dfrac{\sin x}{x}$ is an even function, the result follows.]

14. $\lim\dfrac{1-\cos x}{x^2}=\frac12$. 15. $\lim\dfrac{\sin ax}{x}=a$. Is this true if $a=0$?

16. $\lim\dfrac{\arcsin x}{x}=1$. [Put $x=\sin y$.]

17. $\lim\dfrac{\tan ax}{x}=a$, $\lim\dfrac{\arctan ax}{x}=a$.

18. $\lim\dfrac{\operatorname{cosec} x-\cot x}{x}=\frac12$. 19. $\lim\limits_{x\to 1}\dfrac{1+\cos\pi x}{\tan^2\pi x}=\frac12$.

* The proofs of the inequalities which are used here depend on certain properties of the 'area' of a sector of a circle which are usually taken as geometrically intuitive; for example, that the area of the sector is greater than that of the triangle inscribed in the sector. The justification of these assumptions must be postponed to Ch. VII.

20. How do the functions $\sin(1/x)$, $(1/x)\sin(1/x)$, $x\sin(1/x)$ behave as $x \to 0$? [The first oscillates finitely, the second infinitely, the third tends to the limit 0. None is defined when $x=0$. See Exs. xv. 6, 7, 8.]

21. Does the function

$$y = \left(\sin\frac{1}{x}\right) \Big/ \left(\sin\frac{1}{x}\right)$$

tend to a limit as x tends to 0? [*No.* The function is equal to 1 except when $\sin(1/x)=0$; *i.e.* when $x = 1/\pi, 1/2\pi, \ldots, -1/\pi, -1/2\pi, \ldots$. For these values the formula for y assumes the meaningless form 0/0, and y is therefore not defined for an infinity of values of x near $x=0$.]

22. Prove that if m is any integer then $[x] \to m$ and $x - [x] \to 0$ as $x \to m+0$, and $[x] \to m-1$, $x-[x] \to 1$ as $x \to m-0$.

98. Continuous functions of a real variable. The reader has no doubt some idea as to what is meant by a *continuous curve.* Thus he would call the curve C in Fig. 29 continuous, the curve C' generally continuous but discontinuous for $x = \xi'$ and $x = \xi''$.

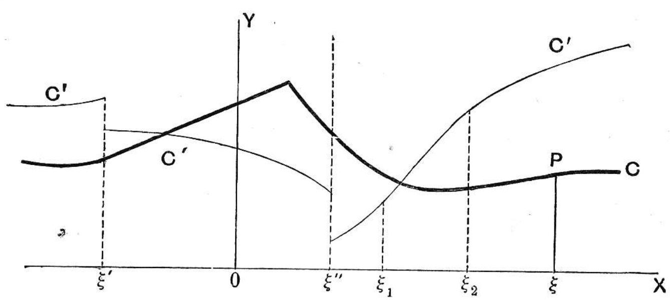

Fig. 29.

Either of these curves may be regarded as the graph of a function $\phi(x)$. It is natural to call a function *continuous* if its graph is a continuous curve, and otherwise discontinuous. Let us take this as a provisional definition and try to distinguish more precisely some of the properties which are involved in it.

In the first place it is evident that the property of the function $y = \phi(x)$ of which C is the graph may be analysed into some property possessed by the curve at each of its points. To be able to define continuity *for all values of x* we must first define continuity *for any particular value of x.* Let us therefore fix on some particular value of x, say the value $x = \xi$

corresponding to the point P of the graph. What are the characteristic properties of $\phi(x)$ associated with this value of x?

In the first place $\phi(x)$ *is defined for* $x = \xi$. This is obviously essential. If $\phi(\xi)$ were not defined there would be a point missing from the curve.

Secondly $\phi(x)$ *is defined for all values of x near* $x = \xi$; *i.e.* we can find an interval, including $x = \xi$ in its interior, for all points of which $\phi(x)$ is defined.

Thirdly *if x approaches the value* ξ *from either side then* $\phi(x)$ *approaches the limit* $\phi(\xi)$.

The properties thus defined are far from exhausting those which are possessed by the curve as pictured by the eye of common sense. This picture of a curve is a generalisation from particular curves such as straight lines and circles. But they are the simplest and most fundamental properties: and the graph of any function which has these properties would, so far as drawing it is practically possible, satisfy our geometrical feeling of what a continuous curve should be. We therefore select these properties as embodying the mathematical notion of continuity. We are thus led to the following

Definition. *The function* $\phi(x)$ *is said to be continuous for* $x = \xi$ *if it tends to a limit as x tends to* ξ *from either side, and each of these limits is equal to* $\phi(\xi)$.

We can now define *continuity throughout an interval.* The function $\phi(x)$ is said to be continuous throughout a certain interval of values of x if it is continuous for all values of x in that interval. It is said to be *continuous everywhere* if it is continuous for every value of x. Thus $[x]$ is continuous in the interval $(\epsilon, 1-\epsilon)$, where ϵ is any positive number less than $\frac{1}{2}$; and 1 and x are continuous everywhere. [x] = greatest integer not greater than x

If we recur to the definitions of a limit we see that our definition is equivalent to '*$\phi(x)$ is continuous for $x = \xi$ if, given δ, we can choose $\epsilon(\delta)$ so that $|\phi(x) - \phi(\xi)| < \delta$ if $0 \leqq |x - \xi| \leqq \epsilon(\delta)$*'.

We have often to consider functions defined only in an interval (a, b). In this case it is convenient to make a slight and obvious change in our definition of continuity in so far as it concerns the particular points a and b. We shall then say that $\phi(x)$ is continuous for $x = a$ if $\phi(a+0)$ exists and is equal to $\phi(a)$, and for $x = b$ if $\phi(b-0)$ exists and is equal to $\phi(b)$.

99. The definition of continuity given in the last section may be illustrated geometrically as follows. Draw the two horizontal lines $y = \phi(\xi) - \delta$ and $y = \phi(\xi) + \delta$. Then $|\phi(x) - \phi(\xi)| < \delta$ expresses the fact that the point on the curve corresponding to x lies

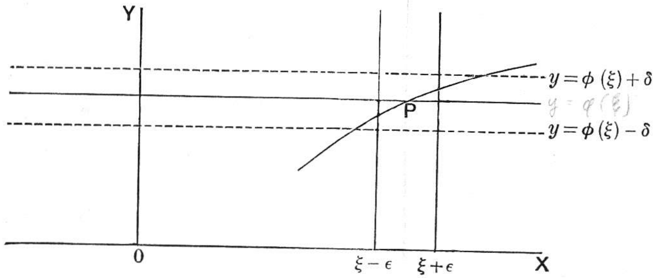

Fig. 30.

between these two lines. Similarly $|x - \xi| \leqq \epsilon$ expresses the fact that x lies in the interval $(\xi - \epsilon, \xi + \epsilon)$. Thus our definition asserts that if we draw two such horizontal lines, no matter how close together, we can always cut off a vertical strip of the plane by two vertical lines in such a way that all that part of the curve which is contained in the strip lies between the two horizontal lines. This is evidently true of the curve C (Fig. 29), whatever value ξ may have.

We shall now discuss the continuity of some special types of functions. Some of the results which follow were (as we pointed out at the time) tacitly assumed in Ch. II.

Examples XXXVII. 1. The sum or product of two functions continuous at a point is continuous at that point. The quotient is also continuous unless the denominator vanishes at the point. [This follows at once from Ex. XXXV. 1.]

2. Any polynomial is continuous for all values of x. Any rational fraction is continuous except for values of x for which the denominator vanishes. [This follows from Exs. XXXV. 6, 7.]

3. $\sqrt{x}$ is continuous for all positive values of x (Ex. XXXV. 8). It is not defined when $x < 0$, but is continuous for $x = 0$ in virtue of the remark made at the end of § 98. The same is true of $x^{m/n}$, where m and n are any positive integers of which n is even.

