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VOLUME I

INTRODUCTION TO MATHEMATICAL LOGIC

BY ALONZO CHURCH

PRINCETON, NEW JERSEY

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Preface

This is a revised and much enlarged edition of *Introduction to Mathematical Logic, Part I,* which was published in 1944 as one of the Annals of Mathematics Studies. In spite of extensive additions, it remains an introduction rather than a comprehensive treatise. It is intended to be used as a textbook by students of mathematics, and also within limitations as a reference work. As a textbook it offers a beginning course in mathematical logic, but presupposes some substantial mathematical background.

An added feature in the new edition is the inclusion of many exercises for the student. Some of these are of elementary character, straightforward illustrations serving the purpose of practice; others are in effect brief sketches of difficult developments to which whole sections of the main text might have been devoted; and still others occupy various intermediate positions between these extremes. No attempt has been made to classify exercises systematically according to difficulty. But for routine use by beginning students the following list is tentatively suggested as a basis for selection: 12.3–12.9, 14.0–14.8, 15.0–15.3, 15.9, 15.10, 18.0–18.3, 19.0–19.7, 19.9, 19.10, 23.1–23.6, 24.0–24.5, 30.0–30.4 (with assistance if necessary), 34.0, 34.3–34.6, 35.1, 35.2, 38.0–38.5, 39.0, 41.0, 43.0, 43.1, 43.4, 45.0, 45.1, 48.0–48.11, 52.0, 52.1, 54.2–54.6, 55.1, 55.2, 55.22, 56.0–56.2, 57.0–57.2.

The book has been cut off rather abruptly in the middle, in order that Volume I may be published, and at many places there are references forward to passages in the still unwritten Volume II. In order to make clear at least the general intent of such references, a tentative table of contents of Volume II has been added at the end of the table of contents of the present volume, and references to Volume II should be understood in the light of this.

Volume I has been written over a period of years, beginning in 1947, and as portions of the work were completed they were made available in manuscript form in the Fine Hall Library of Princeton University. The work was carried on during regular leave of absence from Princeton University from September, 1947, to February 1, 1948, and then under a contract of Princeton University with the United States Office of Naval Research from February 1 to June 30, 1948. To this period should be credited the Introduction and Chapters I and II — although some minor changes have been made n this material since then, including the addition of exercises 15.4, 18.3, 9.12, 24.10, 26.3(2), 26.3(3), 26.8, 29.2, 29.3, 29.4, 29.5, as well as changes esigned to correct errors or to take into account newly published papers. The remainder of the work was done during 1948–1951 with the aid of rants from the Scientific Research Fund of Princeton University and the Eugene Higgins Trust Fund, and credit is due to these Funds for making possible the writing of the latter half of the volume.

For individual assistance, I am indebted still to the persons named in the Preface of the edition of 1944, especially to C. A. Truesdell — whose notes on the lectures of 1943 have continued to be of great value, both in the writing of Volume I and in the preliminary work which has been done towards the writing of Volume II, and notwithstanding the extensive changes which have been made from the content and plan of the original lectures. I am also indebted to many who have read the new manuscript or parts of it and have supplied valuable suggestions and corrections, including especially E. Adler, A. F. Bausch, W. W. Boone, Leon Henkin, J. G. Kemeny, Maurice L'Abbé, E. A. Maier, Paul Meier, I. L. Novak, and Rulon Wells.

ALONZO CHURCH

Princeton,, New Jersey
August 31, 1951

(*Added November 28, 1955.*) For suggestions which could be taken into account only in the proof I am indebted further to A. N. Prior, T. T. Robinson, Hartley Rogers, Jr., J. C. Shepherdson, F. O. Wyse, and G. Zubieta Russi; for assistance in the reading of the proof itself, to Michael Rabin and to Zubieta; and especially for their important contribution in preparing the indexes, to Robinson and Zubieta.

(*Added January 17, 1958.*) In the second printing, additional corrections which were necessary have been made in the text as far as possible, and those which could not be fitted into the text have been included in a list of *Errata* at the end of the book. For some of these corrections I am indebted to Max Black, S. C. Kleene, E. J. Lemmon, Walter Stuermann, John van Heijenoort; for the observation that exercise 55.3(3) would be better placed as 55.2(3), to D. S. Geiger; and for important corrections to 38.8(10) and footnote 550, to E. W. Beth. For assistance in connection with Wajsberg's paper (see the correction to page 142) I am further indebted to T. T. Robinson.

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Contents

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Introduction

This introduction contains a number of preliminary explanations, which it seems are most suitably placed at the beginning, though many will become clearer in the light of the formal development which follows. The reader to whom the subject is new is advised to read the introduction through once, then return to it later after a study of the first few chapters of the book. Footnotes may in general be omitted on a first reading.

00. Logic. Our subject is *logic*—or, as we may say more fully, in order to distinguish from certain topics and doctrines which have (unfortunately) been called by the same name, it is *formal logic.*

Traditionally, (formal) logic is concerned with the analysis of sentences or of propositions[1] and of proof[2] with attention to the *form* in abstraction from the *matter*. This distinction between form and matter is not easy to make precise immediately, but it may be illustrated by examples.

To take a relatively simple argument for illustrative purposes, consider the following:

I Brothers have the same surname; Richard and Stanley are brothers; Stanley has surname Thompson; therefore Richard has surname Thompson.

Everyday statement of this argument would no doubt leave the first of the three premisses[3] tacit, at least unless the reasoning were challenged; but

[1]See §04.

[2]In the light both of recent work and of some aspects of traditional logic we must add here, besides proof, such other relationships among sentences or propositions as can be treated in the same manner, i.e., with regard to form in abstraction from the matter. These include (e.g.) disproof, compatibility; also partial confirmation, which is important in connection with inductive reasoning (cf. C. G. Hempel in *The Journal of Symbolic Logic*, vol. 8 (1943), pp. 122—143).

But no doubt these relationships both can and should be reduced to that of proof, by making suitable additions to the object language (§07) if necessary. E.g., in reference to an appropriate formalized language as object language, disproof of a proposition or sentence may be identified with proof of its negation. The corresponding reduction of the notions of compatibility and confirmation to that of proof apparently requires modal logic—a subject which, though it belongs to formal logic, is beyond the scope of this book.

[3]Following C. S. Peirce (and others) we adopt the spelling *premiss* for the logical term to distinguish it from *premise* in other senses, in particular to distinguish the plural from the legal term *premises.*

for purposes of logical analysis all premisses must be set down explicitly. The argument, it may be held, is valid from its form alone, independently of the matter, and independently in particular of the question whether the premisses and the conclusion are in themselves right or wrong. The reasoning may be right though the facts be wrong, and it is just in maintaining this distinction that we separate the form from the matter.

For comparison with the foregoing example consider also:

II Complex numbers with real positive ratio have the same amplitude; $i - \sqrt{3}/3$ and ω are complex numbers with real positive ratio; ω has amplitude $2\pi/3$; therefore $i - \sqrt{3}/3$ has amplitude $2\pi/3$.

This may be held to have the same form as I, though the matter is different, and therefore to be, like I, valid from the form alone.

Verbal similarity in the statements of I and II, arranged at some slight cost of naturalness in phraseology, serves to highlight the sameness of form. But, at least in the natural languages, such linguistic parallelism is not in general a safe guide to sameness of logical form. Indeed, the natural languages, including English, have been evolved over a long period of history to serve practical purposes of facility of communication, and these are not always compatible with soundness and precision of logical analysis.

To illustrate this last point, let us take two further examples:

III I have seen a portrait of John Wilkes Booth; John Wilkes Booth assassinated Abraham Lincoln; thus I have seen a portrait of an assassin of Abraham Lincoln.

IV I have seen a portrait of somebody; somebody invented the wheeled vehicle; thus I have seen a portrait of an inventor of the wheeled vehicle.

The argument III will be recognized as valid, and presumably from the logical form alone, but IV as invalid. The superficial linguistic analogy of the two arguments as stated is deceptive. In this case the deception is quickly dispelled upon going beyond the appearance of the language to consider the meaning, but other instances are more subtle, and more likely to generate real misunderstanding. Because of this, it is desirable or practically necessary for purposes of logic to employ a specially devised language, a *formalized language* as we shall call it, which shall reverse the tendency of the natural languages and shall follow or reproduce the logical form—at the expense,

where necessary, of brevity and facility of communication. To adopt a particular formalized language thus involves adopting a particular theory or system of logical analysis. (This must be regarded as the essential feature of a formalized language, not the more conspicuous but theoretically less important feature that it is found convenient to replace the spelled words of most (written) natural languages by single letters and various special symbols.)

01. Names. One kind of expression which is familiar in the natural languages, and which we shall carry over also to formalized languages, is the *proper name*. Under this head we include not only proper names which are arbitrarily assigned to denote in a certain way—such names, e.g., as "Rembrandt," "Caracas," "Sirius," "the Mississippi," "The Odyssey," "eight"—but also names having a structure that expresses some analysis of the way in which they denote.[4] As examples of the latter we may cite: "five hundred nine," which denotes a certain prime number, and in the way expressed by the linguistic structure, namely as being five times a hundred plus nine; "the author of *Waverley*," which denotes a certain Scottish novelist, namely Sir Walter Scott, and in the particular way expressed by the linguistic structure, namely as having written *Waverley*; "Rembrandt's birthplace"; "the capital of Venezuela"; "the cube of 2."

The distinction is not always clear in the natural languages between the two kinds of proper names, those which are arbitrarily assigned to have a certain meaning (primitive proper names, as we shall say in the case of a formalized language), and those which have a linguistic structure of meaningful parts. E.g., "The Odyssey" has in the Greek a derivation from "Odysseus," and it may be debated whether this etymology is a mere matter of past history or whether it is still to be considered in modern English that the name "The Odyssey" has a structure involving the name "Odysseus." This uncertainty is removed in the case of a formalized language by fixing and making explicit the formation rules of the language (§07).

There is not yet a theory of the meaning of proper names upon which

[4]We extend the usual meaning of *proper name* in this manner because such alternative terms as *singular name* or *singular term* have traditional associations which we wish to avoid. The single word *name* would serve the purpose except for the necessity of distinguishing from the *common names* (or *general names*) which occur in the natural languages, and hereafter we shall often say simply *name*.

We do use the word *term*, but in its everyday meaning of an item of terminology, and not with any reference to the traditional doctrine of "categorical propositions" or the like.

general agreement has been reached as the best. Full discussion of the question would take us far beyond the intended scope of this book. But it is necessary to outline briefly the theory which will be adopted here, due in its essentials to Gottlob Frege.[5]

The most conspicuous aspect of its meaning is that a proper name always is, or at least is put forward as, a *name of* something. We shall say that a proper name *denotes*[6] or *names*[7] that of which it is a name. The relation between a proper name and what it denotes will be called the *name relation*,[8]

---

[5]See his paper, "Ueber Sinn und Bedeutung," in *Zeitschrift für Philosophie und philosophische Kritik*, vol. 100 (1892), pp. 25–50. (There are an Italian translation of this by L. Geymonat in *Gottlob Frege, Aritmetica e Logica* (1948), pp. 215–252, and English translations by Max Black in *The Philosophical Review*, vol. 57 (1948), pp. 207–230, and by Herbert Feigl in *Readings in Philosophical Analysis* (1949), pp. 85–102. See reviews of these in *The Journal of Symbolic Logic*, vol. 13 (1948), pp. 152–153, and vol. 14 (1949), pp. 184–185.)

A similar theory, but with some essential differences, is proposed by Rudolf Carnap in his recent book *Meaning and Necessity* (1947).

A radically different theory is that of Bertrand Russell, developed in a paper in *Mind*, vol. 14 (1905), pp. 479–493; in the Introduction to the first volume of *Principia Mathematica* (by A. N. Whitehead and Bertrand Russell, 1910); and in a number of more recent publications, among them Russell's book, *An Inquiry into Meaning & Truth* (1940). The doctrine of Russell amounts very nearly to a rejection of proper names as irregularities of the natural languages which are to be eliminated in constructing a formalized language. It falls short of this by allowing a narrow category of proper names which must be names of sense qualities that are known by acquaintance, and which, in Fregean terms, have *Bedeutung* but not *Sinn*.

[6]In the usage of J. S. Mill, and of others following him, not only a singular name (proper name in our terminology) but also a common or general name is said to denote, with the difference that the former denotes only one thing, the latter, many things. E.g., the common name "man" is said to denote Rembrandt; also to denote Scott; also to denote Frege; etc.

In the formalized languages which we shall study, the nearest analogues of the common name will be the *variable* and the *form* (see §02). And we prefer to use a different terminology for variables and forms than that of denoting—in particular because we wish to preserve the distinction of a proper name, or constant, from a form which is concurrent to a constant (in the sense of § 02), and from a variable which has one thing only in its range. In what follows, therefore, we shall speak of *proper names only* as denoting.

From another point of view common names may be thought of as represented in the formalized languages, not by variables or forms, but by proper names of classes (class constants). Hence the usage has also arisen according to which a proper name of a class is said to denote the various members of the class. We shall not follow this, but shall speak of a proper name of a class as denoting the class itself. (Here we agree with Mill, who distinguishes a singular collective name, or proper name of a class, from a common or general name, calling the latter a "name of a class" only in the distributive sense of being a name of each individual.)

[7]We thus translate Frege's *bedeuten* by *denote* or *name*. The verb to *mean* we reserve for general use, in reference to possible different kinds of meaning.

[8]The name relation is properly a ternary relation, among a language, a word or phrase of the language, and a denotation. But it may be treated as binary by fixing the language in a particular context. Similarly one should speak of the denotation of a name *with respect to a language*, omitting the latter qualification only when the language has been fixed or when otherwise no misunderstanding can result.

and the thing[9] denoted will be called the *denotation*. For instance, the proper name "Rembrandt" will thus be said to denote or name the Dutch artist Rembrandt, and he will be said to be the denotation of the name "Rembrandt." Similarly, "the author of *Waverley*" denotes or names the Scottish author, and he is the denotation both of this name and of the name "Sir Walter Scott."

That the meaning of a proper name does not consist solely in its denotation may be seen by examples of names which have the same denotation though their meanings are in some sense different. Thus "Sir Walter Scott" and "the author of *Waverley*" have the same denotation; it is contained in the meaning of the first name, but not of the second, that the person named is a knight or baronet and has the given name "Walter" and surname "Scott";[10] and it is contained in the meaning of the second name, but not of the first, that the person named wrote *Waverley* (and indeed as sole author, in view of the definite article and of the fact that the phrase is put forward as a proper name). To bring out more sharply the difference in meaning of the two names let us notice that, if two names are *synonymous* (have the same meaning in all respects), then one may always be substituted for the other without change of meaning. The sentence, "Sir Walter Scott is the author of *Waverley*," has, however, a very different meaning from the sentence, "Sir Walter Scott is Sir Walter Scott": for the former sentence conveys an important fact of literary history of which the latter gives no hint. This difference in meaning may lead to a difference in truth when the substitution of one name for the other occurs within certain contexts.[11] E.g., it is true that "George IV once demanded to know whether Scott was the author of *Waverley*"; but false that "George IV once demanded to know whether Scott was Scott."[12]

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[9]The word *thing* is here used in its widest sense, in short for anything namable.

[10]The term *proper name* is often restricted to names of this kind, i.e., which have as part of their meaning that the denotation is so called or is or was entitled to be so called. As already explained, we are not making such a restriction.

Though it is, properly speaking, irrelevant to the discussion here, it is of interest to recall that Scott did make use of "the author of *Waverley*" as a pseudonym during the time that his authorship of the Waverley Novels was kept secret.

[11]Contexts, namely, which render the occurrences of the names *oblique* in the sense explained below.

[12]The particular example is due to Bertrand Russell; the point which it illustrates, to Frege.

This now famous question, put to Scott himself in the indirect form of a toast "to the author of *Waverley*" at a dinner at which Scott was present, was met by him with a flat denial, "Sire, I am not the author of *Waverley*." We may therefore enlarge on the example by remarking that Scott, despite a pardonable departure from the truth, did not mean to go so far as to deny his self-identity (as if he had said "I am not I"). And his hearers surely did not so understand him, though some must have shrewdly guessed the deception as to his authorship of *Waverley*.