4. The function $x^{m/n}$, where n is odd, is continuous for all values of x.

5. $1/x$ is not continuous for $x = 0$. It has no value for $x = 0$, nor does it tend to a limit as $x \to 0$. In fact $1/x \to +\infty$ or $1/x \to -\infty$ according as $x \to 0$ by positive or negative values.

6. Discuss the continuity of $x^{-m/n}$, where m and n are positive integers, for $x = 0$.

7. The standard rational function $R(x) = P(x)/Q(x)$ is discontinuous for $x = a$, where a is any root of $Q(x) = 0$. Thus $(x^2+1)/(x^2-3x+2)$ is discontinuous for $x = 1$. It will be noticed that in the case of rational functions a discontinuity is always associated with (a) a failure of the definition for a particular value of x and (b) a tending of the function to $+\infty$ or $-\infty$ as x approaches this value from either side. Such a particular kind of point of discontinuity is usually described as an **infinity** of the function. An 'infinity' is the kind of discontinuity of most common occurrence in ordinary work.

8. Discuss the continuity of

$$\sqrt{\{(x-a)(b-x)\}},\quad \sqrt[3]{\{(x-a)(b-x)\}},\quad \sqrt{\{(x-a)/(b-x)\}},\quad \sqrt[3]{\{(x-a)/(b-x)\}}.$$

9. $\sin x$ and $\cos x$ are continuous for all values of x.

[We have $\qquad \sin(x+h) - \sin x = 2\sin\tfrac{1}{2}h\cos(x+\tfrac{1}{2}h)$,

which is numerically less than the numerical value of h.]

10. For what values of x are $\tan x$, $\cot x$, $\sec x$, and $\operatorname{cosec} x$ continuous or discontinuous?

11. If $f(y)$ is continuous for $y = \eta$, and $\phi(x)$ is a continuous function of x which is equal to η when $x = \xi$, then $f\{\phi(x)\}$ is continuous for $x = \xi$.

12. If $\phi(x)$ is continuous for any particular value of x, then any polynomial in $\phi(x)$, such as $a\{\phi(x)\}^m + \dots$, is so too.

13. Discuss the continuity of

$$1/(a\cos^2 x + b\sin^2 x),\quad \sqrt{(2+\cos x)},\quad \sqrt{(1+\sin x)},\quad 1/\sqrt{(1+\sin x)}.$$

14. $\sin(1/x)$, $x\sin(1/x)$, and $x^2\sin(1/x)$ are continuous except for $x = 0$.

15. The function which is equal to $x\sin(1/x)$ except when $x = 0$, and to zero when $x = 0$, is continuous for all values of x.

16. $[x]$ and $x - [x]$ are discontinuous for all integral values of x.

17. For what (if any) values of x are the following functions discontinuous: $[x^2]$, $[\sqrt{x}]$, $\sqrt{(x-[x])}$, $[x] + \sqrt{(x-[x])}$, $[2x]$, $[x] + [-x]$?

18. **Classification of discontinuities.** Some of the preceding examples suggest a classification of different types of discontinuity.

(1) Suppose that $\phi(x)$ tends to a limit as $x \to a$ either by values less than or by values greater than a. Denote these limits, as in § 95, by $\phi(a-0)$ and $\phi(a+0)$ respectively. Then, for continuity, it is necessary and sufficient that $\phi(x)$ should be defined for $x=a$, and that $\phi(a-0)=\phi(a)=\phi(a+0)$. Discontinuity may arise in a variety of ways.

(α) $\phi(a-0)$ may be equal to $\phi(a+0)$, but $\phi(a)$ may not be defined, or may differ from $\phi(a-0)$ and $\phi(a+0)$. Thus if $\phi(x)=x\sin(1/x)$ and $a=0$, $\phi(0-0)=\phi(0+0)=0$, but $\phi(x)$ is not defined for $x=0$. Or if $\phi(x)=[1-x^2]$ and $a=0$, $\phi(0-0)=\phi(0+0)=0$, but $\phi(0)=1$.

(β) $\phi(a-0)$ and $\phi(a+0)$ may be unequal. In this case $\phi(a)$ may be equal to one or to neither, or be undefined. The first case is illustrated by $\phi(x)=[x]$, for which $\phi(0-0)=-1$, $\phi(0+0)=\phi(0)=0$; the second by $\phi(x)=[x]-[-x]$, for which $\phi(0-0)=-1$, $\phi(0+0)=1$, $\phi(0)=0$; and the third by $\phi(x)=[x]+x\sin(1/x)$, for which $\phi(0-0)=-1$, $\phi(0+0)=0$, and $\phi(0)$ is undefined.

In any of these cases we say that $\phi(x)$ has a **simple discontinuity** at $x=a$. And to these cases we may add those in which $\phi(x)$ is defined only on one side of $x=a$, and $\phi(a-0)$ or $\phi(a+0)$, as the case may be, exists, but $\phi(x)$ is either not defined when $x=a$ or has when $x=a$ a value different from $\phi(a-0)$ or $\phi(a+0)$.

It is plain from § 95 that *a function which increases or decreases steadily in the neighbourhood of $x=a$ can have at most a simple discontinuity for $x=a$.*

(2) It may be the case that only one (or neither) of $\phi(a-0)$ and $\phi(a+0)$ exists, but that, supposing for example $\phi(a+0)$ not to exist, $\phi(x)\to+\infty$ or $\phi(x)\to-\infty$ as $x\to a+0$, so that $\phi(x)$ tends to a limit or to $+\infty$ or to $-\infty$ as x approaches a from either side. Such is the case, for instance, if $\phi(x)=1/x$ or $\phi(x)=1/x^2$, and $a=0$. In such cases we say (cf. Ex. 7) that $x=a$ is **an infinity** of $\phi(x)$. And again we may add to these cases those in which $\phi(x)\to+\infty$ or $\phi(x)\to-\infty$ as $x\to a$ from one side, but $\phi(x)$ is not defined at all on the other side of $x=a$.

(3) Any point of discontinuity which is not a point of simple discontinuity nor an infinity is called a point of **oscillatory discontinuity.** Such is the point $x=0$ for the functions $\sin(1/x)$, $(1/x)\sin(1/x)$.

19. What is the nature of the discontinuities at $x=0$ of the functions $(\sin x)/x$, $[x]+[-x]$, $\operatorname{cosec} x$, $\sqrt{(1/x)}$, $\sqrt[3]{(1/x)}$, $\operatorname{cosec}(1/x)$, $\sin(1/x)/\sin(1/x)$?

20. The function which is equal to 1 when x is rational and to 0 when x is irrational (Ch. II, Ex. XVI. 10) is discontinuous for all values of x. So too is any function which is defined only for rational or for irrational values of x.

21. The function which is equal to x when x is irrational and to $\sqrt{\{(1+p^2)/(1+q^2)\}}$ when x is a rational fraction p/q (Ch. II, Ex. XVI. 11) is discontinuous for all negative and for positive rational values of x, but continuous for positive irrational values.

22. For what points are the functions considered in Ch. IV, Exs. XXXI discontinuous, and what is the nature of their discontinuities? [Consider, *e.g.*, the function $y=\lim x^n$ (Ex. 5). Here y is only defined when $-1<x\leq 1$: it is equal to 0 when $-1<x<1$ and to 1 when $x=1$. The points $x=1$ and $x=-1$ are points of simple discontinuity.]

100. The fundamental property of a continuous function. It may perhaps be thought that the analysis of the idea of a continuous curve given in § 98 is not the simplest or most natural possible. Another method of analysing our idea of continuity is the following. Let A and B be two points on the graph of $\phi(x)$ whose coordinates are x_0, $\phi(x_0)$ and x_1, $\phi(x_1)$ respectively. Draw any straight line λ which passes between A and B. Then common sense certainly declares that if the graph of $\phi(x)$ is continuous it must cut λ.