Therefore, besides the denotation, we ascribe to every proper name another kind of meaning, the *sense*,[13] saying, e.g., that "Sir Walter Scott" and "the author of *Waverley*" have the same denotation but different senses.[14] Roughly, the sense is what is grasped when one understands a name,[15] and it may be possible thus to grasp the sense of a name without having knowledge of its denotation except as being determined by this sense. If, in particular, the question "Is Sir Walter Scott the author of *Waverley*?" is used in an intelligent demand for new information, it must be that the questioner knows the senses of the names "Sir Walter Scott" and "the author of *Waverley*" without knowing of their denotations enough to identify them certainly with each other.

We shall say that a name *denotes* or *names* its denotation and *expresses*[16] its sense. Or less explicitly we may speak of a name just as *having* a certain denotation and *having* a certain sense. Of the sense we say that it *determines* the denotation, or *is a concept*[17] of the denotation.

Concepts[17] we think of as non-linguistic in character—since synonymous names, in the same or different languages, express the same sense or concept—and since the same name may also express different senses, either in different languages or, by equivocation, in the same language. We are even

[13]We adopt this as the most appropriate translation of Frege's *Sinn*, especially since the technical meaning given to the word *sense* thus comes to be very close indeed to the ordinary acceptation of the sense of an expression. (Russell and some others following him have used "meaning" as a translation of Frege's *Sinn*.)

[14]A similar distinction is made by J. S. Mill between the denotation and the connotation of a name. And in fact we are prepared to accept *connotation* as an alternative translation of *Sinn*, although it seems probable that Frege did not have Mill's distinction in mind in making his own. We do not follow Mill in admitting names which have denotation without connotation, but rather hold that a name must always point to its denotation *in some way*, i.e., through some sense or connotation, though the sense may reduce in special cases just to the denotation's being called so and so (e.g., in the case of personal names), or to its being what appears here and now (as sometimes in the case of the demonstrative "this"). Because of this and other differences, and because of the more substantial content of Frege's treatment, we attribute the distinction between sense and denotation to Frege rather than to Mill. Nevertheless the discussion of names in Mill's *A System of Logic* (1843) may profitably be read in this connection.

[15]It is not meant by this to imply any psychological element in the notion of sense. Rather, a sense (or a concept) is a postulated abstract object, with certain postulated properties. These latter are only briefly indicated in the present informal discussion; and in particular we do not discuss the assumptions to be made about equality of senses, since this is unnecessary for our immediate purpose.

[16]This is our translation of Frege's *drückt aus*. Mill's term *connotes* is also acceptable here, provided that care is taken not to confuse Mill's meaning of this term with other meanings which it has since acquired in common English usage.

[17]This use of *concept* is a departure from Frege's terminology. Though not identical with Carnap's use of *concept* in recent publications, it is closely related to it, and was suggested to the writer by correspondence with Carnap in 1943. It also agrees well with Russell's use of *class-concept* in *The Principles of Mathematics* (1903)—cf. §69 thereof.

prepared to suppose the existence of concepts of things which have no name in any language in actual use. But every concept of a thing is a sense of some name of it in some (conceivable) language.

The possibility must be allowed of concepts which are not concepts of any actual thing, and of names which express a sense but have no denotation. Indeed such names, at least on one very plausible interpretation, do occur in the natural languages such as English: e.g., "Pegasus,"[18] "the king of France in A.D. 1905." But, as Frege has observed, it is possible to avoid such names in the construction of formalized languages.[19] And it is in fact often convenient to do this.

To understand a language fully, we shall hold, requires knowing the senses of all names in the language, but not necessarily knowing which senses determine the same denotation, or even which senses determine denotations at all.

In a well constructed language of course every name should have just one sense, and it is intended in the formalized languages to secure such univ-

[18]While the exact sense of the name "Pegasus" is variable or uncertain, it is, we take it, roughly that of the winged horse who took such and such a part in such and such supposed events—where only such minimum essentials of the story are to be included as it would be necessary to verify in order to justify saying, despite the common opinion, that "Pegasus did after all exist."

We are thus maintaining that, in the present actual state of the English language, "Pegasus" is not just a personal name, having the sense of who or what was called so and so, but has the more complex sense described. However, such questions regarding the natural languages must not be supposed always to have one final answer. On the contrary, the present actual state (at any time) tends to be indeterminate in a way to leave much debatable.

[19]For example, in the case of a formalized language obtained from one of the logistic systems of Chapter X (or of a paper by the writer in *The Journal of Symbolic Logic*, vol. 5 (1940), pp. 56–68) by an interpretation retaining the principal interpretation of the variables and of the notations λ (abstraction) and ( ) (application of function to argument), it is sufficient to take the following precautions in assigning senses to the primitive constants. For a primitive constant of type o or ι the sense must be such as—on the basis of accepted presuppositions—to assure the existence of a denotation in the appropriate domain, $\mathfrak{D}$ (of truth-values) or $\mathfrak{J}$ (of individuals). For a primitive constant of type $\alpha\beta$ the sense must be such as—on the same basis—to assure the existence of a denotation which is in the domain $\mathfrak{A}\mathfrak{B}$, i.e., which is a function from the (entire) domain $\mathfrak{B}$ which is taken as the range of variables of type β, to the domain $\mathfrak{A}$ which is taken as the range of variables of type α.

Then every well-formed formula without free variables will have a denotation, as indeed it must if such interpretation of the logistic system is to accord with formal properties of the system.

As in the case, e.g., of $\iota_{\alpha(o\alpha)}$, it may happen that the most immediate or naturally suggested interpretation of a primitive constant of type $\alpha\beta$ makes it denote a function from a proper part of the domain $\mathfrak{B}$ to the domain $\mathfrak{A}$. In such a case the definition of the function must be extended, by artificial means if necessary, over the remainder of the domain $\mathfrak{B}$, so as to obtain a function having the entire domain $\mathfrak{B}$ as its range. The sense assigned to the primitive constant must then be such as to determine this latter function as denotation, rather than the function which had only a proper part of $\mathfrak{B}$ as its range.

ocacy. But this is far from being the case in the natural languages. In particular, as Frege has pointed out, the natural languages customarily allow, besides the *ordinary* (*gewöhnlich*) use of a name, also an *oblique* (*ungerade*) use of the name, the sense which the name would express in its ordinary use becoming the denotation when the name is used obliquely.[20]

Supposing univocacy in the use of names to have been attained (this ultimately requires eliminating the oblique use of names by introducing special names to denote the senses which other names express[21]), we make, with Frege, the following assumptions, about names which have a linguistic structure and contain other names as constituent parts: (1) when a con-

[20]For example, in "Scott is the author of *Waverley*" the names "Scott," "*Waverley*," "the author of *Waverley*" have ordinary occurrences. But in "George IV wished to know whether Scott was the author of *Waverley*" the same three names have oblique occurrences (while "George IV" has an ordinary occurrence). Again, in "Schliemann sought the site of Troy" the names "Troy" and "the site of Troy" occur obliquely. For to seek the site of some other city, determined by a different concept, is not the same as to seek the site of Troy, not even if the two cities should happen as a matter of fact (perhaps unknown to the seeker) to have had the same site.

According to the Fregean theory of meaning which we are advocating, "Schliemann sought the site of Troy" asserts a certain relation as holding, not between Schliemann and the site of Troy (for Schliemann might have sought the site of Troy though Troy had been a purely fabulous city and its site had not existed), but between Schliemann and a certain concept, namely that of the site of Troy. This is, however, not to say that "Schliemann sought the site of Troy" means the same as "Schliemann sought the concept of the site of Troy." On the contrary, the first sentence asserts the holding of a certain relation between Schliemann and the concept of the site of Troy, and is true; but the second sentence asserts the holding of a like relation between Schliemann and the concept of the concept of the site of Troy, and is very likely false. The relation holding between Schliemann and the concept of the site of Troy is not quite that of having sought, or at least it is misleading to call it that—in view of the way in which the verb *to seek* is commonly used in English.

(W. V. Quine—in *The Journal of Philosophy*, vol. 40 (1943), pp. 113–127, and elsewhere—introduces a distinction between the "meaning" of a name and what the name "designates" which parallels Frege's distinction between sense and denotation, also a distinction between "purely designative" occurrences of names and other occurrences which coincides in many cases with Frege's distinction between ordinary and oblique occurrences. For a discussion of Quine's theory and its differences from Frege's see a review by the present writer, in *The Journal of Symbolic Logic*, vol. 8 (1943), pp. 45–47; also a note by Morton G. White in *Philosophy and Phenomenological Research*, vol. 9, no. 2 (1948), pp. 305–308.)

[21]As an indication of the distinction in question we shall sometimes (as we did in the second paragraph of footnote 20) use such phrases as "the concept of Sir Walter Scott," "the concept of the author of *Waverley*," "the concept of the site of Troy" to *denote* the same concepts which are *expressed* by the respective names "Sir Walter Scott," "the author of *Waverley*," "the site of Troy." The definite article "the" sufficiently distinguishes the phrase (e.g.) "the concept of the site of Troy" from the similar phrase "a concept of the site of Troy," the latter phrase being used as a common name to refer to any one of the many different concepts of this same spot.

This device is only a rough expedient to serve the purpose of informal discussion. It does not do away with the oblique use of names because, when the phrase "the concept of the site of Troy" is used in the way described, it contains an oblique occurrence of "the site of Troy."

stituent name is replaced by another having the same sense, the sense of the entire name is not changed; (2) when a constituent name is replaced by another having the same denotation, the denotation of the entire name is not changed (though the sense may be).[22]

We make explicit also the following assumption (of Frege), which, like (1) and (2), has been implicit in the foregoing discussion: (3) The denotation of a name (if there is one) *is a function of* the sense of the name, in the sense of §03 below; i.e., given the sense, the existence and identity of the denotation are thereby fixed, though they may not necessarily therefore be known to every one who knows the sense.

02. Constants and variables. We adopt the mathematical usage according to which a proper name of a number is called a *constant*, and in connection with formalized languages we extend this usage by removing the restriction to numbers, so that the term *constant* becomes synonymous with *proper name having a denotation.*

However, the term *constant* will often be applied also in the construction of uninterpreted calculi—logistic systems in the sense of §07—some of the symbols or expressions being distinguished as constants just in order to treat them differently from others in giving the rules of the calculus. Ordinarily the symbols or expressions thus distinguished as constants will in fact become proper names (with denotation) in at least one of the possible interpretations of the calculus.

As already familiar from ordinary mathematical usage, a *variable* is a symbol whose meaning is like that of a proper name or constant except that the single denotation of the constant is replaced by the possibility of various *values* of the variable.

Because it is commonly necessary to restrict the values which a variable may take, we think of a variable as having associated with it a certain non-empty range of possible values, the *range of* the variable as we shall call it. Involved in the meaning of a variable, therefore, are the kinds of meaning which belong to a proper name of the range.[23] But a variable must not be

[22]To avoid serious difficulties, we must also assume when a constituent name has no denotation that the entire name is then likewise without denotation. In the natural languages such apparent examples to the contrary as "the myth of *Pegasus*," "the search by Ponce de Leon for *the fountain of youth*" are to be explained as exhibiting oblique occurrences of the italicized constituent name.

[23]Thus the distinction of sense and denotation comes to have an analogue for variables Two variables with ranges determined by different concepts have to be considered as variables of different kinds, even if the ranges themselves should be identical. However, because of the restricted variety of ranges of variables admitted, this question does not arise in connection with any of the formalized languages which are actually considered below.

identified with a proper name of its range, since there are also differences of meaning between the two.[24]

The meaning which a variable does possess is best explained by returning to the consideration of complex names, containing other names as constituent parts. In such a complex name, having a denotation, let one of the constituent names be replaced at one or more (not necessarily all) of its occurrences by a variable, say x. To avoid complications, we suppose that x is a variable which does not otherwise occur,[25] and that the denotation of the constituent name which x replaces is in the range of x. The resulting expression (obtained from the complex name by thus replacing one of the constituent names by a variable) we shall call a *form*.[26] Such a form, for each value of x within the range of x, or at least for certain such values of x, has a *value*. Namely, the value of the form, for a given value of x, is the same as the denotation of the expression obtained from the form by substituting everywhere for x a name of the given value of x (or, if the expression so obtained

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[24]That such an identification is impossible may be quickly seen from the point of view of the ordinary mathematical use of variables. For two proper names of the range are fully interchangeable if only they have the same sense; but two distinct variables must be kept distinct even if they have the same range determined by the same concept. E.g., if each of the letters x and y is a variable whose range is the real numbers, we are obliged to distinguish the two inequalities $x(x + y) \geqq 0$ and $x(x + x) \geqq 0$ as different —indeed the second inequality is universally true, the first one is not.

[25]This is for momentary convenience of explanation. We shall apply the name *form* also to expressions which are similarly obtained but in which the variable x may otherwise occur, provided the expression has at least one occurrence of x as a free variable (see footnote 28 and the explanation in §06 which is there referred to).

[26]This is a different use of the word *form* from that which appeared in §00 in the discussion of form and matter. We shall distinguish the latter use, when necessary, by speaking more explicitly of *logical form*.

Our present use of the word *form* is similar to that which is familiar in algebra, and in fact may be thought of as obtained from it by removing the restriction to a special kind of expressions (polynomials, or homogeneous polynomials). For the special case of propositional forms (see §04), the word is already usual in logic in this sense, independently of its use by algebraists—see, e.g., J. N. Keynes, *Formal Logic*, 4th edn., 1906, p. 53; Hugh MacColl in *Mind*, vol. 19 (1910), p. 193; Susanne K. Langer, *Introduction to Symbolic Logic*, 1937, p. 91; also Heinrich Scholz, *Vorlesungen über Grundzüge der Mathematischen Logik*, 1949 (for the use of *Aussageform* in German).

Instead of the word *form*, we might plausibly have used the word *variable* here, by analogy with the way in which we use *constant*. I.e., just as we apply the term *constant* to a complex name containing other names (constants) as constituent parts, so we might apply the term *variable* to an appropriate complex expression containing variables as constituent parts. This usage may indeed be defended as having some sanction in mathematical writing. But we prefer to preserve the better established usage according to which a variable is always a single symbol (usually a letter or letter with subscripts).

The use, by some recent authors, of the word *function* (with or without a qualifying adjective) for what we here call a form is, in our opinion, unfortunate, because it tends to conflict with and obscure the abstract notion of a function which will be explained in §03.

has no denotation, then the form has no value for that value of x).[27]

A variable such as x, occurring in the manner just described, is called a *free variable*[28] of the expression (form) in which it occurs.

Likewise suppose a complex name, having a denotation, to contain two constituent names neither of which is a part of the other, and let these two constituent names be replaced by two variables, say x and y respectively, each at one or more (not necessarily all) of its occurrences. For simplicity suppose that x and y are variables which do not occur in the original complex name, and that the denotations of the constituent names which x and y replace are in the ranges of x and y respectively. The resulting expression (obtained by the substitution described) is a *form*, with two *free variables* x and y. For certain pairs of values of x and y, within the ranges of x and y respectively, the form has a *value*. Namely, the value of the form, for given values of x and y, is the same as the denotation of the expression obtained from the form by substituting everywhere for x and y names of their re-

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[27]It follows from assumption (2), at the end of §01, that the value thus obtained for the form is independent of the choice of a particular name of the given value of x.

The distinction of sense and denotation is, however, relevant here. For in addition to a *value* of the form in the sense explained in the text (we may call it more explicitly a *denotation value*), a complete account must mention also what we may call a *sense value* of the form. Namely, a sense value of the form is determined by a concept of some value of x, and is the same as the sense of the expression obtained from the form by substituting everywhere for x a name having this concept as its sense.

It should also be noted that a form, in a particular language, may have a value even for a value of x which is without a name in that language: it is sufficient that the given value of x shall have a name in some suitable extension of the language—say, that obtained by adding to the vocabulary of the language a name of the given value of x, and allowing it to be substitutable for x wherever x occurs as a free variable. Likewise a form may have a sense value for a given concept of a value of x if some suitable extension of the language contains a name having that concept as its sense.

It is indeed possible, as we shall see later by particular examples, to construct languages of so restricted a vocabulary as to contain no constants, but only variables and forms. But it would seem that the most natural way to arrive at the meaning of forms which occur in these languages is by contemplating languages which are extensions of them and which do contain constants—or else, what is nearly the same thing, by allowing a temporary change in the meaning of the variables ("fixing the values of the variables") so that they become constants.

[28]We adopt this term from Hilbert (1922), Wilhelm Ackermann (1924), J. v. Neumann (1927), Hilbert and Ackermann (1928), Hilbert and Bernays (1934). For what we here call a free variable the term *real variable* is also familiar, having been introduced by Giuseppe Peano in 1897 and afterward adopted by Russell (1908), but is less satisfactory because it conflicts with the common use of "real variable" to mean a variable whose range is the real numbers.

As we shall see later (§06), a free variable must be distinguished from a *bound variable* (in the terminology of the Hilbert school) or *apparent variable* (Peano's terminology). The difference is that an expression containing x as a free variable has values for various values of x, but an expression, containing x as a bound or apparent variable only, has a meaning which is independent of x—not in the sense of having the same value for every value of x, but in the sense that the assignment of particular values to x is not a relevant procedure.

spective values (or, if the expression so obtained has no denotation, then the form has no value for these particular values of x and y).

In the same way forms with three, four, and more free variables may be obtained. If a form contains a single free variable, we shall call it a *singulary*[29] form, if just two free variables, *binary*, if three, *ternary*, and so on. A form with exactly n different free variables is an *n-ary* form.

Two forms will be called *concurrent* if they agree in value—i.e., either have the same value or both have no value—for each assignment of values to their free variables. (Since the two forms may or may not have the same free variables, all the variables are to be considered together which have free occurrences in either form, and the forms are concurrent if they agree in value for every assignment of values to these variables.) A form will be called *concurrent* to a constant if, for every assignment of values to its free variables, its value is the same as the denotation of the constant. And two constants will be called *concurrent* if they have the same denotation.

Using the notion of concurrence, we may now add a fourth assumption, or principle of meaning, to the assumptions (1)–(3) of the last two paragraphs of §01. This is an extension of (2) to the case of forms, as follows: (4) In any constant or form, when a constituent constant or form is replaced by another concurrent to it, the entire resulting constant or form is concurrent to the original one.[30] The significance of this principle will become clearer in connection with the use of operators and bound variables, explained in §06 below. It is to be taken, like (2), as a part of our explanation of the name relation, and thus a part of our theory of meaning.

As in the case of *constant*, we shall apply the terms *variable* and *form* also in the construction of uninterpreted calculi, introducing them by special definition for each such calculus in connection with which they are to be used. Ordinarily the symbols and expressions so designated will be ones which become variables and forms in our foregoing sense under one of the principal interpretations of the calculus as a language (see §07).

It should be emphasized that a variable, in our usage, is a symbol of a

[29]We follow W. V. Quine in adopting this etymologically more correct term, rather than the presently commoner "unary."

[30]For completeness—using the notion of sense value explained in footnote 27 and extending it in obvious fashion to n-ary forms—we must also extend the assumption (1) to the case of forms, as follows. Let two forms be called *sense-concurrent* if they agree in sense value for each system of concepts of values of their free variables; let a form be called *sense-concurrent* to a constant if, for every system of concepts of values of its free variables, its sense value is the same as the sense of the constant; and let two constants be called *sense-concurrent* if they express the same sense. Then: (5) In any constant or form, when a constituent constant or form is replaced by another which is sense-concurrent to it, the entire resulting constant or form is sense-concurrent to the original one.

certain kind[31] rather than something (e.g., a number) which is denoted or otherwise meant by such symbol. Mathematical writers do speak of "variable real numbers," or oftener "variable quantities," but it seems best not to interpret these phrases literally. Objections to the idea that real numbers are to be divided into two sorts or classes, "constant real numbers" and "variable real numbers," have been clearly stated by Frege[32] and need not be repeated here at length.[33] The fact is that a satisfactory theory has never been developed on this basis, and it is not easy to see how it might be done.

The mathematical theory of real numbers provides a convenient source of examples in a system of notation[34] whose general features are well established. Turning to this theory to illustrate the foregoing discussion, we cite as particular examples of constants the ten expressions:

$$0, \quad -\frac{1}{2}, \quad e, \quad -\frac{1}{2\pi}, \quad \frac{1-4+1}{4\pi}, \quad 4e^4, \quad e^e, \quad e-e, \quad -\frac{\pi}{2\pi}, \quad \frac{\sin \pi/7}{\pi/7}.$$

Let us say that x and y are variables whose range is the real numbers, and m, n, r are variables whose range is the positive integers.[35] The following are examples of forms:

[31]Therefore, a variable (or more precisely, particular instances or occurrences of a variable) can be written on paper—just as the figure 7 can be written on paper, though the number 7 cannot be so written except in the indirect sense of writing something which denotes it.

And similarly constants and forms are symbols or expressions of certain kinds. It is indeed usual to speak also of numbers and physical quantities as "constants"—but this usage is not the same as that in which a constant can be contrasted with a variable, and we shall avoid it in this book.

[32]See his contribution to *Festschrift Ludwig Boltzmann Gewidmet*, 1904. (Frege's theory of functions as "ungesättigt," mentioned at the end of his paper, is another matter, not necessarily connected with his important point about variables. It will not be adopted in this book, but rather we shall take a function—see §03—to be more nearly what Frege would call "Werthverlauf einer Function.")

[33]However, we mention the following parallel to one of Frege's examples. Shall we say that the usual list of seventeen names is a complete list of the Saxon kings of England, or only that it is a complete list of the constant Saxon kings of England, and that account must be taken in addition of an indefinite number of variable Saxon kings? One of these variable Saxon kings would appear to be a human being of a very striking sort, having been, say, a grown man named Alfred in A.D. 876, and a boy named Edward in A.D. 976.

According to the doctrine we would advocate (following Frege), there are just seventeen Saxon kings of England, from Egbert to Harold, and neither a variable Saxon king nor an indeterminate Saxon king is to be admitted to swell the number. And the like holds for the positive integers, for the real numbers, and for all other domains abstract and concrete. Variability or indeterminacy, where such exists, is a matter of language and attaches to symbols or expressions.

[34]We say "system of notation" rather than "language" because only the specifically numerical notations can be regarded as well established in ordinary mathematical writing. They are usually supplemented (for the statement of theorems and proofs) by one or another of the natural languages, according to the choice of the particular writer.

[35]Every positive integer is also a real number. I.e., the terms must be so understood for purposes of these illustrations.

$$y, \; -\frac{1}{y}, \; -\frac{1}{x}, \; -\frac{1}{2x}, \; \frac{1-4+1}{4x}, \; 4e^x, \; xe^x, \; x^x,$$

$$x-x, \; n-n, \; -\frac{x}{2x}, \; -\frac{r}{2r}, \; \frac{\sin x}{x}, \; \frac{\sin r}{r},$$

$$ye^x, \; -\frac{y}{xy}, \; -\frac{r}{xr}, \; \frac{x-m+1}{m\pi}.$$

The forms on the first two lines are singulary, each having one free variable, y, x, n, or r as the case may be. The forms on the third line are binary, the first two having x and y as free variables, the third one x and r, the fourth one x and m.[36]

The constants

$$-\frac{1}{2\pi} \text{ and } \frac{1-4+1}{4\pi}$$

are not identical. But they are concurrent, since each denotes the same number.[37] Similarly the constants $e - e$ and 0, though not identical, are concurrent because the numbers $e - e$ and 0 are identical. Similarly $-\pi/2\pi$ and $-1/2$.

The form xe^x, for the value 0 of x, has the value 0. (Of course it is the number 0 that is here in question, not the constant 0, so that it is equally correct to say that the form xe^x, for the value 0 of x, has the value $e - e$; or that, for the value $e - e$ of x, it has the value 0; etc.) For the value 1 of x the form xe^x has the value e. For the value 4 of x its value is $4e^4$, a real number for which (as it happens) no simpler name is in standard use.

The form ye^x, for the values 0 and 4 of x and y respectively, has the value 4.

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[36]To illustrate the remark of footnote 28, following are some examples of expressions containing bound variables:

$$\int_0^2 x^x dx, \qquad \lim_{x \to 0} \frac{\sin x}{x}, \qquad \sum_{n=1}^{\infty} \prod_{m=1}^{m=n} \frac{x-m+1}{m\pi}.$$

The first two of these are constants, containing x as a bound variable. The third is a singulary form, with x as a free variable and m and n as bound variables.

A variable may have both free and bound occurrences in the same expression. An example is $\int_0^x x^x dx$, the double use of the letter x constituting no ambiguity. Other examples are the variable Δx in $(D_x \sin x)\Delta x$ and the variable x in $xE(k)$, if the notations $D_x \sin x$ and $E(k)$ are replaced by their equivalents

$$\lim_{\Delta x \to 0} \frac{\sin(x + \Delta x) - \sin x}{\Delta x}$$

and $$\int_0^1 \frac{\sqrt{1 - k^2x^2}}{\sqrt{1 - x^2}} dx$$ respectively.

[37]Whether these two constants have the same sense (as well as the same denotation) is a question which depends for its answer on a general theory of equality of senses, such as we have not undertaken to discuss here—cf. footnote 15. It is clear that Frege, though he formulates no complete theory of equality of senses, would regard these two constants as having different senses. But a plausible case might be made out for supposing that the two constants have the same sense, on some such ground as that the equation between them expresses a necessary proposition or is true on logical grounds alone or the like. No doubt there is more than one meaning of "sense," according to the criterion adopted for equality of senses, and the decision among them is a matter of convention and expediency.

For the values 1 and 1 of x and y it has the value $1e^1$; or, what is the same thing, it has the value e.

The form $-y/xy$, for the values e and 2 of x and y respectively, has the value $-1/e$. For the values e and e of x and y, it has again the value $-1/e$. For the values e and 0 of x and y it has no value, because of the non-existence of a quotient of 0 by 0.

The form $-r/xr$, for the values e and 2 of x and r respectively, has the value $-1/e$. But there is no value for the values e and e of x and r, because e is not in the range of r (e is not one of the possible values of r).

The forms

$$-\frac{1}{2x} \text{ and } \frac{1-4+1}{4x}$$

are concurrent, since they are both without a value for the value 0 of x, and they have the same value for all other values of x. The forms $-1/x$ and $-y/xy$ fail to be concurrent, since they disagree for the value 0 of y (if the value of x is not 0). But the forms $-1/x$ and $-r/xr$ are concurrent.

The forms $-1/y$ and $-1/x$ are not concurrent, as they disagree, e.g., for the values 1 and 2 of x and y respectively.

The forms $x - x$ and $n - n$ are concurrent to the same constant, namely 0,[38] and are therefore also concurrent to each other.

The forms $-x/2x$ and $-r/2r$ are non-concurrent because of disagreement for the value 0 of x. The latter form, but not the former, is concurrent to a constant, namely to $-1/2$.

03. Functions. By a *function*—or, more explicitly, a *one-valued singulary* function—we shall understand an operation[39] which, when applied to something as *argument*, yields a certain thing as the *value* of the function *for* that argument. It is not required that the function be applicable to every possible thing as argument, but rather it lies in the nature of any given function to be applicable to certain things and, when applied to one of them as argument, to yield a certain value. The things to which the function is applicable constitute the *range of* the function (or the *range of arguments of* the function) and the values constitute the *range of values of* the function. The function itself consists in the yielding or determination[39] of a value from each argument in the range of the function.

As regards equality or identity of functions we make the decision which is

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[38]Or also to any other constant which is concurrent to 0.

[39]Of course the words "operation," "yielding," "determination" as here used are near-synonyms of "function" and therefore our statement, if taken as a definition, would be open to the suspicion of circularity. Throughout this Introduction, however, we are engaged in informal explanation rather than definition, and, for this purpose, elaboration by means of synonyms may be a useful procedure. Ultimately, it seems, we must take the notion of function as primitive or undefined, or else some related notion, such as that of a class. (We shall see later how it is possible to think of a class as a special case of a function, and also how classes may be used, in certain connections or for certain purposes, to replace and do the work of functions in general.)

usual in mathematics. Namely, functions are identical if they have the same range and have, for each argument in the range, the same value. In other words, we take the word "function" to mean what may otherwise be called a *function in extension*. If the way in which a function yields or produces its value from its argument is altered without causing any change either in the range of the function or in the value of the function for any argument, then the function remains the same; but the associated *function concept*, or concept determining the function (in the sense of §01), is thereby changed.

We shall speak of a function *from* a certain class *to* a certain class to mean a function which has the first class as its range and has all its values in the second class (though the second class may possibly be more extensive than the range of values of the function).

To denote the value of a function for a given argument, it is usual to write a name of the function, followed by a name of the argument between parentheses. And of course the same notation applies (*mutatis mutandis*) with a variable or a form in place of either one or both of the names. Thus if f is a function and x belongs to the range of f, then $f(x)$ is the value of the function f for the argument x.[40]

This is the usual notation for application of a function to an argument, and we shall often employ it. In some contexts (see Chapter X) we find it convenient to alter the notation by changing the position of the parentheses, so that we may write in the altered notation: if f is a function and x belongs to the range of f, then (fx) is the value of the function f for the argument x.

So far we have discussed only *one-valued singulary functions* (and have used the word "function" in this sense). Indeed no use will be made in this book of many-valued functions,[41] and the reader must always understand

[40]This sentence exemplifies the use of variables to make general statements, which we assume is understood from familiar mathematical usage, though it has not yet been explained in this Introduction. (See the end of §06.)

[41]It is the idea of a many-valued (singulary) function that, for a fixed argument, there may be more than one value of the function. If a name of the function is written, followed by a name of an argument between parentheses, the resulting expression is a common name (see footnote 6) denoting the values of the function for that argument.

Though many-valued functions seem to arise naturally in the mathematical theories of real and complex numbers, objections immediately suggest themselves to the idea as just explained and are not easily overcome. Therefore it is usual to replace such many-valued functions in one way or another by one-valued functions. One method is to replace a many-valued singulary function by a corresponding one-valued binary propositional function or relation (§04). Another method is to replace the many-valued function by a one-valued function whose values are classes, namely, the value of the one-valued function for a given argument is the class of the values of the many-valued function for that argument. Still another method is to change the range of the function, an argument for which the function has n values giving way to n different arguments for each of which the function has a different one of those n values (this is the standard role of the Riemann surface in the theory of complex numbers).

"function" to mean a one-valued function. But we go on to explain functions of more than one argument.

A *binary function*, or function *of two arguments*,[42] is characterized by being applicable to two arguments in a certain order and yielding, when so applied, a certain value, the *value of* the function *for* those two arguments in that order. It is not required that the function be applicable to every two things as arguments; but rather, the function is applicable in certain cases to an ordered pair of things as arguments, and all such ordered pairs constitute the *range of* the function. The values constitute the *range of values of* the function.

Binary functions are identical (i.e., are the same function) if they have the same range and have, for each ordered pair of arguments which lies in that range, the same value.

To denote the value of a binary function for given arguments, it is usual to write a name of the function and then, between parentheses and separated by a comma, names of the arguments in order. Thus if f is a binary function and the ordered pair of x and y belongs to the range of f, then $f(x, y)$ is the value of the function f for the arguments x and y in that order.

In the same way may be explained the notion of a ternary function, of a quaternary function, and so on. In general, an n-ary function is applied to n arguments in an order, and when so applied yields a value, provided the ordered system of n arguments is in the range of the function. The value of an n-ary function for given arguments is denoted by a name of the function followed, between parentheses and separated by commas, by names of the arguments in order.