If we consider this property as an intrinsic geometrical property of continuous curves it is clear that there is no real loss of generality in supposing λ to be parallel to the axis of x. In this case the ordinates of A and B cannot be equal: let us suppose, for definiteness, that $\phi(x_1)>\phi(x_0)$. And let λ be the line $y=\eta$, where $\phi(x_0)<\eta<\phi(x_1)$. Then to say that the graph of $\phi(x)$ must cut λ is the same thing as to say that there is a value of x between x_0 and x_1 for which $\phi(x)=\eta$.

We conclude then that a continuous function $\phi(x)$ must possess the following property: *if*

$$\phi(x_0)=y_0, \quad \phi(x_1)=y_1,$$

and $y_0<\eta<y_1$, then there is a value of x between x_0 and x_1 for which $\phi(x)=\eta$. In other words *as x varies from x_0 to x_1, y must assume at least once every value between y_0 and y_1.*

We shall now prove that if $\phi(x)$ is a continuous function of x in the sense defined in § 98 then it does in fact possess this property. There is a certain range of values of x, to the right of x_0, for which $\phi(x)<\eta$. For $\phi(x_0)<\eta$, and so $\phi(x)$ is certainly less than η if

$\phi(x) - \phi(x_0)$ is numerically less than $\eta - \phi(x_0)$. But since $\phi(x)$ is continuous for $x = x_0$, this condition is certainly satisfied if x is near enough to x_0. Similarly there is a certain range of values, to the left of x_1, for which $\phi(x) > \eta$.

Let us divide the values of x between x_0 and x_1 into two classes L, R as follows:

(1) in the class L we put all values ξ of x such that $\phi(x) < \eta$ when $x = \xi$ and for all values of x between x_0 and ξ;

(2) in the class R we put all the other values of x, *i.e.* all numbers ξ such that either $\phi(\xi) \geqq \eta$ or there is a value of x between x_0 and ξ for which $\phi(x) \geqq \eta$.

Then it is evident that these two classes satisfy all the conditions imposed upon the classes L, R of § 17, and so constitute a section of the real numbers. Let ξ_0 be the number corresponding to the section.

First suppose $\phi(\xi_0) > \eta$, so that ξ_0 belongs to the upper class: and let $\phi(\xi_0) = \eta + k$, say. Then $\phi(\xi') < \eta$ and so

$$\phi(\xi_0) - \phi(\xi') > k,$$

for all values of ξ' less than ξ_0, which contradicts the condition of continuity for $x = \xi_0$.

Next suppose $\phi(\xi_0) = \eta - k < \eta$. Then, if ξ' is any number greater than ξ_0, either $\phi(\xi') \geqq \eta$ or we can find a number ξ'' between ξ_0 and ξ' such that $\phi(\xi'') \geqq \eta$. In either case we can find a number as near to ξ_0 as we please and such that the corresponding values of $\phi(x)$ differ by more than k. And this again contradicts the hypothesis that $\phi(x)$ is continuous for $x = \xi_0$.

Hence $\phi(\xi_0) = \eta$, and the theorem is established. It should be observed that we have proved more than is asserted explicitly in the theorem; we have proved in fact that ξ_0 is the *least* value of x for which $\phi(x) = \eta$. It is not obvious, or indeed generally true, that there is a least among the values of x for which a function assumes a given value, though this is true for continuous functions.

It is easy to see that the converse of the theorem just proved is not true. Thus such a function as the function $\phi(x)$ whose graph is represented by Fig. 31 obviously assumes at least once every value between $\phi(x_0)$ and $\phi(x_1)$: yet $\phi(x)$ is discontinuous. Indeed it is not even true that $\phi(x)$ must be continuous when it assumes each value *once and once only*. Thus let $\phi(x)$ be defined as follows from $x=0$ to $x=1$. If $x=0$ let $\phi(x)=0$; if $0<x<1$ let $\phi(x)=1-x$; and if $x=1$ let $\phi(x)=1$. The graph of the function is shown in Fig. 32; it includes the points O, C but *not* the points A, B. It is clear that, as x varies from 0 to 1, $\phi(x)$ assumes once and once only every value between $\phi(0)=0$ and $\phi(1)=1$; but $\phi(x)$ is discontinuous for $x=0$ and $x=1$.

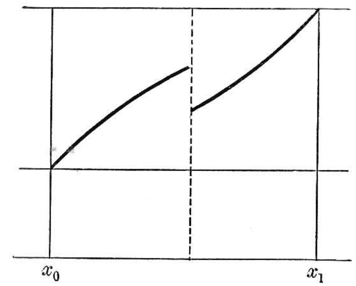

Fig. 31.

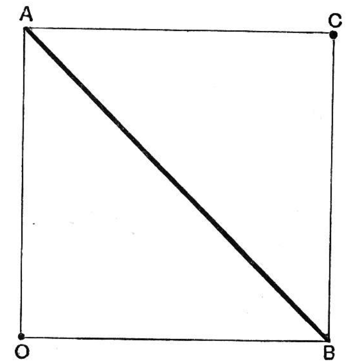

Fig. 32.

As a matter of fact, however, the curves which usually occur in elementary mathematics are composed of *a finite number of pieces along which y always varies in the same direction*. It is easy to show that if $y=\phi(x)$ always varies in the same direction, *i.e.* steadily increases or decreases, as x varies from x_0 to x_1, then the two notions of continuity are really equivalent, *i.e.* that if $\phi(x)$ takes every value between $\phi(x_0)$ and $\phi(x_1)$ then it must be a continuous function in the sense of § 98. For let ξ be any value of x between x_0 and x_1. As $x \to \xi$ through values less than ξ, $\phi(x)$ tends to the limit $\phi(\xi-0)$ (§ 95). Similarly as $x \to \xi$ through values greater than ξ, $\phi(x)$ tends to the limit $\phi(\xi+0)$. The function will be continuous for $x=\xi$ if and only if

$$\phi(\xi-0)=\phi(\xi)=\phi(\xi+0).$$

But if either of these equations is untrue, say the first, then it is evident that $\phi(x)$ never assumes any value which lies between $\phi(\xi-0)$ and $\phi(\xi)$, which is contrary to our assumption. Thus $\phi(x)$ must be continuous. The net result of this and the last section is consequently to show that our common-sense notion of what we mean by continuity is substantially accurate, and capable of precise statement in mathematical terms.

101. In this and the following paragraphs we shall state and prove some general theorems concerning continuous functions.

THEOREM 1. *Suppose that $\phi(x)$ is continuous for $x=\xi$, and that $\phi(\xi)$ is positive. Then we can determine a positive number ϵ such that $\phi(\xi)$ is positive throughout the interval $(\xi-\epsilon, \xi+\epsilon)$.*

For, taking $\delta = \frac{1}{2}\phi(\xi)$ in the fundamental inequality of p. 175, we can choose ϵ so that

$$|\phi(x)-\phi(\xi)| < \tfrac{1}{2}\phi(\xi)$$

throughout $(\xi-\epsilon, \xi+\epsilon)$, and then

$$\phi(x) \geqq \phi(\xi) - |\phi(x)-\phi(\xi)| > \tfrac{1}{2}\phi(\xi) > 0,$$

so that $\phi(x)$ is positive. There is plainly a corresponding theorem referring to negative values of $\phi(x)$.

THEOREM 2. *If $\phi(x)$ is continuous for $x=\xi$, and $\phi(x)$ vanishes for values of x as near to ξ as we please, or assumes, for values of x as near to ξ as we please, both positive and negative values, then $\phi(\xi)=0$.*

This is an obvious corollary of Theorem 1. If $\phi(\xi)$ is not zero, it must be positive or negative; and if it were, for example, positive, it would be positive for all values of x sufficiently near to ξ, which contradicts the hypotheses of the theorem.