Two binary functions ϕ and ψ are called *converses*, each of the other, in case the two following conditions are satisfied: (1) the ordered pair of x and y belongs to the range of ϕ if and only if the ordered pair of y and x belongs to the range of ψ; (2) for all x, y such that the ordered pair of x and y belongs to the range of ϕ,[43]

$$\phi(x, y) = \psi(y, x).$$

A binary function is called *symmetric* if it is identical with its converse. The notions of converse and of symmetry may also be extended to n-ary functions, several different converses and several different kinds of symme-

[42]Though it is in common use we shall avoid the phrase "function of two variables" (and "function of three variables" etc.) because it tends to make confusion between *arguments* to which a function is applied and *variables* taking such arguments as values.

[43]The use of the sign = to express that things are identical is assumed familiar to the reader. We do not restrict this notation to the special case of numbers, but use it for identity generally.

try appearing when the number of arguments is three or more (we need not stop over details of this).

We shall speak of a function *of* things of a certain kind to mean a function such that all the arguments to which it is applicable are of that kind. Thus a singulary function of real numbers, for instance, is a function from some class of real numbers to some (arbitrary) class. A binary function of real numbers is a binary function whose range consists of ordered pairs of real numbers (not necessarily all ordered pairs of real numbers).

We shall use the phrase "____ is a function of ____," filling the blanks with forms,[44] to mean what is more fully expressed as follows: "There exists a function f such that

$$\underline{\qquad} = f(\underline{\qquad})$$

for all ____," where the first two blanks are filled, in order, with the same forms as before, and the third blank is filled with a complete list of the free variables of those forms. Similarly we shall use "____ is a function of ____ and ____," filling the three blanks with forms, to stand for: "There exists a binary function f such that

$$\underline{\qquad} = f(\underline{\qquad}, \underline{\qquad})$$

for all ____," where the first three blanks are filled, in order, with the same forms as before, and the last blank is filled with a complete list of the free variables of those forms.[45] And similar phraseology will also be used where the reference is to a function f of more than two arguments.

The phraseology just explained will also be used with the added statement of a condition or restriction. For example, "____ is a function of ____ and ____ if ____," where the first three blanks are filled with forms, and the fourth is filled with the statement of a condition involving some or all of the free variables of those forms,[46] stands for: "There exists a binary function f such that

$$\underline{\qquad} = f(\underline{\qquad}, \underline{\qquad})$$

for all ____ for which ____," where the first three blanks are filled, in order,

[44]Our explanation assumes that neither of these forms has the particular letter f as one of its free variables. In the contrary case, the explanation is to be altered by using in place of the letter f as it appears in the text some variable (with appropriate range) which is not a free variable of either form.

[45]The theory of real numbers again serving as a source of examples, it is thus true that $x^3 + y^3$ is a function of $x + y$ and xy. But it is false that $x^3 + x^2y - xy^2 + y^3$ is a function of $x + y$ and xy (as is easily seen on the ground that the form $x^3 + x^2y - xy^2 + y^3$ is not symmetric). Again, $x^4 + y^4 + z^4 + 4x^3y + 4xy^3 + 4x^3z + 4xz^3 + 4y^3z + 4yz^3$ is a function of $x + y + z$ and $xy + xz + yz$. But $x^4 + y^4 + z^4$ is not a function of $x + y + z$ and $xy + xz + yz$.

[46]Thus with a *propositional form* in the sense of §04 below.

with the same forms as before, the fourth blank is filled with a complete list of the free variables of those forms, and the fifth blank is filled in the same way as the fourth blank was before.[47]

Also the same phraseology, explained in the two preceding paragraphs, will be used with common names[48] in place of forms. In this case the forms which the common names represent have to be supplied from the context. For example, the statement that "*The density of helium gas* is a function of *the temperature* and *the pressure*" is to be understood as meaning the same as "*The density of h* is a function of *the temperature of h* and *the pressure of h*," where the three italicized forms replace the three original italicized common names, and where h is a variable whose values are instantaneous bits of helium gas (and whose range consists of all such). Or to avoid introducing the variable h with so special a range, we may understand instead: "The density of b is a function of the temperature of b and the pressure of b if b is an instantaneous bit of helium gas." Similarly the statement at the end of §01 that the denotation of a name is a function of the sense means more explicitly (the reference being to a fixed language) that there exists a function f such that

$$\text{denotation of } N = f(\text{sense of } N)$$

for all names N for which there is a denotation.

It remains now to discuss the relationship between *functions*, in the abstract sense that we have been explaining, and *forms*, in the sense of the preceding section (§02).

If we suppose the language fixed, every singulary form has corresponding to it a function f (which we shall call the *associated function* of the form) by the rule that the value of f for an argument x is the same as the value of the form for the value x of the free variable of the form, the range of f consisting of all x's such that the form has a value for the value x of its free variable.[49]

[47]Accordingly it is true, for example, that: $x^3 + x^2y - xy^2 + y^3$ is a function of $x + y$ and xy if $x \geqq y$. For the special case that the variables have a range consisting of real or complex numbers, a geometric terminology is often used, thus: $x^3 + x^2y - xy^2 + y^3$ is a function of $x + y$ and xy in the half-plane $x \geqq y$.

[48]See footnotes 4, 6.

[49]For example, in the theory of real numbers, the form $\frac{1}{2}(e^x - e^{-x})$ determines the function sinh as its associated function, by the rule that the value of sinh for an argument x is $\frac{1}{2}(e^x - e^{-x})$. The range of sinh then consists of all x's (i.e., all real numbers x) for which $\frac{1}{2}(e^x - e^{-x})$ has a value. In other words, as it happens in this particular case, the range consists of all real numbers.

Of course the free variable of the form need not be the particular letter x, and indeed it may be clearer to take an example in which the free variable is some other letter. Thus the form $\frac{1}{2}(e^y - e^{-y})$ determines the function sinh as its associated function, by the rule that the value of sinh for an argument x is the same as the value of the form $\frac{1}{2}(e^y - e^{-y})$ for the value x of the variable y. (I.e., in particular, the value of sinh for

But, still with reference to a fixed language, not every function is necessarily the associated function of some form.[50]

It follows that two concurrent singulary forms with the same free variable have the same associated function. Also two singulary forms have the same associated function if they differ only by alphabetic change of the free variable,[51] i.e., if one is obtained from the other by substituting everywhere for its free variable some other variable with the same range—with, however, the proviso (the need of which will become clearer later) that the substituted variable must remain a free variable at every one of its occurrences resulting from the substitution.

As a notation for (i.e., to denote) the associated function of a singulary form having, say, x as its free variable, we write the form itself with the letters λx prefixed. And of course likewise with any other variable in place of x.[52] Parentheses are to be supplied as necessary.[53]

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the argument 2 is the same as the value of the form $\frac{1}{2}(e^y - e^{-y})$ for the value 2 of the variable y; and so on for each different argument x that may be assigned.)

Ordinarily, just the equation

$$\sinh(x) = \tfrac{1}{2}(e^x - e^{-x})$$

is written as sufficient indication of the foregoing. And this equation may even be called a *definition* of sinh, in the sense of footnote 168, (1) or (3).

[50]According to classical real-number theory, the singulary functions from real numbers to real numbers (or even just the analytic singulary functions) are non-enumerable. Since the forms in a particular language are always enumerable, it follows that there is no language or system of notation in which every singulary function from real numbers to real numbers is the associated function of some form.

Because of the non-enumerability of the real numbers themselves, it is even impossible in any language to provide proper names of all the real numbers. (Such a thing as, e.g., an infinite decimal expansion must not be considered a *name* of the corresponding real number, as of course an infinite expansion cannot ever be written out in full, or included as a part of any actually written or spoken sentence.)

[51]E.g., as appears in footnote 49, the forms $\frac{1}{2}(e^x - e^{-x})$ and $\frac{1}{2}(e^y - e^{-y})$ have the same associated function.

[52]Thus the expressions $\lambda x(\frac{1}{2}(e^x - e^{-x}))$, $\lambda y(\frac{1}{2}(e^y - e^{-y}))$, sinh are all three synonymous, having not only the same denotation (namely the function sinh), but also the same sense, even under the severest criterion of sameness of sense.

(In saying this we are supposing a language or system of notation in which the two different expressions sinh and $\lambda x(\frac{1}{2}(e^x - e^{-x}))$ both occur. However, the very fact of synonymy shows that the expression sinh is dispensable in principle: except for considerations of convenience, it could always be replaced by the longer expression $\lambda x(\frac{1}{2}(e^x - e^{-x}))$. In constructing a formalized language, we prefer to avoid such duplications of notation so far as readily possible. See §11.)

The expressions $\lambda x(\frac{1}{2}(e^x - e^{-x}))$ and $\lambda y(\frac{1}{2}(e^y - e^{-y}))$ contain the variables x and y respectively, as *bound* variables in the sense of footnotes 28, 36 (and of §06 below). For, according to the meaning just explained for them, these expressions are constants, not singulary forms. But of course the expression $\frac{1}{2}(e^x - e^{-x})$ is a singulary form, with x as a free variable.

The meaning of such an expression as $\lambda x(ye^x)$, formed from the binary form ye^x by prefixing λx, now follows as a consequence of the explanation about variables and forms in §02. In this expression, x is a bound variable and y is a free variable, and the

As an obvious extension of this notation, we shall also prefix the letters λx (λy, etc.) to any constant as a notation for the function whose value is the same for all arguments and is the denotation of the constant, the range of the function being the same as the range of the variable x.[54] This function will be called an *associated singulary function* of the constant, by analogy with the terminology "associated function of a form," though there is the difference that the same constant may have various associated functions with different ranges. Any function whose value is the same for all arguments will be called a *constant function* (without regard to any question whether it is an associated function of a constant, in some particular language under consideration).[55]

Analogous to the associated function of a singulary form, a binary form has two associated binary functions, one for each of the two orders in which the two free variables may be considered—or better, one for each of the two ways in which a pair of arguments of the function may be assigned as values to the two free variables of the form.

The two associated functions of a binary form are identical, and thus reduce to one function, if and only if they are symmetric. In this case the binary form itself is also called *symmetric*.[56]

Likewise an n-ary form has $n!$ associated n-ary functions, one for each of the permutations of its free variables. Some of these associated functions are identical in certain cases of symmetry.

Likewise a constant has associated m-ary functions, for $m = 1, 2, 3, \ldots$, by an obvious extension of the explanation already made for the special case $m = 1$. And by a still further extension of this we may speak of the associated m-ary functions of an n-ary form, when $m > n$. In particular a

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expression is a singulary form whose values are singulary functions. From it, by prefixing λy, we obtain a constant, denoting a singulary function, and the range of values of this singulary function consists of singulary functions.

[53]In constructing a formalized language, the manner in which parentheses are to be put in has to be specified with more care. As a matter of fact this will be done, as we shall see, not by associating parentheses with the notation λx, but by suitable provision for parentheses (or brackets) in connection with various other notations which may occur in the form to which λx is prefixed.

[54]Thus in connection with real-number theory we use $\lambda x 2$ as a notation for the function whose range consists of all real numbers and whose value is 2 for every argument.

[55]Note should also be taken of expressions in which the variable after λ is not the same as the free variable of the form which follows; thus, for example, $\lambda y(\frac{1}{2}(e^x - e^{-x}))$. As is seen from the explanation in §02, this expression is a singulary form with x as its free variable, the values of the form being constant functions. For the value 0 of x, e.g., the form $\lambda y(\frac{1}{2}(e^x - e^{-x}))$ has as its value the constant function $\lambda y 0$.

In both expressions, $\lambda y(\frac{1}{2}(e^x - e^{-x}))$ and $\lambda y 0$, y is a bound or apparent variable.

[56]We have already used this term, as applied to forms, in footnote 45, assuming the reader's understanding of it as familiar mathematical terminology.

singulary form has not only an associated singulary function but also associated binary functions, associated ternary functions, and so on. (When, however, we speak simply of *the* associated function of a singulary form, we shall mean the associated singulary function.)

The notation by means of λ for the associated functions of a form, as introduced above for singulary functions, is readily extended to the case of m-ary functions,[57] but we shall not have occasion to use such extension in this book. The passage from a form to an associated function (for which the λ-notation provides a symbolism) we shall speak of as *abstraction* or, more explicitly, *m-ary functional abstraction* (if the associated function is m-ary).

Historically the notion of a function was of gradual growth in mathematics, and its beginning is difficult to trace. The particular word "function" was first introduced by G. W. v. Leibniz and was adopted from him by Jean Bernoulli. The notation $f(x)$, or fx, with a letter such as f in the role of a function variable, was introduced by A. C. Clairaut and by Leonhard Euler. But early accounts of the notion of *function* do not sufficiently separate it from that of an expression containing free variables (or a *form*). Thus Euler explains a *function of a variable quantity* by identifying it with an analytic expression,[58] i.e., a form in some standard system of mathematical notation. The abstract notion of a function is usually attributed by historians of mathematics to G. Lejeune Dirichlet, who in 1837 was led by his study of Fourier series to a major generalization in freeing the idea of a function from its former dependence on a mathematical expression or law of circumscribed kind.[59] Dirichlet's notion of a function was adopted by Bernhard Riemann (1851),[60] by Hermann Hankel (1870),[61] and indeed by mathematicians generally. But two important steps remained to be taken by

[57]This has been done by Carnap in *Notes for Symbolic Logic* (1937) and elsewhere.

[58]"*Functio quantitatis variabilis est expressio analytica quomodocunque composita ex illa quantitate variabili et numeris seu quantitatibus constantibus.* Omnis ergo expressio analytica, in qua praeter quantitatem variabilem z omnes quantitates illam expressionem componentes sunt constantes, erit functio ipsius z . . . *Functio ergo quantitatis variabilis ipsa erit quantitas variabilis.*" *Introductio in Analysin Infinitorum* (1748), p. 4; *Opera*, ser. 1, vol. 8, p. 18. See further footnote 62.

[59]See his *Werke*, vol. 1, p. 135. It is not important that Dirichlet restricts his statement at this particular place to continuous functions, since it is clear from other passages in his writings that the same generality is allowed to discontinuous functions. On page 132 of the same volume is his well-known example of a function from real numbers to real numbers which has exactly two values, one for rational arguments and one for irrational arguments.

Dirichlet's generalization had been partially anticipated by Euler in 1749 (see an account by H. Burkhardt in *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 10 part 2 (1908), pp. 13–14) and later by J. B. J. Fourier (see his *Oeuvres*, vol. 1, pp. 207, 209, 230–232).

[60]*Werke*, pp. 3–4.

[61]In a paper reprinted in the *Mathematische Annalen*, vol. 20 (1882), pp. 63–112.

Frege (in his *Begriffsschrift* of 1879 and later publications): (i) the elimination of the dubious notion of a variable quantity in favor of the variable as a kind of symbol;[62] (ii) the admission of functions of arbitrary range by removing the restriction that the arguments and values of a function be numbers. Closely associated with (ii) is Frege's introduction of the *propositional function* (in 1879), a notion which we go on to explain in the next section.

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04. Propositions and propositional functions. According to grammarians, the unit of expression in the natural languages is the *sentence*, an aggregation of words which makes complete sense or expresses a complete thought. When the complete thought expressed is that of an assertion, the sentence is called a *declarative sentence.* In what follows we shall have occasion to refer only to declarative sentences, and the simple word "sentence" is to be understood always as meaning a declarative sentence.[63]

We shall carry over the term *sentence* from the natural languages also to the formalized languages. For logistic systems in the sense of §07—uninterpreted calculi—the term *sentence* will be introduced by special definition in each case, but always with the intention that the expressions defined to be sentences are those which will become sentences in our foregoing sense under interpretations of the calculus as a formalized language.[64]

In order to give an account of the meaning of sentences, we shall adopt a theory due to Frege according to which sentences are names of a certain kind. This seems unnatural at first sight, because the most conspicuous use of sentences (and indeed the one by which we have just identified or

[62]The passage quoted from Euler in footnote 58 reads as if his *variable quantity* were a kind of symbol or expression. But this is not consistent with statements made elsewhere in the same work which are essential to Euler's use of the notion of function —e.g., "*Si fuerit y functio quaecunque ipsius z, tum vicissim z erit functio ipsius y*" (*Opera*, p. 24), "Sed omnis transformatio consistit in alio modo eandem functionem exprimendi, quemadmodum ex Algebra constat eandem quantitatem per plures diversas formas exprimi posse" (*Opera*, p. 32).

[63]The question may be raised whether, say, an interrogative or an imperative logic is possible, in which interrogative or imperative sentences and what they express (questions or commands) have roles analogous to those of declarative sentences and propositions in logic of ordinary kind. And some tentative proposals have in fact been made towards an imperative logic, and also towards an optative logic or logic of wishes. But these matters are beyond the scope of this book.

[64]Cf. the explanation in §02 regarding the use in connection with logistic systems of the terms *constant, variable, form.* An analogous explanation applies to a number of terms of like kind to be introduced below—in particular, *propositional variable, propositional form, operator, quantifier, bound variable, connective.*

described them) is not barely to name something but to make an assertion. Nevertheless it is possible to regard sentences as names by distinguishing between the assertive use of a sentence on the one hand, and its non-assertive use, on the other hand, as a name and a constituent of a longer sentence (just as other names are used). Even when a sentence is simply asserted, we shall hold that it is still a name, though used in a way not possible for other names.[65]

An important advantage of regarding sentences as names is that all the ideas and explanations of §§01–03 can then be taken over at once and applied to sentences, and related matters, as a special case. Else we should have to develop independently a theory of the meaning of sentences; and in the course of this, it seems, the developments of these three sections would be so closely paralleled that in the end the identification of sentences as a kind of names (though not demonstrated) would be very forcefully suggested as a means of simplifying and unifying the theory. In particular we shall require variables for which sentences may be substituted, forms which become sentences upon replacing their free variables by appropriate constants, and associated functions of such forms—things which, on the theory of sentences as names, fit naturally into their proper place in the scheme set forth in §§02–03.

Granted that sentences are names, we go on, in the light of the discussion in §01, to consider the denotation and the sense of sentences.

As a consequence of the principle (2), stated in the next to last paragraph of §01, examples readily present themselves of sentences which, though in some sense of different meaning, must apparently have the same denotation. Thus the denotation (in English) of "Sir Walter Scott is the author of *Waverley*" must be the same as that of "Sir Walter Scott is Sir Walter Scott," the name "the author of *Waverley*" being replaced by another which has the same denotation. Again the sentence "Sir Walter Scott is the author of *Waverley*" must have the same denotation as the sentence "Sir Walter Scott is the man who wrote twenty-nine Waverley Novels altogether," since the name "the author of *Waverley*" is replaced by another name of the same person; the latter sentence, it is plausible to suppose, if it is not synonymous with "The number, such that Sir Walter Scott is the man who wrote that many Waverley Novels altogether, is twenty-nine," is at least so nearly so as to ensure its having the same denotation; and from this last sentence in turn, replacing the complete subject by another name of the same number, we obtain, as still having the same denotation, the sentence "The number of counties in Utah is twenty-nine."

Now the two sentences, "Sir Walter Scott is the author of *Waverley*" and "The number of counties in Utah is twenty-nine," though they have the same denotation according to the preceding line of reasoning, seem actually to have very little in common. The most striking thing that they do have in common is that both are true. Elaboration of examples of this kind leads us quickly to the conclusion, as at least plausible, that all true sentences have the same denotation. And parallel examples may be used in the same way to suggest that all false sentences have the same denotation (e.g., "Sir Walter Scott is not the author of *Waverley*" must have the same denotation as "Sir Walter Scott is not Sir Walter Scott").

Therefore, with Frege, we postulate[66] two abstract objects called *truth-values*, one of them being *truth* and the other one *falsehood*. And we declare all true sentences to denote the truth-value truth, and all false sentences to denote the truth-value falsehood. In alternative phraseology, we shall also speak of a sentence as *having* the truth-value truth (if it is true) or *having* the truth-value falsehood (if it is false).[67]

The sense of a sentence may be described as that which is grasped when one understands the sentence, or as that which two sentences in different languages must have in common in order to be correct translations each of the other. As in the case of names generally, it is possible to grasp the sense

[65]To distinguish the non-assertive use of a sentence and the assertive use, especially in a formalized language, Frege wrote a horizontal line, —, before the sentence in the former case, and the character ⊢ before it in the latter case, the addition of the vertical line thus serving as a sign of assertion. Russell, and Whitehead and Russell in *Principia Mathematica*, did not follow Frege's use of the horizontal line before non-asserted sentences, but did take over the character ⊢ in the role of an assertion sign.

(Frege also used the horizontal line before names other than sentences, the expression so formed being a false sentence. But this is a feature of his notation which need not concern us here.)

In this book we shall not make use of a special assertion sign, but (in a formalized language) shall employ the mere writing of a sentence displayed on a separate line or lines as sufficient indication of its assertion. This is possible because sentences used non-assertively are always constituent parts of asserted sentences, and because of the availability of a two-dimensional arrangement on the printed page. (In a one-dimensional arrangement the assertion sign would indeed be necessary, if only as punctuation.)

The sign ⊢ which is employed below, in Chapter I and later chapters, is not the Frege-Russell assertion sign, but has a wholly different use.

[66]To Frege, as a thoroughgoing Platonic realist, our use of the word "postulate" here would not be acceptable. It would represent his position better to say that the situation indicates that *there are* two such things as truth and falsehood (*das Wahre* and *das Falsche*).

[67]The explicit use of two truth-values appears for the first time in a paper by C. S. Peirce in the *American Journal of Mathematics*, vol. 7 (1885), pp. 180–202 (or see his *Collected Papers*, vol. 3, pp. 210–238). Frege's first use of truth-values is in his *Funktion und Begriff* of 1891 and in his paper of 1892 which is cited in footnote 5; it is in these that the account of sentences as names of truth-values is first put forward.

of a sentence without therefore necessarily having knowledge of its denotation (truth-value) otherwise than as determined by this sense. In particular, though the sense is grasped, it may sometimes remain unknown whether the denotation is truth.

Any concept of a truth-value, provided that *being a truth-value* is contained in the concept, and whether or not it is the sense of some actually available sentence in a particular language under consideration, we shall call a *proposition*, translating thus Frege's *Gedanke*.

Therefore a proposition, as we use the term, is an abstract object of the same general category as a class, a number, or a function. It has not the psychological character of William of Ockham's *propositio mentalis* or of the traditional *judgment*: in the words of Frege, explaining his term *Gedanke*, it is "nicht das subjective Thun des Denkens, sondern dessen objectiven Inhalt, der fähig ist, gemeinsames Eigenthum von Vielen zu sein."

Traditional (post-Scholastic) logicians were wont to define a proposition as a judgment expressed in words, thus as a linguistic entity, either a sentence or a sentence taken in association with its meaning.[68] But in non-technical English the word has long been used rather for the meaning (in our view the sense) of a sentence,[69] and logicians have latterly come to accept this as the technical meaning of "proposition." This is the happy result of a process which, historically, must have been due in part to sheer confusion between the sentence in itself and the meaning of the sentence. It provides in English a distinction not easily expressed in some other languages, and makes possible a translation of Frege's *Gedanke* which is less misleading than the word "thought."[70]

According to our usage, every proposition determines or is a concept of

[68]E.g., in Isaac Watts's *Logick*, 1725: "A *Proposition* is a Sentence wherein two or more *Ideas* or *Terms* are joined or disjoined by one Affirmation or Negation. . . . In describing a *Proposition* I use the Word *Terms* as well as *Ideas*, because when mere Ideas are join'd in the Mind without Words, it is rather called a *Judgment*; but when clothed with Words, it is called a *Proposition*, even tho' it be in the Mind only, as well as when it is expressed by speaking or Writing." Again in Richard Whately's *Elements of Logic*, 1826: "The second part of Logic treats of the *proposition*; which is, '*Judgment expressed in words*.' A Proposition is defined logically 'a sentence indicative,' i.e. affirming or denying; (this excludes *commands* and *questions*.)" Here Whately is following in part the Latin of Henry Aldrich (1691). In fact these passages show no important advance over Petrus Hispanus, who wrote a half millennium earlier, but they are quoted here apropos of the history of the word "proposition" in English.

[69]Consider, for example, the incongruous result obtained by substituting the words "declarative sentence" for the word "proposition" in Lincoln's Gettysburg Address.

[70]For a further account of the history of the matter, we refer to Carnap's *Introduction to Semantics*, 1942, pp. 235–236; and see also R. M. Eaton, *General Logic*, 1931.

(or, as we shall also say, has) some truth-value. It is, however, a somewhat arbitrary decision that we deny the name *proposition* to senses of such sentences (of the natural languages) as express a sense but have no truth-value.[71] To this extent our use of *proposition* deviates from Frege's use of *Gedanke*. But the question will not arise in connection with the formalized languages which we shall study, as these languages will be so constructed that every name—and in particular every sentence—has a denotation.

A proposition is then *true* if it determines or has the truth-value truth, *false* if it has the truth-value falsehood. When a sentence expressing a proposition is asserted we shall say that the proposition itself is thereby *asserted*.[72]

A variable whose range is the two truth-values—thus a variable for which sentences (expressing propositions) may appropriately be substituted—is called a *propositional variable*. We shall not have occasion to use variables

[71]By the remark of footnote 22, such are sentences which contain non-obliquely one or more names that express a sense but lack a denotation—or so, following Frege, we shall take them. Examples are: "The present king of France is bald"; "The present king of France is not bald"; "The author of *Principia Mathematica* was born in 1861." (As to the last example, it is true that the phrase "the author of *Principia Mathematica*" in some appropriate supporting context may be an ellipsis for something like "the author of *Principia Mathematica* who was just mentioned" and therefore have a denotation; but we here suppose that there is no such supporting context, so that the phrase can only mean "the one and only author of *Principia Mathematica*" and therefore have no denotation.)

To sentences as a special case of names, of course the second remark of footnote 22 also applies. Thus we understand as true (and containing oblique occurrences of names) each of the sentences: "Lady Hamilton was like Aphrodite in beauty"; "The fountain of youth is not located in Florida"; "The present king of France does not exist." Cases of doubt whether a sentence has a truth-value or not are also not difficult to find in this connection, the exact meaning of various phraseologies in the natural languages being often insufficiently determinate for a decision.

[72]Notice the following distinction. The statement that a certain proposition was asserted (say on such and such an occasion) need not reveal what language was used nor make any reference to a particular language. But the statement that a certain sentence was asserted does not convey the meaning of the transaction unless it is added what language was used. For not only may the same proposition be expressed by different sentences in different languages, but also the same sentence may be used to assert different propositions according to what language the user intends. It is beside the point that the latter situation is comparatively rare in the principal known natural languages; it is not rare when all possible languages are taken into account.

Thus, if the language is English, the statement, "Seneca said that man is a rational animal," conveys the proposition that Seneca asserted but not the information what language he used. On the other hand the statement, "Seneca wrote, 'Rationale enim animal est homo,' " gives only the information what succession of letters he set down, not what proposition he asserted. (The reader may guess or know from other sources that Seneca used Latin, but this is neither said nor implied in the given statement—for there are many languages besides Latin in which this succession of letters spells a declarative sentence and, for all that thou and I know, one of them may once have been in actual use.)

whose values are propositions, but we would suggest the term *intensional propositional variable* for these.

A form whose values are truth-values (and which therefore becomes a sentence when its free variables are replaced by appropriate constants) is a *propositional form*. Usage sanctions this term[73] rather than "truth-value form," thus naming the form rather by what is expressed, when constants replace the variables, than by what is denoted.

A propositional form is said to be *satisfied by* a value of its free variable, or a system of values of its free variables, if its value for those values of its free variables is truth. (More explicitly, we should speak of a system of values of variables as satisfying a given propositional form *in a given language*, but the reference to the particular language may often be omitted as clear from the context.) A propositional form may also be said to be *true* or *false* for a given value of its free variable, or system of values of its free variables, according as its value for those values of its free variables is truth or falsehood.

A function whose range of values consists exclusively of truth-values, and thus in particular any associated function of a propositional form, is a *propositional function*. Here again, established usage sanctions "propositional function"[74] rather than "truth-value function," though the latter term would be the one analogous to, e.g., the term "numerical function" for a function whose values are numbers.

A propositional function is said to be *satisfied by* an argument (or ordered system of arguments) if its value for that argument (or ordered system of arguments) is truth. Or synonymously we may say that a propositional function *holds for* a particular argument or ordered system of arguments.

From its use in mathematics, we assume that the notion of a *class* is already at least informally familiar to the reader. (The words *set* and *aggregate* are ordinarily used as synonymous with *class*, but we shall not follow this usage, because in connection with the Zermelo axiomatic set

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[73]Cf. footnote 26.

[74]This statement seems to be on the whole just, though the issue is much obscured by divergencies among different writers as to the theory of meaning adopted and in the accounts given of the notions of function and proposition. The idea of the propositional function as an analogue of the numerical function of mathematical analysis originated with Frege, but the term "propositional" function is originally Russell's. Russell's early use of this term is not wholly clear. In his introduction to the second edition of *Principia Mathematica* (1925) he decides in favor of the meaning which we are adopting here, or very nearly that.

theory[75] we shall wish later to give the word *set* a special meaning, somewhat different from that of *class*.) We recall that a class is something which has or may have *members*, and that classes are considered identical if and only if they have exactly the same members. Moreover it is usual mathematical practice to take any given singulary propositional form as having associated with it a class, namely the class whose members are those values of the free variable for which the form is true.

In connection with the functional calculi of Chapters III–VI, or rather, with the formalized languages obtained from them by adopting one of the indicated principal interpretations (§07), it turns out that we may secure everything necessary about classes by just identifying a class with a singulary propositional function, and membership in the class with satisfaction of the singulary propositional function. We shall consequently make this identification, on the ground that no purpose is served by maintaining a distinction between classes and singulary propositional functions.

We must add at once that the notion of a class obtained by thus identifying classes with singulary propositional functions does not quite coincide with the informal notion of a class which we first described, because it does not fully preserve the principle that classes are identical if they have the same members. Rather, it is necessary to take into account also the *range-members* of a class (constituting, i.e., the range of the singulary propositional function). And only when the range-members are given to be the same is the principle preserved that classes are identical if they have the same members. This or some other departure from the informal notion of a class is in fact necessary, because, as we shall see later,[76] the informal notion—in the presence of some other assumptions difficult to avoid—is self-inconsistent and leads to antinomies. (The *sets* of Zermelo set theory preserve the principle that sets having the same members are identical, but at the sacrifice of the principle that an arbitrary singulary propositional form has an associated set.)

Since, then, a class is a singulary propositional function, we speak of the *range of* the class just as we do of the propositional function (i.e., it is the same thing). We think of the range as being itself a class, having as members the range-members of the given class, and having the same range-members.

(In any particular discussion hereafter in which classes are introduced,

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[75]Chapter XI.

[76]In Chapter VI.

and in the absence of any indication to the contrary, it is to be understood that there is a fixed range determined in advance and that all classes have this same range.)

Relations may be similarly accounted for by identifying them with binary propositional functions, the relation being said to *hold between* an ordered pair of things (or the things being said to *stand in* that relation, or to *bear* that relation one to the other) if the binary propositional function is satisfied by the ordered pair. Given that the ranges are the same, this makes two relations identical if and only if they hold between the same ordered pairs, and to indicate this we may speak more explicitly of a *relation in extension*—using this term as synonymous with *relation*.

A *property*, as ordinarily understood, differs from a class only or chiefly in that two properties may be different though the classes determined by them are the same (where the class determined by a property is the class whose members are the things that have that property). Therefore we identify a property with a *class concept*, or concept of a class in the sense of §01. And two properties are said to *coincide in extension* if they determine the same class.

Similarly, a *relation in intension* is a *relation concept*, or concept of a relation in extension.