102. The range of values of a continuous function. Let us consider a function $\phi(x)$ about which we shall only assume at present that it is defined for every value of x in an interval (a, b).

The values assumed by $\phi(x)$ for values of x in (a, b) form an aggregate S to which we can apply the arguments of § 80, as we applied them in § 81 to the aggregate of values of a function of n. If there is a number K such that $\phi(x) \leqslant K$, for all values of x in question, we say that $\phi(x)$ is *bounded above*. In this case $\phi(x)$ possesses an *upper bound* M: no value of $\phi(x)$ exceeds M, but any number less than M is exceeded by at least one value of $\phi(x)$. Similarly we define '*bounded below*', '*lower bound*', '*bounded*', as applied to functions of a continuous variable x.

THEOREM 1. *If $\phi(x)$ is continuous throughout (a, b), then it is bounded in (a, b).*

We can certainly determine an interval (a, ξ), extending to the right from a, in which $\phi(x)$ is bounded. For since $\phi(x)$ is continuous for $x=a$, we can, given any positive number δ however small, determine an interval (a, ξ) throughout which $\phi(x)$ lies between $\phi(a)-\delta$ and $\phi(a)+\delta$; and obviously $\phi(x)$ is bounded in this interval.

Now divide the points ξ of the interval (a, b) into two classes L, R, putting ξ in L if $\phi(\xi)$ is bounded in (a, ξ), and in R if this is not the case. It follows from what precedes that L certainly exists: what we propose to prove is that R does not. Suppose that R does exist, and let β be the number corresponding to the section whose lower and upper classes are L and R. Since $\phi(x)$ is continuous for $x=\beta$, we can, however small δ may be, determine an interval $(\beta-\eta, \beta+\eta)$* throughout which

$$\phi(\beta)-\delta < \phi(x) < \phi(\beta)+\delta.$$

Thus $\phi(x)$ is bounded in $(\beta-\eta, \beta+\eta)$. Now $\beta-\eta$ belongs to L. Therefore $\phi(x)$ is bounded in $(a, \beta-\eta)$: and therefore it is bounded in the whole interval $(a, \beta+\eta)$. But $\beta+\eta$ belongs to R and so $\phi(x)$ is *not* bounded in $(a, \beta+\eta)$. This contradiction shows that R does not exist. And so $\phi(x)$ is bounded in the whole interval (a, b).

THEOREM 2. *If $\phi(x)$ is continuous throughout (a, b), and M and m are its upper and lower bounds, then $\phi(x)$ assumes the values M and m at least once each in the interval.*

For, given any positive number δ, we can find a value of x for which $M-\phi(x) < \delta$ or $1/\{M-\phi(x)\} > 1/\delta$. Hence $1/\{M-\phi(x)\}$ is not bounded, and therefore, by Theorem 1, is not continuous. But $M-\phi(x)$ is a continuous function, and so $1/\{M-\phi(x)\}$ is continuous at any point at which its denominator does not vanish (Ex. XXXVII. 1). There must therefore be one point at which the denominator vanishes: at this point $\phi(x)=M$. Similarly it may be shown that there is a point at which $\phi(x)=m$.

The proof just given is somewhat subtle and indirect, and it may be well, in view of the great importance of the theorem, to indicate alternative lines of proof. It will however be convenient to postpone these for a moment†.

* If $\beta=b$ we must replace this interval by $(\beta-\eta, \beta)$, and $\beta+\eta$ by β, throughout the argument which follows.

† See § 104.

Examples XXXVIII. 1. If $\phi(x)=1/x$ except when $x=0$, and $\phi(x)=0$ when $x=0$, then $\phi(x)$ has neither an upper nor a lower bound in any interval which includes $x=0$ in its interior, as *e.g.* the interval $(-1, +1)$.

2. If $\phi(x)=1/x^2$ except when $x=0$, and $\phi(x)=0$ when $x=0$, then $\phi(x)$ has the lower bound 0, but no upper bound, in the interval $(-1, +1)$.

3. Let $\phi(x)=\sin(1/x)$ except when $x=0$, and $\phi(x)=0$ when $x=0$. Then $\phi(x)$ is discontinuous for $x=0$. In any interval $(-\delta, +\delta)$ the lower bound is -1 and the upper bound $+1$, and each of these values is assumed by $\phi(x)$ an infinity of times.

4. Let $\phi(x)=x-[x]$. This function is discontinuous for all integral values of x. In the interval $(0, 1)$ its lower bound is 0 and its upper bound 1. It is equal to 0 when $x=0$ or $x=1$, but it is never equal to 1. Thus $\phi(x)$ never assumes a value equal to its upper bound.

5. Let $\phi(x)=0$ when x is irrational, and $\phi(x)=q$ when x is a rational fraction p/q. Then $\phi(x)$ has the lower bound 0, but no upper bound, in any interval (a, b). But if $\phi(x)=(-1)^p q$ when $x=p/q$, then $\phi(x)$ has neither an upper nor a lower bound in any interval.

103. The oscillation of a function in an interval. Let $\phi(x)$ be any function bounded throughout (a, b), and M and m its upper and lower bounds. We shall now use the notation $M(a, b)$, $m(a, b)$ for M, m, in order to exhibit explicitly the dependence of M and m on a and b, and we shall write

$$O(a, b) = M(a, b) - m(a, b).$$

This number $O(a, b)$, the difference between the upper and lower bounds of $\phi(x)$ in (a, b), we shall call the **oscillation** *of* $\phi(x)$ *in* (a, b). The simplest of the properties of the functions $M(a, b)$, $m(a, b)$, $O(a, b)$ are as follows.

(1) *If* $a \leqq c \leqq b$ *then* $M(a, b)$ *is equal to the greater of* $M(a, c)$ *and* $M(c, b)$, *and* $m(a, b)$ *to the lesser of* $m(a, c)$ *and* $m(c, b)$.

(2) $M(a, b)$ *is an increasing,* $m(a, b)$ *a decreasing, and* $O(a, b)$ *an increasing function of* b.

(3) $O(a, b) \leqq O(a, c) + O(c, b)$.

The first two theorems are almost immediate consequences of our definitions. Let μ be the greater of $M(a, c)$ and $M(c, b)$, and let δ be any positive number. Then $\phi(x) \leqq \mu$ throughout (a, c) and (c, b), and therefore throughout (a, b); and $\phi(x) > \mu - \delta$ somewhere in (a, c) or in (c, b), and therefore somewhere in (a, b).

Hence $M(a, b) = \mu$. The proposition concerning m may be proved similarly. Thus (1) is proved, and (2) is an obvious corollary.

Suppose now that M_1 is the greater and M_2 the less of $M(a, c)$ and $M(c, b)$, and that m_1 is the less and m_2 the greater of $m(a, c)$ and $m(c, b)$. Then, since c belongs to both intervals, $\phi(c)$ is not greater than M_2 nor less than m_2. Hence $M_2 \geqq m_2$, whether these numbers correspond to the same one of the intervals (a, c) and (c, b) or not, and

$$O(a, b) = M_1 - m_1 \leqq M_1 + M_2 - m_1 - m_2.$$

But $$O(a, c) + O(c, b) = M_1 + M_2 - m_1 - m_2;$$

and (3) follows.

104. Alternative proofs of Theorem 2 of § 102. The most straightforward proof of Theorem 2 of § 102 is as follows. Let ξ be any number of the interval (a, b). The function $M(a, \xi)$ increases steadily with ξ and never exceeds M. We can therefore construct a section of the numbers ξ by putting ξ in L or in R according as $M(a, \xi) < M$ or $M(a, \xi) = M$. Let β be the number corresponding to the section. If $a < \beta < b$, we have

$$M(a, \beta - \eta) < M, \quad M(a, \beta + \eta) = M$$

for all positive values of η, and so

$$M(\beta - \eta, \beta + \eta) = M,$$

by (1) of § 103. Hence $\phi(x)$ assumes, for values of x as near as we please to β, values as near as we please to M, and so, since $\phi(x)$ is continuous, $\phi(\beta)$ must be equal to M.