To turn once more for illustrative purposes to the theory of real numbers and its notations, the following are examples of propositional forms:

$$\sin x = 0, \qquad \sin x = 2,$$

$$e^x > 0, \qquad e^x > 1, \qquad x > 0,$$

$$\varepsilon > 0, \qquad \varepsilon < 0,$$

$$x^3 + y^3 = 3xy, \qquad x \neq y,$$

$$|x - y| < t, \qquad |x - y| < \varepsilon,$$

$$\text{If } |x - y| < \delta \text{ then } |\sin x - \sin y| < \varepsilon.$$

Here we are using x, y, t as variables whose range is the real numbers, and ε and δ as variables whose range is the positive real numbers. The seven forms on the first three lines are examples of singulary propositional forms. Those on the fourth line are binary, on the fifth line ternary, while on the last line is an example of a quaternary propositional form.

Each of the singulary propositional forms has an associated class. Thus with the form $\sin x = 0$ is associated the class of those real numbers whose sine is 0, i.e., the class whose range is the real numbers and whose members are $0, \pi, -\pi, 2\pi, -2\pi, 3\pi$, and so on. As explained, we identify this class with the propositional function $\lambda x(\sin x = 0)$, or in other words the function from real numbers to truth-values which has for any argument x the value $\sin x = 0$.

The two propositional forms $e^x > 1$ and $x > 0$ have the same associated class, namely, the class whose range is the real numbers and whose members are the positive real numbers. This class is identified with either $\lambda x(e^x > 1)$ or $\lambda x(x > 0)$, these two propositional functions being identical with each other by the convention about identity of functions adopted in §03.

Since the propositional form $\sin x = 2$ has the value falsehood for every value of x, the associated class $\lambda x(\sin x = 2)$ has no members.

A class which has no members is called a *null class* or an *empty class*. From our conventions about identity of propositional functions and of classes, if the range is given, it follows that there is only one null class. But, e.g., the range of the null class associated with the form $\sin x = 2$ and the range of the null class associated with the form $\varepsilon < 0$ are not the same: the former range is the real numbers, and the latter range is the positive real numbers.[77] We shall speak respectively of the "null class of real numbers" and of the "null class of positive real numbers."

A class which coincides with its range is called a *universal class*. For example, the class associated with the form $e^x > 0$ is the universal class of real numbers; and the class associated with the form $\varepsilon > 0$ is the universal class of positive real numbers.

The binary propositional forms $x^3 + y^3 = 3xy$ and $x \neq y$ are both symmetric and therefore each have one associated binary propositional function or relation. In particular, the associated relation of the form $x \neq y$ is the relation of diversity between real numbers; or in other words the relation which has the pairs of real numbers as its range, which any two different real numbers bear to each other, and which no real number bears to itself.

The ternary propositional forms $|x - y| < t$ and $|x - y| < \varepsilon$ have each three associated ternary propositional functions[78] (being symmetric in x and y). All six of these propositional functions are different; but an appropriately chosen pair of them, one associated with each form, will be found to agree in value for all ordered triples of arguments which are in the range of both, differing only in that the first one has the value falsehood for certain ordered triples of arguments which are not in the range of the other.

05. Improper symbols, connectives. When the expressions, especially the sentences, of a language are analyzed into the single symbols of which they consist, symbols which may be regarded as indivisible in the sense that

[77] According to the informal notion that classes with the same members are identical, it would be true absolutely that there is only one null class. The distinction of null classes with different ranges was introduced by Russell in 1908 as a part of his theory of types (see Chapter VI). The same thing had previously been done by Ernst Schröder in the first volume of his *Algebra der Logik* (1890), though with a very different motivation.

[78] We may also occasionally use the term *ternary relation* (and *quaternary relation* etc). But the simple term *relation* will be reserved for the special case of a binary relation, or binary propositional function.

no division of them into parts has relevance to the meaning,[79] we have seen that there are two sorts of symbols which may in particular appear, namely primitive proper names and variables. These we call *proper symbols*, and we regard them as having meaning in isolation, the primitive names as denoting (or at least purporting to denote) something, the variables as having (or at least purporting to have) a non-empty range. But in addition to proper symbols there must also occur symbols which are *improper*—or in traditional (Scholastic and pre-Scholastic) terminology, *syncategorematic*—i.e., which have no meaning in isolation but which combine with proper symbols (one or more) to form longer expressions that do have meaning in isolation.[80]

Conspicuous among improper symbols are parentheses and brackets of various kinds, employed (as familiar in mathematical notation) to show the way in which parts of an expression are associated. These parentheses and brackets occur as constituents in certain combinations of improper symbols such as we now go on to consider—either exclusively to show association and in connection with other improper symbols which carry the burden of showing the particular character of the notation,[81] or else sometimes in a way that combines the showing of association with some special meaning-producing character.[82]

Connectives are combinations of improper symbols which may be used together with one or more constants to form or produce a new constant.

[79]The formalized languages are to be so constructed as to make such analysis into single symbols precisely possible. In general it is possible in the natural languages only partially and approximately—or better, our thinking of it as possible involves a certain idealization.

In written English (say), the single symbols obtained are not just the letters with which words are spelled, since the division of a word into letters has or may have no relevance to the meaning. Frequently the single symbols are words. In other cases they are parts of words, since the division, e.g., of "books" into "book" and "s" or of "colder" into "cold" and "er" does have relevance to the meaning. In still other cases the linguistic structure of meaningful parts is an idealization, as when "worse" is taken to have an analysis parallel to that of "colder," or "I went" an analysis parallel to that of "I shall go," or "had I known" parallel to that of "if I should hear." (Less obvious and more complex examples may be expected to appear if analysis is pressed more in detail.)

[80]Apparently the case may be excluded that several improper symbols combine without any proper symbols to form an expression that has meaning in isolation. For the division of that expression into the improper symbols as parts could then hardly be said to have relevance to the meaning.

[81]Thus in the expression $(t - (x - y))$ we may say that the inner parentheses serve exclusively to show the association together of the part $x - y$ of the expression, and that they are used in connection with the sign $-$, which serves to show subtraction.

[82]In real number theory, the usual notation $|\ \ |$ for the absolute value is an obvious example of this latter. Again it may be held that the parentheses have such a double use in either of the two notations introduced in §03 for application of a singulary function to its argument.

Then, as follows from the discussion in §02, if we replace one or more of the constants each by a form which has the denotation of that constant among its values, the resulting expression becomes a form (instead of a constant); and the free variables of this resulting form are the free variables of all the forms (one or more) which were united by means of the connective (with each other and possibly also with some constants) to produce the resulting form. In order to give completely the meaning-producing character of a particular connective in a particular language, not only is it necessary to give the denotation[83] of the new constant in every permissible case that the connective is used together with one or more constants to form such a new constant, but also, for every case that the connective may be used with forms or forms and constants to produce a resulting form, it is necessary to give the complete scheme of values of this resulting form for values of its free variables. And this must all be done in a way to conform to the assumptions about sense and denotation at the end of §01, and to the conventions about meaning and values of variables and forms as these were described in §02. Connectives may then be used not only in languages which contain constants but also in languages whose only proper symbols are variables.[84]

The constants or forms, united by means of a connective to produce a new constant or form, are called the *operands*. A connective is called *singulary*, *binary*, *ternary*, etc., according to the number of its operands.

A singulary connective may be used with a variable of appropriate range as the operand (this falls under our foregoing explanation since, of course, a variable is a special case of a form). The form so produced is called an *associated form* of the connective if the range of the variable includes the denotations of all constants which may be used as operands of the connective and all the relevant values of all the forms which may be used as operands of the connective (where by a *relevant* value of a form used as operand is meant a value corresponding to which the entire form, consisting of connective and operand, has a value). And the *associated function* of a singulary connective is the associated function of any associated form. The associated function as thus defined is clearly unique.

[83]It is not necessary (or possible) to give the sense of the new constant separately, since the way in which the denotation is given carries with it a sense—the same phrase which is used to name the denotation must also express a sense.

Further questions arise if, besides constants, names having a sense but no denotation are allowed. Such names seem to be used with connectives in the natural languages and in usual systems of mathematical notation, and indeed some illustrations which we have employed depend on this. However, as already explained, we avoid this in the formalized languages which we shall consider.

[84]Cf. footnote 27.

The notion of the associated function of a singulary connective is possible also in the case of a language containing no variable with a range of the kind required to produce an associated form, namely we may consider an extension of the language obtained by adding such a variable.

In the same way an n-ary connective may be used together with n different variables as operands to produce a form; and this is called an *associated form* of the connective if, for each variable, the range includes both the denotations of all constants and all relevant values of all forms which may be used as operands at that place. The *associated function* of the connective is that one of the associated n-ary functions of an associated form which is obtained by assigning the arguments of the function, in their order, as values to the free variables of the form in their left-to-right order of occurrence in the form.

In general the meaning-producing character of a connective is most readily given by just giving the associated function, this being sufficient to fix the use of the connective completely.[85]

Indeed there is a close relationship between connectives and *functional constants* or proper names of functions. Differences are that (a) a functional constant *denotes* a function whereas a connective *is associated with* a function, (b) a connective is never replaced by a variable, and (c) the notation for application of a function to its arguments may be paralleled by a different notation when a corresponding connective takes the place of a functional constant. But these differences are from some points of view largely non-essential because (a) notations of course have such meaning as we choose to give them (within limitations imposed by requirements of consistency and adequacy), (b) languages are possible which do not contain variables with functions as values and in which functional constants are never replaced by variables, and (c) the notation for application of a function to its arguments may, like any other, be changed—or even duplicated

[85]For example, the familiar notation (—) for subtraction of real numbers may be held to be a connective. That is, the combination of symbols which consists of a left parenthesis, a minus sign, and a right parenthesis, in that order, may be considered as a connective—where the understanding is that an appropriate constant or form is to be filled in at each of two places, namely immediately before and immediately after the minus sign. To give completely the meaning-producing character of this connective, it is necessary to give the denotation of the resulting constant when constants are filled in at the two places, and also to give the complete scheme of values of the resulting form when forms are filled in at the two places, or a form at one place and a constant at the other. In order to do this in a way to conform to §§01, 02, it may often be most expeditious first to introduce (by whatever means may be available in the particular context) the binary function of real numbers that is called *subtraction*, and then to declare this to be the associated function of the connective.

by introducing several synonymous notations into the same language.[86]

In the case of a language having notations for application of a function to its arguments, it is clear that a connective may often be eliminated or dispensed with altogether by employing instead a name of the associated function—by modifying the language, if necessary, to the extent of adding such a name to its vocabulary. However, the complete elimination of all connectives from a language can never be accomplished in this way. For the notations for application of a singulary function to its argument, for application of a binary function to its arguments, and so on (e.g., the notations for these which were introduced in §03) are themselves connectives. And though these connectives, like any other, no doubt have their associated functions,[87] nevertheless not all of them can ever be eliminated by the device in question.[88]

[86]Thus, to use once more the example of the preceding footnote, we may hold that the notation (—) is a connective and that the minus sign has no meaning in isolation. Or alternatively we may hold that the minus sign denotes (is a name of) the binary function, *subtraction*, and that in such expressions as, e.g., $(x - y)$ or $(5 - 2)$ we have a special notation for application of a binary function to its arguments, different from the notation for this which was introduced in §03. The choice would seem to be arbitrary between these two accounts of the meaning of the minus sign. But from one standpoint it may be argued that, if we are willing to invent some name for the binary function, then this name might just as well, and would most simply, be the minus sign.

[87]As explained below, we are for expository purposes temporarily ignoring difficulties or complications which may be caused by the theory of types or by such alternative to the theory of types as may be adopted. On this basis, for the connective which is the notation for application of a singulary function to its argument, we explain the associated function by saying that it is the binary function whose value for an ordered pair of arguments f, x is $f(x)$. But if a name of this associated function is to be used for the purpose of eliminating the connective, then another connective is found to be necessary, the notation, namely, for application of a binary function to its arguments. If the latter connective is to be eliminated by using a name of its associated function, then the notation for application of a ternary function to its arguments becomes necessary. And so on. Obviously no genuine progress is being made in these attempts.

(After studying the theory of types the reader will see that the foregoing statement, and others we have made, remain in some sense essentially true on the basis of that theory. It is only that the connective, e.g., which is the notation for application of a singulary function to its argument must be thought of as replaced by many different connectives, corresponding to different types, and each of these has its own associated function. Or alternatively, if we choose to retain this connective as always the same connective, regardless of considerations of type, then there may well be no variable in the language with a range of the kind required to produce an associated form: an extension of the language by adding such a variable can be made to provide an associated form, but not so easily a name of the associated function. See Carnap, *The Logical Syntax of Language* (cited in footnote 131), examples at the end of §53, and references there given; also Bernard Notcutt's proposal of "intertypical variables" in *Mind*, n.s. vol. 43 (1934), pp. 63–77; and remarks by Tarski in the appendix to his *Wahrheitsbegriff* (cited in footnote 140).)

[88]There is, however, a device which may be used in appropriate context (cf. Chapter X) to eliminate all the connectives except the notation for application of a singulary function to its argument. This is done by reconstruing a binary function as a singulary function whose values are singular functions; a ternary function as a singulary function whose values are binary functions in the foregoing sense; and so on. For it turns out

Connectives other than notations for application of a function to its arguments are apparently always eliminable in the way described by a sufficient extension of the language in which they occur (including if necessary the addition to the language of notations for application of a function to its arguments). Nevertheless such other connectives are often used—especially in formalized languages of limited vocabulary, where it may be preferred to preserve this limitation of vocabulary, so as to use the language as a means of singling out for separate consideration some special branch of logic (or other subject).

In particular we shall meet with *sentence connectives* in Chapter I. Namely, these are connectives which are used together with one or more sentences to produce a new sentence; or when propositional forms replace some or all of the sentences as operands, then a propositional form is produced rather than a sentence.

The chief *singulary* sentence connective we shall need is one for negation. In this role we shall use, in formalized languages, the single symbol $\sim$, which, when prefixed to a sentence, forms a new sentence that is the negation of the first one. The associated function of this connective is the function from truth-values to truth-values whose value for the argument *falsehood* is *truth*, and whose value for the argument *truth* is *falsehood*. For convenience in reading orally expressions of a formalized language, the symbol $\sim$ may be rendered by the word "not" or by the phrase "it is false that."

The principal *binary* sentence connectives are indicated in the table which follows. The notation which we shall use in formalized languages is shown in the first column of the table, with the understanding that each of the two blanks is to be filled by a sentence of the language in question. In the second column of the table a convenient oral reading of the connective is suggested, or sometimes two alternative readings; here the understanding is that the two blanks are to be filled by oral readings of the same two sentences (in the same order) which filled the two corresponding blanks in the first column; and words which appear between parentheses are words which

that n-ary functions in the sense thus obtained can be made to serve all the ordinary purposes of n-ary functions (in any sense).

The alternative device of reducing (e.g.) a binary function to a singulary function by reconstruing it as a singulary function whose arguments are ordered pairs is also useful in certain contexts (e.g., in axiomatic set theory). This device does not (at least *prima facie*) serve to reduce the number of connectives to one, as besides the notation for application of a singulary function to its argument there will be required also a connective which unites the names of two things to form a name of their ordered pair (or at least some notation for this latter purpose). Nevertheless it is a device which may sometimes be used to accomplish a reduction, especially where other connectives—or operators (§06)—are available.

may ordinarily be omitted for brevity, but which are to be supplied whenever necessary to avoid a misunderstanding or to emphasize a distinction. In the third column the associated function of the connective is indicated by means of a code sequence of four letters: in doing this, t is used for truth and f for falsehood, and the first letter of the four gives the value of the function for the arguments t, t, the second letter gives the value for the arguments t, f, the third letter for the arguments f, t, the fourth letter for the arguments f, f. In many cases there is an English name in standard use, which may denote either the connective or its associated function. This is indicated in a fourth column of the table; where alternative names are in use, both are given, and in some cases where none is in use a suggested name is supplied.