If $\beta = a$ then $M(a, a+\eta) = M$. And if $\beta = b$ then $M(a, b-\eta) < M$, and so $M(b-\eta, b) = M$. In either case the argument may be completed as before.

The theorem may also be proved by the method of repeated bisection used in § 71. If M is the upper bound of $\phi(x)$ in an interval PQ, and PQ is divided into two equal parts, then it is possible to find a half P_1Q_1 in which the upper bound of $\phi(x)$ is also M. Proceeding as in § 71, we construct a sequence of intervals PQ, P_1Q_1, P_2Q_2, ... in each of which the upper bound of $\phi(x)$ is M. These intervals, as in § 71, converge to a point T, and it is easily proved that the value of $\phi(x)$ at this point is M.

105. Sets of intervals on a line. The Heine-Borel Theorem. We shall now proceed to prove some theorems concerning the oscillation of a function which are of a somewhat abstract character but of very great importance, particularly, as we shall see later, in the theory of integration. These theorems depend upon a general theorem concerning intervals on a line.

Suppose that we are given a *set of intervals* in a straight line, that is to say an aggregate each of whose members is an interval (α, β). We make no restriction as to the nature of these intervals; they may be finite or infinite in number; they may or may not overlap*; and any number of them may be included in others.

It is worth while in passing to give a few examples of sets of intervals to which we shall have occasion to return later.

(i) If the interval (0, 1) is divided into n equal parts then the n intervals thus formed define a finite set of non-overlapping intervals which just cover up the line.

(ii) We take every point ξ of the interval (0, 1), and associate with ξ the interval $(\xi-\epsilon, \xi+\epsilon)$, where ϵ is a positive number less than 1, except that with 0 we associate $(0, \epsilon)$ and with 1 we associate $(1-\epsilon, 1)$, and in general we reject any part of any interval which projects outside the interval (0, 1). We thus define an infinite set of intervals, and it is obvious that many of them overlap with one another.

(iii) We take the rational points p/q of the interval (0, 1), and associate with p/q the interval

$$\left(\frac{p}{q}-\frac{\epsilon}{q^3}, \ \frac{p}{q}+\frac{\epsilon}{q^3}\right),$$

where ϵ is positive and less than 1. We regard 0 as 0/1 and 1 as 1/1: in these two cases we reject the part of the interval which lies outside (0, 1). We obtain thus an infinite set of intervals, which plainly overlap with one another, since there are an infinity of rational points, other than p/q, in the interval associated with p/q.

The Heine-Borel Theorem. *Suppose that we are given an interval* (a, b), *and a set of intervals* I *each of whose members is included in* (a, b). *Suppose further that* I *possesses the following properties:*

(i) *every point of* (a, b), *other than* a *and* b, *lies inside*† *at least one interval of* I;

(ii) a *is the left-hand end point, and* b *the right-hand end point, of at least one interval of* I.

Then it is possible to choose **a finite number** *of intervals from the set* I *which form a set of intervals possessing the properties* (i) *and* (ii).

* The word *overlap* is used in its obvious sense: two intervals overlap if they have points in common which are not end points of either. Thus $(0, \frac{2}{3})$ and $(\frac{1}{3}, 1)$ overlap. A pair of intervals such as $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$ may be said to *abut*.

† That is to say 'in and not at an end of'.

We know that a is the left-hand end point of at least one interval of I, say (a, a_1). We know also that a_1 lies inside at least one interval of I, say (a_1', a_2). Similarly a_2 lies inside an interval (a_2', a_3) of I. It is plain that this argument may be repeated indefinitely, unless after a finite number of steps a_n coincides with b.

If a_n does coincide with b after a finite number of steps then there is nothing further to prove, for we have obtained a finite set of intervals, selected from the intervals of I, and possessing the properties required. If a_n never coincides with b, then the points $a_1, a_2, a_3, \ldots$ must (since each lies to the right of its predecessor) tend to a limiting position, but this limiting position may, so far as we can tell, lie anywhere in (a, b).

Let us suppose now that the process just indicated, starting from a, is performed in all possible ways, so that we obtain all possible sequences of the type $a_1, a_2, a_3, \ldots$. Then we can prove that *there must be at least one such sequence which arrives at* b *after a finite number of steps.*

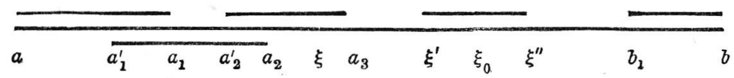

Fig. 33.

There are two possibilities with regard to any point ξ between a and b. Either (i) ξ lies to the left of *some* point a_n of *some* sequence or (ii) it does not. We divide the points ξ into two classes L and R according as to whether (i) or (ii) is true. The class L certainly exists, since all points of the interval (a, a_1) belong to L. We shall now prove that R does not exist, so that every point ξ belongs to L.

If R exists then L lies entirely to the left of R, and the classes L, R form a section of the real numbers between a and b, to which corresponds a number ξ_0. The point ξ_0 lies inside an interval of I, say (ξ', ξ''), and ξ' belongs to L, and so lies to the left of some term a_n of some sequence. But then we can take (ξ', ξ'') as the interval (a_n', a_{n+1}) associated with a_n in our construction of the sequence $a_1, a_2, a_3, \ldots$; and all points to the left of ξ'' lie to the left of a_{n+1}. There are therefore points of L to the right of ξ_0, and this contradicts the definition of R. It is therefore impossible that R should exist.

Thus every point ξ belongs to L. Now b is the right-hand end point of an interval of I, say (b_1, b), and b_1 belongs to L. Hence there is a member a_n of a sequence $a_1, a_2, a_3, \dots$ such that $a_n > b_1$. But then we may take the interval (a_n', a_{n+1}) corresponding to a_n to be (b_1, b), and so we obtain a sequence in which the term after the nth coincides with b, and therefore a finite set of intervals having the properties required. Thus the theorem is proved.

It is instructive to consider the examples of p. 186 in the light of this theorem.

(i) Here the conditions of the theorem are not satisfied; the points $1/n, 2/n, 3/n, \dots$ do not lie inside any interval of I.

(ii) Here the conditions of the theorem are satisfied. The set of intervals

$$(0, 2\epsilon), (\epsilon, 3\epsilon), (2\epsilon, 4\epsilon), \dots, (1-2\epsilon, 1),$$

associated with the points $\epsilon, 2\epsilon, 3\epsilon, \dots, 1-\epsilon$, possesses the properties required.

(iii) In this case we can prove, by using the theorem, that there are, if ϵ is small enough, points of $(0, 1)$ which do not lie in any interval of I.

If every point of $(0, 1)$ lay inside an interval of I (with the obvious reservation as to the end points), then we could find a finite number of intervals of I possessing the same property and having therefore a total length greater than 1. Now there are two intervals, of total length 2ϵ, for which $q=1$, and $q-1$ intervals, of total length $2\epsilon(q-1)/q^3$, associated with any other value of q. The sum of any finite number of intervals of I can therefore not be greater than 2ϵ times that of the series

$$1+\frac{1}{2^3}+\frac{2}{3^3}+\frac{3}{4^3}+\dots,$$

which will be shown to be convergent in Ch. VIII. Hence it follows that, if ϵ is small enough, the supposition that every point of $(0, 1)$ lies inside an interval of I leads to a contradiction.

The reader may be tempted to think that this proof is needlessly elaborate, and that the existence of points of the interval, not in any interval of I, follows at once from the fact that the sum of all these intervals is less than 1. But the theorem to which he would be appealing is (when the set of intervals is infinite) far from obvious, and can only be proved rigorously by some such use of the Heine-Borel Theorem as is made in the text.

106. We shall now apply the Heine-Borel Theorem to the proof of two important theorems concerning the oscillation of a continuous function.

THEOREM I. *If $\phi(x)$ is continuous throughout the interval (a, b), then we can divide (a, b) into a finite number of sub-intervals $(a, x_1), (x_1, x_2), \dots (x_n, b)$, in each of which the oscillation of $\phi(x)$ is less than an assigned positive number δ.*

Let ξ be any number between a and b. Since $\phi(x)$ is continuous for $x=\xi$, we can determine an interval $(\xi-\epsilon, \xi+\epsilon)$ such that the oscillation of $\phi(x)$ in this interval is less than δ. It is indeed obvious that there are an infinity of such intervals corresponding to every ξ and every δ, for if the condition is satisfied for any particular value of ϵ, then it is satisfied *a fortiori* for any smaller value. What values of ϵ are admissible will naturally depend upon ξ; we have at present no reason for supposing that a value of ϵ admissible for one value of ξ will be admissible for another. We shall call the intervals thus associated with ξ *the δ-intervals of ξ*.

If $\xi=a$ then we can determine an interval $(a, a+\epsilon)$, and so an infinity of such intervals, having the same property. These we call the δ-intervals of a, and we can define in a similar manner the δ-intervals of b.

Consider now the set I of intervals formed by taking all the δ-intervals of all points of (a, b). It is plain that this set satisfies the conditions of the Heine-Borel Theorem; every point interior to the interval is interior to at least one interval of I, and a and b are end points of at least one such interval. We can therefore determine a set I' which is formed by a finite number of intervals of I, and which possesses the same property as I itself.

The intervals which compose the set I' will in general overlap, as in Fig. 34. But their end points obviously divide up (a, b) into a finite set of intervals I'' each of which is included in an interval of I', and in each of which the oscillation of $\phi(x)$ is less than δ. Thus Theorem I is proved.

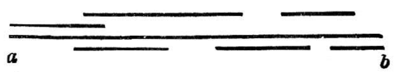

Fig. 34.

THEOREM II. *Given any positive number δ, we can find a number η such that, if the interval (a, b) is divided in any manner into sub-intervals of length less than η, then the oscillation of $\phi(x)$ in each of them will be less than δ.*

Take $\delta_1 < \frac{1}{2}\delta$, and construct, as in Theorem I, a finite set of sub-intervals j in each of which the oscillation of $\phi(x)$ is less than δ_1. Let η be the length of the least of these sub-intervals j. If now we divide (a, b) into parts each of length less than η, then any such part must lie entirely within at most two successive sub-intervals j. Hence, in virtue of (3) of § 103, the oscillation of $\phi(x)$, in one of the parts of length less than η, cannot exceed twice the greatest oscillation of $\phi(x)$ in a sub-interval j, and is therefore less than $2\delta_1$, and therefore than δ.

This theorem is of fundamental importance in the theory of definite integrals (Ch. VII). It is impossible, without the use of this or some similar theorem, to prove that a function continuous throughout an interval necessarily possesses an integral over that interval.

107. Continuous functions of several variables. The notions of continuity and discontinuity may be extended to functions of several independent variables (Ch. II, §§ 31 *et seq.*). Their application to such functions, however, raises questions much more complicated and difficult than those which we have considered in this chapter. It would be impossible for us to discuss these questions in any detail here; but we shall, in the sequel, require to know what is meant by a continuous function of two variables, and we accordingly give the following definition. It is a straightforward generalisation of the last form of the definition of § 98.

The function $\phi(x, y)$ of the two variables x and y is said to be **continuous** *for $x = \xi$, $y = \eta$ if, given any positive number δ, however small, we can choose $\epsilon(\delta)$ so that*

$$|\phi(x, y) - \phi(\xi, \eta)| < \delta$$

*when $0 \leqq |x - \xi| \leqq \epsilon(\delta)$ and $0 \leqq |y - \eta| \leqq \epsilon(\delta)$; that is to say if we can draw a square, whose sides are parallel to the axes of coordinates and of length $2\epsilon(\delta)$, whose centre is the point (ξ, η), and which is such that the value of $\phi(x, y)$ at any point inside it or on its boundary differs from $\phi(\xi, \eta)$ by less than δ.**

This definition of course presupposes that $\phi(x, y)$ is defined at all points of the square in question, and in particular at the point (ξ, η). Another method of stating the definition is this: *$\phi(x, y)$ is continuous for $x = \xi$, $y = \eta$ if $\phi(x, y) \to \phi(\xi, \eta)$ when $x \to \xi$, $y \to \eta$ in any manner.* This statement is apparently simpler; but it contains phrases the precise meaning of which has not yet been explained and can only be explained by the help of inequalities like those which occur in our original statement.

* The reader should draw a figure to illustrate the definition.

It is easy to prove that the sums, the products, and in general the quotients of continuous functions of two variables are themselves continuous. A polynomial in two variables is continuous for all values of the variables; and the ordinary functions of x and y which occur in every-day analysis are *generally* continuous, *i.e.* are continuous except for pairs of values of x and y connected by special relations.

The reader should observe carefully that to assert the continuity of $\phi(x, y)$ with respect to the two variables x and y is to assert much more than its continuity with respect to each variable considered separately. It is plain that if $\phi(x, y)$ is continuous with respect to x and y then it is certainly continuous with respect to x (or y) when any fixed value is assigned to y (or x). But the converse is by no means true. Suppose, for example, that

$$\phi(x, y) = \frac{2xy}{x^2 + y^2}$$

when neither x nor y is zero, and $\phi(x, y) = 0$ when either x or y is zero. Then if y has any fixed value, zero or not, $\phi(x, y)$ is a continuous function of x, and in particular continuous for $x = 0$; for its value when $x = 0$ is zero, and it tends to the limit zero as $x \to 0$. In the same way it may be shown that $\phi(x, y)$ is a continuous function of y. But $\phi(x, y)$ is *not* a continuous function of x *and* y for $x = 0$, $y = 0$. Its value when $x = 0$, $y = 0$ is zero; but if x and y tend to zero along the straight line $y = ax$, then

$$\phi(x, y) = \frac{2a}{1 + a^2}, \quad \lim \phi(x, y) = \frac{2a}{1 + a^2},$$

which may have any value between -1 and 1.

108. Implicit functions. We have already, in Ch. II, met with the idea of an *implicit function*. Thus, if x and y are connected by the relation

$$y^5 - xy - y - x = 0 \quad \text{.................................(1),}$$

then y is an 'implicit function' of x.

But it is far from obvious that such an equation as this does really define a function y of x, or several such functions. In Ch. II we were content to take this for granted. We are now in a position to consider whether the assumption we made then was justified.

We shall find the following terminology useful. Suppose that it is possible to surround a point (a, b), as in § 107, with a square throughout which a certain condition is satisfied. We shall call such a square a *neighbourhood* of (a, b), and say that the condition in question is satisfied *in the neighbourhood of* (a, b), or *near* (a, b), meaning by this simply that it is possible to find *some* square throughout which the condition is satisfied. It is obvious that similar language may be used when we are dealing with a single variable, the square being replaced by an interval on a line.

THEOREM. *If* (i) $f(x, y)$ *is a continuous function of* x *and* y *in the neighbourhood of* (a, b),

(ii) $f(a, b)=0$,

(iii) $f(x, y)$ *is, for all values of* x *in the neighbourhood of* a, *a steadily increasing function of* y, *in the stricter sense of* § 95,

then (1) *there is a unique function* $y=\phi(x)$ *which, when substituted in the equation* $f(x, y)=0$, *satisfies it identically for all values of* x *in the neighbourhood of* a,

(2) $\phi(x)$ *is continuous for all values of* x *in the neighbourhood of* a.