[____ $\vee$ ____]	____ or ____ (or both).	tttf	(Inclusive) disjunction, alternation.
[____ $\subset$ ____]	____ if ____.[89]	ttft	Converse implication.
[____ $\supset$ ____]	If ____ then ____,[89]	tftt	The (truth-functional) conditional,[90]
	____ (materially) implies ____.[89]		(material) implication.
[____ $\equiv$ ____]	____ if and only if ____,[89]	tfft	The (truth-functional) biconditional,[90]
	____ is (materially) equivalent to ____.[89]		(material) equivalence.
[____ ____]	____ and ____.	tfff	Conjunction.
[____ $\mid$ ____]	Not both ____ and ____.	fttt	Non-conjunction, Sheffer's stroke.
[____ $\not\equiv$ ____]	____ or ____ but not both, ____ is not (materially) equivalent to ____.[89]	fttf	Exclusive disjunction, (material) non-equivalence.
[____ $\not\supset$ ____]	____ but not ____.	ftff	(Material) non-implication.
[____ $\not\subset$ ____]	Not ____ but ____.	fftf	Converse non-implication.
[____ $\bar{\vee}$ ____]	Neither ____ nor ____.	ffft	Non-disjunction.

[89] The use of the English words "if," "implies," "equivalent" in these oral readings must not be taken as indicating that the meanings of these English words are faithfully

The notations which we use as sentence connectives—and those which we use as quantifiers (see below)—are adaptations of those in Whitehead and Russell's *Principia Mathematica* (some of which in turn were taken from Peano). Various other notations are in use,[91] and the student who would

rendered by the corresponding connectives in all, or even in most, cases. On the contrary, the meaning-producing character of the connectives is to be learned with accuracy from the third column of the table, where the associated functions are given, and the oral readings supply at best a rough approximation.

As a matter of fact, the words "if . . . then" and "implies" as used in ordinary non-technical English often seem to denote a relation between propositions rather than between truth-values. Their possible meanings when employed in this way are difficult to fix precisely and we shall make no attempt to do so. But we select the one use of the words "if . . . then" (or "implies")—their material use, we shall call it—in which they may be construed as denoting a relation between truth-values, and we assign this relation as the associated function for the connective $[\ \supset\ ]$.

As examples of the material use of "if . . . then," consider the four following English sentences:

(i) If Joan of Arc was a patriot then Nathan Hale was a patriot.
(ii) If Joan of Arc was a patriot then Vidkun Quisling was a patriot.
(iii) If Vidkun Quisling was a patriot then attar of roses is a perfume.
(iv) If Vidkun Quisling was a patriot then Limburger cheese is a perfume.

For the sake of the illustration let us suppose examination of the historical facts to reveal that Joan of Arc and Nathan Hale were indeed patriots and that Vidkun Quisling was not a patriot. Then (i), (iii), and (iv) are true, and (ii) is false; and to reach these conclusions no examination is necessary of the characteristics of either attar of roses or Limburger cheese. (If the reader is inclined to question the truth of, e.g., (iii) on the ground of complete lack of connection between Vidkun Quisling and attar of roses, then this means that he has in mind some other use of "if . . .. then" than the material use.)

[90]These terms were introduced by Quine, who uses them for "the mode of composition described in" the list of truth-values as given in the third column of the table—i.e., in effect, and in our terminology, for the associated function of the connective rather than for the connective itself. See his *Mathematical Logic*, 1940, pp. 15, 20.

We prefer the better established terms *material implication* and *material equivalence*, from which the adjective *material* may be omitted whenever there is no danger of confusion with other kinds of implication or equivalence—as, for example, with formal implication and formal equivalence (§06), or with kinds of implication and equivalence (belonging to modal logic) which are relations between propositions rather than between truth-values.

[91]Worthy of special remark is the parenthesis-free notation of Jan Łukasiewicz. In this, the letters N, A, C, E, K are used in the roles of negation, disjunction, implication, equivalence, conjunction respectively. Further letters may be introduced if desired (R has been employed as non-equivalence, D as non-conjunction). In use as a sentence connective, the letter is written first and then in order the sentences or propositional forms together with which it is used. No parentheses or brackets or other notations specially to show association are necessary. E.g., the propositional form

$$[[p \supset [q \vee r]] \supset {\sim}p]$$

(where p, q, r are propositional variables) becomes, in the Łukasiewicz notation,

$$CCpAqrNp.$$

It is of course possible to apply the same idea to other connectives, in particular to the notation for application of a singulary function to its argument. Hence (see footnote 88) parentheses and brackets may be avoided altogether in a formalized language. The possibility of this is interesting. But the notation so obtained is unfamiliar, and less perspicuous than the usual one.

compare the treatments of different authors must learn a certain facility in shifting from one system of notation to another.

The brackets which we indicate as constituents in these notations may in actual use be found unnecessary at certain places, and we may then just omit them at such places (though only as a practically convenient abbreviation).

We shall use the term *truth-function*[92] for a propositional function of truth-values which has as range, if it is n-ary, all ordered systems of n truth-values. Thus every associated function of a sentence connective is a truth-function. And likewise every associated function of a form built up from propositional variables solely by iterated use of sentence connectives.[93]

06. Operators, quantifiers. An *operator* is a combination of improper symbols which may be used together with one or more variables—the *operator variables* (which must be fixed in number and all distinct)—and one or more constants or forms or both—the *operands*—to produce a new constant or form. In this new constant or form, however, the operator variables are at certain determinate places not free variables, though they may have been free variables at those places in the operands.

To be more explicit, we remark that, in any application of an operator, the operator variables may (and commonly will) occur as free variables in some of the operands. In the new constant or form produced we distinguish three possible kinds of occurrences of the operator variables, viz.: an occurrence in one of the operands which, when considered as an occurrence in that operand alone, is an occurrence as a free variable; an occurrence in one of the operands, not of this kind; and an occurrence which is an occurrence *as* an operator variable, therefore not in any of the operands. In the new constant or form, an occurrence of one of the two latter kinds is never an occurrence as a free variable, and each occurrence of the first kind is an occurrence as a free variable or not, according to some rule associated with the particular operator.[94] The simplest case is that, in the new constant or form, none of the occurrences of the operator variables are occurrences as free variables. And this is the only case with which we shall meet in the following chapters

[92]We adopt this term from *Principia Mathematica*, giving it substantially the meaning which it acquires through changes in that work that were made (or rather, proposed) by Russell in his introduction to the second edition of it.

[93]For example, the associated function of the propositional form mentioned in footnote 91.

[94]We do require in the case of each operator variable that all occurrences of the first kind shall be occurrences as free variables or else all not, *in any one occurrence of a particular operand* in the new constant or form produced. For operators violating this requirement are not found among existing standard mathematical and logical notations, and it is clear that they would involve certain anomalies of meaning which it is preferable to avoid.

(though many operators which are familiar as standard mathematical notation fail to fall under this simplest case).

Variables thus having occurrences in a constant or form which are not occurrences as free variables of it are called *bound variables* of the constant or form.[95] The difference is that a form containing a particular variable, say x, *as a free variable* has values for various values of the variable, but a constant or form which contains x *as a bound variable only* has a meaning which is independent of x—not in the sense of having the same value for every value of x, but in the sense that the assignment of particular values to x is not a relevant procedure.[96]

It may happen that a form contains both free and bound occurrences of the same variable. This case will arise, for example, if a form containing a particular variable as a free variable and a form or constant containing that same variable as a bound variable are united by means of a binary connective.[97]

As in the case of connectives, we require that operators be such as to conform to the principles (1)–(3) at the end of §01; also that they conform to the conventions about meaning and values of variables as these were described in §02, and in particular to the principle (4) of §02.[98]

An operator is called *m-ary-n-ary* if it is used with m distinct operator variables and n operands.[99] The most common case is that of a singulary-singulary operator—or, as we shall also call it, a *simple* operator.

In particular, the notation for singulary functional abstraction, which

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[95]Cf. footnote 28.

[96]Therefore a constant or form which contains a particular variable as a bound variable is unaltered in meaning by alphabetic change of that variable, at all of its bound occurrences, to a new variable (not previously occurring) which has the same range. The condition in parentheses is included only as a precaution against identifying two variables which should be kept distinct, and indeed it may be weakened somewhat—cf. the remark in §03 about alphabetic change of free variables.

E.g., the constant $\int_0^2 x^x dx$ (see footnote 36) is unaltered in meaning by alphabetic change of the variable x to the variable y: it has not only the same denotation but also the same sense as $\int_0^2 y^y dy$.

[97]See illustrations in the second paragraph of footnote 36.

[98]And also to the principle (5) of footnote 30.

[99]Thus, in the theory of real numbers, the usual notation for definite integration is a singulary-ternary operator. And in, e.g., the form $\int_0^x x^x dx$ (see footnote 36) the operator variable is x and the three operands are the constant 0, the form x, and the form x^x.

Again, the large $\prod$ (product sign), as used in the third example at the beginning of footnote 36, is part of a singulary-ternary operator. The signs $=$ above and below the $\prod$ are not to be taken as equality signs in the ordinary sense (namely that of footnote 43) but as improper symbols, and also part of the operator. In the particular application of the operator, as it appears in this example, the operator variable is m and the operands are 1, n, and

$$\frac{x - m + 1}{m\pi}.$$

was introduced in §03, is a simple operator (the variable which is placed immediately after the letter λ being the operator variable). We shall call this the *abstraction operator* or, more explicitly, the *singulary functional abstraction operator*. In appropriate context, as we shall see in Chapter X, all other operators can in fact be reduced to this one.[100]

Another operator which we shall use—also a simple operator—is the *description operator*, $(\iota\ )$. To illustrate, let the operator variable be x. Then the notation (ιx) is to have as its approximate reading in words, "the x such that"; or more fully, the notation is explained as follows. It may happen that a singulary propositional form whose free variable is x has the value truth for one and only one value of x, and in this case a name of that value of x is produced by prefixing (ιx) to the form. In case there is no value of x or more than one for which the form has the value truth, there are various meanings which might be assigned to the name produced by prefixing (ιx) to the form: the analogy of English and other natural languages would suggest giving the name a sense which determines no denotation; but we prefer to select some fixed value of x and to assign this as the denotation of the name in all such cases (this selection is arbitrary, but is to be made once for all for each range of variables which is used).

Of especial importance for our purposes are the *quantifiers*. These are namely operators for which both the operands and the new constant or form produced by application of the operator are sentences or propositional forms.

As the *universal quantifier* (when, e.g., the operator variable is x) we use

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As another example of application of the same operator, showing both bound and free occurrences of m, we cite

$$\prod_{m=m+1}^{m=m+n+1} \frac{x - m + 1}{m\pi}.$$

Examples of operators taking more than one operator variable are found in familiar notations for double and multiple limits, double and multiple integrals.

It should also be noted that n-ary connectives may, if we wish, be regarded as 0-ary-n-ary operators.

[100]In the combinatory logic of H. B. Curry (based on an idea due to M. Schönfinkel) a more drastic reduction is attempted, namely the complete elimination of operators, of variables, and of all connectives, except a notation for application of a singulary function to its argument, so as to obtain a formalized language in which, with the exception of the one connective, all single symbols are constants, and which is nevertheless adequate for some or all of the purposes for which variables are ordinarily used. This is a matter beyond the scope of this book, and the present status of the undertaking is too complex for brief statement. The reader may be referred to a monograph by the present writer, *The Calculi of Lambda-Conversion* (1941), which is concerned with a related topic; also to papers by Schönfinkel, Curry, and J. B. Rosser which are there cited, to several papers by Curry and by Rosser in *The Journal of Symbolic Logic* in 1941 and 1942, to an expository paper by Robert Feys in *Revue Philosophique de Louvain*, vol. 44 (1946), pp. 74–103, 237–270, and to a paper by Curry in *Synthese*, vol. 7 (1949), pp. 391–399.

the notation $(\forall x)$ or (x), prefixing this to the operand. The universal quantifier is thus a simple operator, and we may explain its meaning as follows (still using the particular variable x as an example). (x)____ is true if the value of ____ is truth for all values of x, and (x)____ is false if there is any value of x for which the value of ____ is falsehood. Here the blank is to be filled by a singulary propositional form containing x as a free variable, the same one at all four places. Or if as a special case we fill the blank with a sentence, then (x)____ is true if and only if ____ is true. (The meaning in case the blank is filled by a propositional form containing other variables besides x as free variables now follows by the discussion of variables in §02, and may be supplied by the reader.)

Likewise the existential quantifier is a simple operator for which we shall use the notation $(\exists\ )$, filling the blank space with the operator variable and prefixing the whole to the operand. To take the particular operator variable x as an example, $(\exists x)$____ is true if the value of ____ is truth for at least one value of x, and $(\exists x)$____ is false if the value of ____ is falsehood for all values of x. Here again the blank is to be filled by a singulary propositional form containing x as a free variable. Or if as a special case we fill the blank with a sentence, then $(\exists x)$____ is true if and only if ____ is true.

In words, the notations "(x)" and "$(\exists x)$" may be read respectively as "for all x" (or "for every x") and "there is an x such that."

To illustrate the use of the universal and existential quantifiers, and in particular their iterated application, consider the binary propositional form,

$$[xy > 0],$$

where x and y are real variables, i.e., variables whose range is the real numbers. This form expresses about two real numbers x and y that their product is positive, and thus it comes to express a particular proposition as soon as values are given to x and y. If we apply to it the existential quantifier with y as operator variable, we obtain the singulary propositional form,

$$(\exists y)[xy > 0],$$

or as we may also write it, using the device (which we shall find frequently convenient later) of writing a heavy dot to stand for a bracket extending, from the place where the dot occurs, forward,

$$(\exists y) \centerdot xy > 0.$$

This singulary form expresses about a real number x that there is some real number with which its product is positive; and it comes to express a particular proposition as soon as a value is given to x. If we apply to it the universal quantifier with x as operator variable, we obtain the sentence,

$$(x)(\exists y) \centerdot xy > 0.$$

This sentence expresses the proposition that for every real number there is some real number such that the product of the two is positive. It must be distinguished from the sentence,

$$(\exists y)(x) \centerdot xy > 0,$$

expressing the proposition that there is a real number whose product with every real number is positive, though it happens that both are false.[101] To bring out more sharply the difference which is made by the different order of the quantifiers, let us replace product by sum and consider the two sentences:

$$(x)(\exists y) \centerdot x + y > 0$$
$$(\exists y)(x) \centerdot x + y > 0$$

Of these sentences, the first one is true and the second one false.[102]

It should be informally clear to the reader that not both the universal and the existential quantifier are actually necessary in a formalized language, if negation is available. For it would be possible, in place of $(\exists x)$____, to write always $\sim(x)\sim$____; or alternatively, in place of (x)____, to write always $\sim(\exists x)\sim$____. And of course likewise with any other variable in place of the particular variable x.

In most treatments the universal and existential quantifiers, one or both,

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[101]The single counterexample, of the value 0 for x, is of course sufficient to render the first sentence false.

The reader is warned against saying that the sentence $(x)(\exists y) \centerdot xy > 0$ is "nearly always true" or that it is "true with one exception" or the like. These expressions are appropriate rather to the propositional form $(\exists y) \centerdot xy > 0$, and of the sentence it must be said simply that it is false.

[102]A somewhat more complex example of the difference made by the order in which the quantifiers are applied is found in the familiar distinction between continuity and uniform continuity. Using x and y as variables whose range is the real numbers, and ε and δ as variables whose range is the positive real numbers, we may express as follows that the real function f is continuous, on the class F of real numbers (assumed to be an open or a closed interval):

$$(y)(\varepsilon)(\exists\delta)(x) \centerdot F(y) \supset \centerdot F(x) \supset \centerdot |x - y| < \delta \supset \centerdot |f(x) - f(y)| < \varepsilon$$

And we may express as follows that f is uniformly continuous on F:

$$(\varepsilon)(\exists\delta)(x)(y) \centerdot F(y) \supset \centerdot F(x) \supset \centerdot |x - y| < \delta \supset \centerdot |f(x) - f(y)| < \varepsilon$$

To avoid complications that are not relevant to the point being illustrated, we have here assumed not only that the class F is an open or closed interval but also that the range of the function f is all real numbers. (A function with more restricted range may always have its range extended by some arbitrary assignment of values; and indeed it is a common simplifying device in the construction of a formalized language to restrict attention to functions having certain standard ranges (cf. footnote 19).)

are made fundamental, notations being provided for them directly in setting up a formalized language, and other quantifiers are explained in terms of them (in a way similar to that in which, as we have just seen in the preceding paragraph, the universal and existential quantifiers may be explained, either one in terms of the other). No definite or compelling reason can be given for such a preference of these two quantifiers above others that might equally be made fundamental. But it is often convenient.