In the figure the square represents a 'neighbourhood' of (a, b) throughout which the conditions (i) and (iii) are satisfied, and P the point (a, b). If we take Q and R as in the figure, it follows from (iii) that $f(x, y)$ is positive at Q and negative at R. This being so, and $f(x, y)$ being continuous at Q and at R, we can draw lines QQ' and RR' parallel to OX, so that $R'Q'$ is parallel to OY and $f(x, y)$ is positive at all points of QQ' and negative at all points of RR'. In particular $f(x, y)$ is positive at Q' and negative at R', and therefore, in virtue of (iii) and § 100, vanishes once and only once at a point P' on $R'Q'$. The same construction gives us a unique point at which $f(x, y)=0$ on each ordinate between RQ and $R'Q'$. It is obvious, moreover, that the same construction can be carried out to the left of RQ. The aggregate of points such as P' gives us the graph of the required function $y=\phi(x)$.

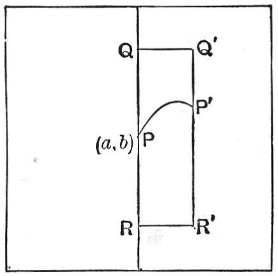

Fig. 35.

It remains to prove that $\phi(x)$ is continuous. This is most simply effected by using the idea of the 'limits of indetermination' of $\phi(x)$ as $x\to a$ (§ 96). Suppose that $x\to a$, and let λ and Λ be the limits of indetermination of $\phi(x)$ as $x\to a$. It is evident that the points (a, λ) and (a, Λ) lie on QR. Moreover, we can find a sequence of values of x such that $\phi(x)\to\lambda$ when $x\to a$ through the values of the sequence; and since $f\{x, \phi(x)\}=0$, and $f(x, y)$ is a continuous function of x and y, we have

$$f(a, \lambda)=0.$$

Hence $\lambda=b$; and similarly $\Lambda=b$. Thus $\phi(x)$ tends to the limit b as $x\to a$, and so $\phi(x)$ is continuous for $x=a$. It is evident that we can show in

exactly the same way that $\phi(x)$ is continuous for any value of x in the neighbourhood of a.

It is clear that the truth of the theorem would not be affected if we were to change 'increasing' to 'decreasing' in condition (iii).

As an example, let us consider the equation (1), taking $a=0$, $b=0$. It is evident that the conditions (i) and (ii) are satisfied. Moreover

$$f(x, y)-f(x, y')=(y-y')(y^4+y^3y'+y^2y'^2+yy'^3+y'^4-x-1)$$

has, when x, y, and y' are sufficiently small, the sign opposite to that of $y-y'$. Hence condition (iii) (with 'decreasing' for 'increasing') is satisfied. It follows that there is one and only one continuous function y which satisfies the equation (1) identically and vanishes with x.

The same conclusion would follow if the equation were

$$y^2-xy-y-x=0.$$

The function in question is in this case

$$y=\tfrac{1}{2}\{1+x-\sqrt{(1+6x+x^2)}\},$$

where the square root is positive. The second root, in which the sign of the square root is changed, does not satisfy the condition of vanishing with x.

There is one point in the proof which the reader should be careful to observe. We supposed that the hypotheses of the theorem were satisfied 'in the neighbourhood of (a, b)', that is to say throughout a certain square $\xi-\epsilon\leqq x\leqq\xi+\epsilon$, $\eta-\epsilon\leqq y\leqq\eta+\epsilon$. The conclusion holds 'in the neighbourhood of $x=a$', that is to say throughout a certain interval $\xi-\epsilon_1\leqq x\leqq\xi+\epsilon_1$. There is nothing to show that the ϵ_1 of the conclusion is the ϵ of the hypotheses, and indeed this is generally untrue.

109. Inverse Functions. Suppose in particular that $f(x, y)$ is of the form $F(y)-x$. We then obtain the following theorem.

If $F(y)$ *is a function of* y, *continuous and steadily increasing* (*or decreasing*), *in the stricter sense of* § 95, *in the neighbourhood of* $y=b$, *and* $F(b)=a$, *then there is a unique continuous function* $y=\phi(x)$ *which is equal to* b *when* $x=a$ *and satisfies the equation* $F(y)=x$ *identically in the neighbourhood of* $x=a$.

The function thus defined is called the *inverse function* of $F(y)$.

Suppose for example that $y^3=x$, $a=0$, $b=0$. Then all the conditions of the theorem are satisfied. The inverse function is $x=\sqrt[3]{y}$.

If we had supposed that $y^2=x$ then the conditions of the theorem would not have been satisfied, for y^2 is not a steadily increasing function of y in any interval which includes $y=0$: it decreases when y is negative and increases when y is positive. And in this case the conclusion of the theorem does not hold, for $y^2=x$ defines *two* functions of x, viz. $y=\sqrt{x}$ and $y=-\sqrt{x}$, both of which vanish when $x=0$, and each of which is defined only for positive values of x, so that the equation has sometimes two solutions and sometimes none. The reader should consider the more general equations

$$y^{2n}=x,\quad y^{2n+1}=x,$$

in the same way. Another interesting example is given by the equation

$$y^5 - y - x = 0,$$

already considered in Ex. XIV. 7.

Similarly the equation $\sin y = x$

has just one solution which vanishes with x, viz. the value of $\arcsin x$ which vanishes with x. There are of course an infinity of solutions, given by the other values of $\arcsin x$ (cf. Ex. XV. 10), which do not satisfy this condition.

So far we have considered only what happens in the neighbourhood of a particular value of x. Let us suppose now that $F(y)$ is positive and steadily increasing (or decreasing) throughout an interval (a, b). Given any point ξ of (a, b), we can determine an interval i including ξ, and a unique and continuous inverse function $\phi_i(x)$ defined throughout i.

From the set I of intervals i we can, in virtue of the Heine-Borel Theorem, pick out a finite sub-set covering up the whole interval (a, b); and it is plain that the finite set of functions $\phi_i(x)$, corresponding to the sub-set of intervals i thus selected, define together a unique inverse function $\phi(x)$ continuous throughout (a, b).

We thus obtain the theorem: *if $x = F(y)$, where $F(y)$ is continuous and increases steadily and strictly from A to B as x increases from a to b, then there is a unique inverse function $y = \phi(x)$ which is continuous and increases steadily and strictly from a to b as x increases from A to B.*

It is worth while to show how this theorem can be obtained directly without the help of the more difficult theorem of § 108. Suppose that $A < \xi < B$, and consider the class of values of y such that (i) $a < y < b$ and (ii) $F(y) \leqq \xi$. This class has an upper bound η, and plainly $F(\eta) \leqq \xi$. If $F(\eta)$ were less than ξ, we could find a value of y such that $y > \eta$ and $F(y) < \xi$, and η would not be the upper bound of the class considered. Hence $F(\eta) = \xi$. The equation $F(y) = \xi$ has therefore a unique solution $y = \eta = \phi(\xi)$, say; and plainly η increases steadily and continuously with ξ, which proves the theorem.

MISCELLANEOUS EXAMPLES ON CHAPTER V.

1. Show that, if neither a nor b is zero, then

$$ax^n + bx^{n-1} + \dots + k = ax^n(1 + \epsilon_x),$$

where ϵ_x is of the first order of smallness when x is large.

2. If $P(x) = ax^n + bx^{n-1} + \dots + k$, and a is not zero, then as x increases $P(x)$ has ultimately the sign of a; and so has $P(x+\lambda) - P(x)$, where λ is any constant.

3. Show that in general

$$(ax^n + bx^{n-1} + \dots + k)/(Ax^n + Bx^{n-1} + \dots + K) = \alpha + (\beta/x)(1 + \epsilon_x),$$

where $\alpha = a/A$, $\beta = (bA - aB)/A^2$, and ϵ_x is of the first order of smallness when x is large. Indicate any exceptional cases.

4. Express $(ax^2 + bx + c)/(Ax^2 + Bx + C)$

in the form $\alpha + (\beta/x) + (\gamma/x^2)(1 + \epsilon_x)$,

where ϵ_x is of the first order of smallness when x is large.

5. Show that $\lim_{x\to\infty} \sqrt{x}\{\sqrt{(x+a)} - \sqrt{x}\} = \frac{1}{2}a$.

[Use the formula $\sqrt{(x+a)} - \sqrt{x} = a/\{\sqrt{(x+a)} + \sqrt{x}\}$.]

6. Show that $\sqrt{(x+a)} = \sqrt{x} + \frac{1}{2}(a/\sqrt{x})(1 + \epsilon_x)$, where ϵ_x is of the first order of smallness when x is large.

7. Find values of α and β such that $\sqrt{(ax^2 + 2bx + c)} - \alpha x - \beta$ has the limit zero as $x \to \infty$; and prove that $\lim x\{\sqrt{(ax^2 + 2bx + c)} - \alpha x - \beta\} = (ac - b^2)/2a$.

8. Evaluate $\lim_{x\to\infty} x\{\sqrt{[x^2 + \sqrt{(x^4+1)}]} - x\sqrt{2}\}$.

9. Prove that $(\sec x - \tan x) \to 0$ as $x \to \frac{1}{2}\pi$.

10. Prove that $\phi(x) = 1 - \cos(1 - \cos x)$ is of the fourth order of smallness when x is small; and find the limit of $\phi(x)/x^4$ as $x \to 0$.

11. Prove that $\phi(x) = x\sin(\sin x) - \sin^2 x$ is of the sixth order of smallness when x is small; and find the limit of $\phi(x)/x^6$ as $x \to 0$.

12. From a point P on a radius OA of a circle, produced beyond the circle, a tangent PT is drawn to the circle, touching it in T, and TN is drawn perpendicular to OA. Show that $NA/AP \to 1$ as P moves up to A.

13. Tangents are drawn to a circular arc at its middle point and its extremities; Δ is the area of the triangle formed by the chord of the arc and the two tangents at the extremities, and Δ' the area of that formed by the three tangents. Show that $\Delta/\Delta' \to 4$ as the length of the arc tends to zero.

14. For what values of a does $\{a + \sin(1/x)\}/x$ tend to (1) ∞, (2) $-\infty$, as $x \to 0$? [To ∞ if $a > 1$, to $-\infty$ if $a < -1$: the function oscillates if $-1 \leqq a \leqq 1$.]

15. If $\phi(x) = 1/q$ when $x = p/q$, and $\phi(x) = 0$ when x is irrational, then $\phi(x)$ is continuous for all irrational and discontinuous for all rational values of x.

16. Show that the function whose graph is drawn in Fig. 32 may be represented by either of the formulae

$$1 - x + [x] - [1 - x], \quad 1 - x - \lim_{n\to\infty}(\cos^{2n+1}\pi x).$$

17. Show that the function $\phi(x)$ which is equal to 0 when $x = 0$, to $\frac{1}{2} - x$ when $0 < x < \frac{1}{2}$, to $\frac{1}{2}$ when $x = \frac{1}{2}$, to $\frac{3}{2} - x$ when $\frac{1}{2} < x < 1$, and to 1 when $x = 1$, assumes every value between 0 and 1 once and once only as x increases from 0 to 1, but is discontinuous for $x = 0$, $x = \frac{1}{2}$, and $x = 1$. Show also that the function may be represented by the formula

$$\tfrac{1}{2} - x + \tfrac{1}{2}[2x] - \tfrac{1}{2}[1 - 2x].$$

18. Let $\phi(x)=x$ when x is rational and $\phi(x)=1-x$ when x is irrational. Show that $\phi(x)$ assumes every value between 0 and 1 once and once only as x increases from 0 to 1, but is discontinuous for every value of x except $x=\frac{1}{2}$.

19. As x increases from $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$, $y=\sin x$ is continuous and steadily increases, in the stricter sense, from -1 to 1. Deduce the existence of a function $x=\arcsin y$ which is a continuous and steadily increasing function of y from $y=-1$ to $y=1$.

20. Show that the numerically least value of $\arctan y$ is continuous for all values of y and increases steadily from $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$ as y varies through all real values.

21. Discuss, on the lines of §§ 108—109, the solution of the equations

$$y^2-y-x=0,\quad y^4-y^2-x^2=0,\quad y^4-y^2+x^2=0$$

in the neighbourhood of $x=0$, $y=0$.

22. If $ax^2+2bxy+cy^2+2dx+2ey=0$ and $\Delta=2bde-ae^2-cd^2$, then one value of y is given by $y=\alpha x+\beta x^2+(\gamma+\epsilon_x)x^3$, where

$$\alpha=-d/e,\quad \beta=\Delta/2e^3,\quad \gamma=(cd-be)\Delta/2e^5,$$

and ϵ_x is of the first order of smallness when x is small.

[If $y-\alpha x=\eta$ then

$$-2e\eta=ax^2+2bx(\eta+\alpha x)+c(\eta+\alpha x)^2=Ax^2+2Bx\eta+C\eta^2,$$

say. It is evident that η is of the second order of smallness, $x\eta$ of the third, and η^2 of the fourth; and $-2e\eta=Ax^2-(AB/e)x^3$, the error being of the fourth order.]

23. If $x=ay+by^2+cy^3$ then one value of y is given by

$$y=\alpha x+\beta x^2+(\gamma+\epsilon_x)x^3,$$

where $\alpha=1/a$, $\beta=-b/a^3$, $\gamma=(2b^2-ac)/a^5$, and ϵ_x is of the first order of smallness when x is small.

24. If $x=ay+by^n$, where n is an integer greater than unity, then one value of y is given by $y=\alpha x+\beta x^n+(\gamma+\epsilon_x)x^{2n-1}$, where $\alpha=1/a$, $\beta=-b/a^{n+1}$, $\gamma=nb^2/a^{2n+1}$, and ϵ_x is of the $(n-1)$th order of smallness when x is small.

25. Show that the least positive root of the equation $xy=\sin x$ is a continuous function of y throughout the interval $(0, 1)$, and decreases steadily from π to 0 as y increases from 0 to 1. [The function is the inverse of $(\sin x)/x$: apply § 109.]

26. The least positive root of $xy=\tan x$ is a continuous function of y throughout the interval $(1, \infty)$, and increases steadily from 0 to $\frac{1}{2}\pi$ as y increases from 1 towards ∞.

CHAPTER VI

DERIVATIVES AND INTEGRALS

110. Derivatives or Differential Coefficients. Let us return to the consideration of the properties which we naturally associate with the notion of a curve. The first and most obvious property is, as we saw in the last chapter, that which gives a curve its appearance of connectedness, and which we embodied in our definition of a continuous function.

The ordinary curves which occur in elementary geometry, such as straight lines, circles and conic sections, have of course many other properties of a general character. The simplest and most noteworthy of these is perhaps that they have a definite *direction* at every point, or what is the same thing, that at every point of the curve we can draw a *tangent* to it. The reader will probably remember that in elementary geometry the tangent to a curve at P is defined to be 'the limiting position of the chord PQ, when Q moves up towards coincidence with P'. Let us consider what is implied in the assumption of the existence of such a limiting position.

In the figure (Fig. 36) P is a fixed point on the curve, and Q a variable point; PM, QN are parallel to OY, and PR to OX. We denote the coordinates of P by x, y and those of Q by $x+h$, $y+k$: h will be positive or negative according as N lies to the right or left of M.

We have assumed that there is a tangent to the curve at P, or that there is a definite 'limiting position' of the chord PQ. Suppose that PT, the tangent at P, makes an angle ψ with OX. Then to say that PT is the limiting position of PQ is equivalent to saying that the limit of the angle QPR is ψ, when Q approaches