The application of one or more quantifiers to an operand (especially universal and existential quantifiers) is spoken of as *quantification*.[103]

Another quantifier is a singulary-binary quantifier for which we shall use the notation $[\_\_\_ \supset \_\_\_]$, with the operands in the two blanks, and the operator variable as a subscript after the sign $\supset$. It may be explained by saying that $[\_\_\_ \supset_x \_\_\_]$ is to mean the same as $(x)[\_\_\_ \supset \_\_\_]$, the two blanks being filled with two propositional forms or sentences, the same two in each case (and in the same order); and of course likewise with any other variable in place of the particular variable x. The name *formal implication*[104] is given to this quantifier—or to the associated binary propositional function, i.e., to an appropriate one of the two associated functions of (say) the form $[F(u) \supset_u G(u)]$, where u is a variable with some assigned range, and F and G are variables whose range is all classes (singulary propositional functions) that have a range coinciding with the range of u.

Another quantifier is that which (or its associated propositional function) is called *formal equivalence*.[104] For this we shall use the notation $[\_\_\_ \equiv \_\_\_]$, with the two operands in the two blanks, and the operator variable as a subscript after the sign $\equiv$. It may be explained by saying that $[\_\_\_ \equiv_x \_\_\_]$ is to mean the same as $(x)[\_\_\_ \equiv \_\_\_]$, the two blanks being filled in each case with the two operands in order; and of course likewise with any other variable in place of x.

We shall also make use of quantifiers similar in character to those just explained but having two or more operator variables. These (or their associated propositional functions) we call *binary formal implication*, *binary formal equivalence*, *ternary formal implication*, etc. E.g., binary formal implication may be explained by saying that $[\_\_\_ \supset_{xy} \_\_\_]$ is to mean the same as $(x)(y)[\_\_\_ \supset \_\_\_]$, the two blanks being filled in each case with the two operands in order; and likewise with any two distinct variables in place of x and y as operator variables. Similarly binary formal equivalence $[\_\_\_ \equiv_{xy} \_\_\_]$, ternary formal implication $[\_\_\_ \supset_{xyz} \_\_\_]$, and so on.[105]

Besides the assertion of a sentence, as contemplated in §04, it is usual also to allow assertion of a propositional form, and to treat such an assertion as a particular fixed assertion (in spite of the presence of free variables in the expression asserted). This is common especially in mathematical contexts; where, for instance, the assertion of the equation $\sin(x + 2\pi) = \sin x$ may

[103]The use of quantifiers originated with Frege in 1879. And independently of Frege the same idea was introduced somewhat later by Mitchell and Peirce. (See the historical account in §49.)

[104]The names *formal implication* and *formal equivalence* are those used by Whitehead and Russell in *Principia Mathematica*, and have become sufficiently well established that it seems best not to change them—though the adjective *formal* is perhaps not very well chosen, and must not be understood here in the same sense that we shall give it elsewhere.

[105]With the aid of the notations that have now been explained, we may return to §00 and rewrite the examples I–IV of that section as they might appear in some appropriate formalized language.

For this purpose let a and b be variables whose range is human beings. Let v be a variable whose range is words (taking, let us say for definiteness, any finite sequence of letters of the English alphabet as a word). Let B denote the relation of being a brother of. Let S denote the relation of having as surname. Let ϱ and σ denote the human beings Richard and Stanley respectively, and let τ denote the word "Thompson." Then the three premisses and the conclusion of I may be expressed as follows:

$$B(a, b) \supset_{ab} . S(a, v) \equiv_v S(b, v)$$
$$B(\varrho, \sigma)$$
$$S(\sigma, \tau)$$
$$S(\varrho, \tau)$$

Further, let z and w be variables whose range is complex numbers, and x a variable whose range is real numbers. Let R denote the relation of having real positive ratio, and let A denote the relation of having as amplitude. Then the premisses and conclusion of II may be expressed as follows:

$$R(z, w) \supset_{zw} . A(z, x) \equiv_x A(w, x)$$
$$R(i - \sqrt{3}/3, \omega)$$
$$A(\omega, 2\pi/3)$$
$$A(i - \sqrt{3}/3, 2\pi/3)$$

Here it is obvious that the relation of having real positive ratio is capable of being analyzed, so that instead of $R(z, w)$ we might have written, e.g.:

$$(\exists x)[x > 0][z = xw]$$

Likewise the relation of having as amplitude or (in I) the relation of being a brother of might have received some analysis. But these analyses are not relevant to the validity of the reasoning in these particular examples. And they are, moreover, in no way final or absolute; e.g., instead of analyzing the relation of having real positive ratio, we might with equal right take it as fundamental and analyze instead the relation of being greater than, in such a way that, in place of $x > y$ would be written $R(x - y, 1)$.

In the same way, for III and IV, we make no analysis of the singulary propositional functions of having a portrait seen by me, of having assassinated Abraham Lincoln, and of having invented the wheeled vehicle, but let them be denoted just by P, L and W respectively. Then if β denotes John Wilkes Booth, the premisses and conclusion of III may be expressed thus:

$$P(\beta) \qquad L(\beta) \qquad (\exists a)[P(a)L(a)]$$

And the premisses and fallacious conclusion of IV thus:

$$(\exists a)P(a) \qquad (\exists a)W(a) \qquad (\exists a)[P(a)W(a)]$$

When so rewritten, the false appearance of analogy between III and IV disappears. It was due to the logically irregular feature of English grammar by which "somebody" is construed as a substantive.

be used as a means to assert this for all real numbers x; or the assertion of the inequality $x^2 + y^2 \geqq 2xy$ may be used as a means to assert that for any real numbers x and y the sum of the squares is greater than or equal to twice the product.

It is clear that, in a formalized language, if universal quantification is available, it is unnecessary to allow the assertion of expressions containing free variables. E.g., the assertion of the propositional form

$$x^2 + y^2 \geqq 2xy$$

could be replaced by assertion of the sentence

$$(x)(y) \mathbin{\centerdot} x^2 + y^2 \geqq 2xy.$$

But on the other hand it is not possible to dispense with quantifiers in a formalized language merely by allowing the assertion of propositional forms, because, e.g., such assertions as that of

$$\sim(x)(y) \mathbin{\centerdot} \sin(x+y) = \sin x + \sin y,$$ [106]

or that of

$$(y)[|x| \leqq |y|] \supset_x \mathbin{\centerdot} x = 0,$$

could not be reproduced.

Consequently it has been urged with some force that the device of asserting propositional forms constitutes an unnecessary duplication of ways of expressing the same thing, and ought to be eliminated from a formalized language.[107] Nevertheless it appears that the retention of this device often facilitates the setting up of a formalized language by simplifying certain details; and it also renders more natural and obvious the separation of such restricted systems as propositional calculus (Chapter I) or functional calculus of first order (Chapter III) out from more comprehensive systems of which they are part. In the development which follows we shall therefore make free use of the assertion of propositional forms. However, in the case of such systems as functional calculus of order ω (Chapter VI) or Zermelo set theory (Chapter XI), after a first treatment employing the device in question we shall sketch briefly a reformulation that avoids it.

[106]This assertion (which is correct, and must sometimes be made to beginners in trigonometry) is of course to be distinguished from the different (and erroneous) assertion of

$$\sim \mathbin{\centerdot} \sin(x+y) = \sin x + \sin y.$$

[107]The proposal to do this was made by Russell in his introduction to the second edition of *Principia Mathematica* (1925). The elimination was actually carried out by Quine in his *Mathematical Logic* (1940), and simplifications of Quine's method were effected in papers by F. B. Fitch and by G. D. W. Berry in *The Journal of Symbolic Logic* (vol. 6 (1941), pp. 18–22, 23–27).

07. The logistic method. In order to set up a formalized language we must of course make use of a language already known to us, say English or some portion of the English language, stating in that language the vocabulary and rules of the formalized language. This procedure is analogous to that familiar to the reader in language study—as, e.g., in the use of a Latin grammar written in English[108]—but differs in the precision with which the rules are stated, in the avoidance of irregularities and exceptions, and in the leading idea that the rules of the language embody a theory or system of logical analysis (cf. §00).

This device of employing one language in order to talk about another is one for which we shall have frequent occasion not only in setting up formalized languages but also in making theoretical statements as to what can be done in a formalized language, our interest in formalized languages being less often in their actual and practical use as languages than in the general theory of such use and in its possibilities in principle. Whenever we employ a language in order to talk about some language (itself or another[109]), we shall call the latter language the *object language*, and we shall call the former the *meta-language*.[110]

In setting up a formalized language we first employ as meta-language a certain portion of English. We shall not attempt to delimit precisely this portion of the English language, but describe it approximately by saying that it is just sufficient to enable us to give general directions for the manip-

[108]It is worth remark in passing that this same procedure also enters into the learning of a first language, being a necessary supplement to the method of learning by example and imitation. Some part of the language must first be learned approximately by the method of example and imitation; then this imprecisely known part of the language is applied in order to state rules of the language (and perhaps to correct initial misconceptions); then the known part of the language may be extended by further learning by example and imitation, and so on in alternate steps, until some precision in knowledge of the language is reached.

There is no reason in principle why a first language, learned in this way, should not be one of the formalized languages of this book, instead of one of the natural languages. (But of course there is the practical reason that these formalized languages are ill adapted to purposes of facility of communication.)

[109]The employment of a language to talk about that same language is clearly not appropriate as a method of setting up a formalized language. But once set up, a formalized language with adequate means of expression may be capable of use in order to talk about that language itself; and in particular the very setting up of the language may afterwards be capable of restatement in that language. Thus it may happen that object language and meta-language are the same, a situation which it will be important later to take into account.

[110]The distinction is due to David Hilbert, who, however, speaks of "Mathematik" (mathematics) and "Metamathematik" (metamathematics) rather than "object language" and "meta-language." The latter terms, or analogues of them in Polish or German, are due to Alfred Tarski and Rudolf Carnap, by whom especially (see footnotes 131, 140) the subjects of *syntax* and *semantics* have been developed.

ulation of concrete physical objects (each instance or occurrence of one of the symbols of the language being such a concrete physical object, e.g., a mass of ink adhering to a bit of paper). It is thus a language which deals with matters of everyday human experience, going beyond such matters only in that no finite upper limit is imposed on the number of objects that may be involved in any particular case, or on the time that may be required for their manipulation according to instructions. Those additional portions of English are excluded which would be used in order to treat of infinite classes or of various like abstract objects which are an essential part of the subject matter of mathematics.

Our procedure is not to define the new language merely by means of translations of its expressions (sentences, names, forms) into corresponding English expressions, because in this way it would hardly be possible to avoid carrying over into the new language the logically unsatisfactory features of the English language. Rather, we begin by setting up, in abstraction from all considerations of meaning, the purely formal part of the language, so obtaining an uninterpreted calculus or *logistic system*. In detail, this is done as follows.

The vocabulary of the language is specified by listing the single symbols which are to be used.[111] These are called the *primitive symbols* of the language,[112] and are to be regarded as indivisible in the double sense that (A) in

[111]Notice that we use the term "language" in such a sense that a given language has a given and uniquely determined vocabulary. E.g., the introduction of one additional symbol into the vocabulary is sufficient to produce a new and different language. (Thus the English of 1849 is not the same language as the English of 1949, though it is convenient to call them by the same name, and to distinguish, by specifying the date, only in cases where the distinction is essential.)

[112]The fourfold classification of the primitive notations of a formalized language into constants, variables, connectives, and operators is due in substance to J. v. Neumann in the *Mathematische Zeitschrift*, vol. 26 (1927), see pp. 4–6. He there adds a fifth category, composed of association-showing symbols such as parentheses and brackets. Our terms "connective" and "operator" correspond to his "Operation" and "Abstraktion" respectively.

Though there is a possibility of notations not falling in any of von Neumann's categories, such have seldom been used, and for nearly all formalized languages that have actually been proposed the von Neumann classification of primitive notations suffices. Many formalized languages have primitive notations of all four (or five) kinds, but it does not appear that this is indispensable, even for a language intended to be adequate for the expression of mathematical ideas generally.

As an interesting example of a (conceivable) notation not in any of the von Neumann categories, we mention the question of a notation by means of which from a name of a class would be formed an expression playing the role of a variable with that class as its range. Provision might perhaps be made for the formation from any class name of an infinite number of expressions playing the roles of different variables with the class as their range. But these expressions would have to differ from variables in the sense of §02 not only in being composite expressions rather than single symbols but also in the

setting up the language no use is made of any division of them into parts and (B) any finite linear sequence of primitive symbols can be regarded *in only one way* as such a sequence of primitive symbols.[113] A finite linear sequence of primitive symbols is called a *formula*. And among the formulas, rules are given by which certain ones are designated as *well-formed formulas* (with the intention, roughly speaking, that only the well-formed formulas are to be regarded as being genuinely expressions of the language).[114] Then certain among the well-formed formulas are laid down as *axioms*. And finally (primitive) *rules of inference* (or *rules of procedure*) are laid down, rules according to which, from appropriate well-formed formulas as *premisses*, a well-formed formula is *immediately inferred*[115] as *conclusion*. (So long as we are dealing only with a logistic system that remains uninterpreted, the terms *premiss*, *immediately infer*, *conclusion* have only such meaning as is conferred upon them by the rules of inference themselves.)

A finite sequence of one or more well-formed formulas is called a *proof* if each of the well-formed formulas in the sequence either is an axiom or is immediately inferred from preceding well-formed formulas in the sequence by means of one of the rules of inference. A proof is called a proof *of* the last well-formed formula in the sequence, and the *theorems* of the logistic system

possibility that the range might be empty. A language containing such a notation has never been set up and studied in detail and it is therefore not certain just what is feasible. (A suggestion which seems to be in this direction was made by Beppo Levi in *Universidad Nacional de Tucumán, Revista*, ser. A vol. 3 no. 1 (1942), pp. 13–78.)

The use in Chapter X of variables with subscripts indicating the range of the variable (the type) is not an example of a notation of the kind just described. For the variable, letter and subscript together, is always treated as a single primitive symbol.

[113]In practice, condition (B) usually makes no difficulty. Though the (written) symbols adopted as primitive symbols may not all consist of a single connected piece, it is ordinarily possible to satisfy (B), if not otherwise, by providing that a sequence of primitive symbols shall be written with spaces between the primitive symbols of fixed width and wider than the space at any place within a primitive symbol.

The necessity for (B), and its possible failure, were brought out by a criticism by Stanisław Leśniewski against the paper of von Neumann cited in the preceding footnote. See von Neumann's reply in *Fundamenta Mathematicae*, vol. 17 (1931), pp. 331–334, and Leśniewski's final word in the matter in an offprint published in 1938 as from *Collectanea Logica*, vol. 1 (cf. *The Journal of Symbolic Logic*, vol. 5, p. 83).

[114]The restriction to one dimension in combining the primitive symbols into expressions of the language is convenient, and non-essential. Two-dimensional arrangements are of course possible, and are familiar especially in mathematical notations, but they may always be reduced to one dimension by a change of notation. In particular the notation of Frege's *Begriffsschrift* relies heavily on a two-dimensional arrangement; but because of the difficulty of printing it this notation was never adopted by any one else and has long since been replaced by a one-dimensional equivalent.

[115]No reference to the so-called immediate inferences of traditional logic is intended. We term the inferences *immediate* in the sense of requiring only one application of a rule of inference—not in the traditional sense of (among other things) having only one premiss.

are those well-formed formulas of which proofs exist.[116] As a special case, each axiom of the system is a theorem, that finite sequence being a proof which consists of a single well-formed formula, the axiom alone.

The scheme just described—viz. the primitive symbols of a logistic system, the rules by which certain formulas are determined as well-formed (following Carnap let us call them the *formation rules* of the system), the rules of inference, and the axioms of the system—is called the *primitive basis* of the logistic system.[117]

In defining a logistic system by laying down a primitive basis, we employ as meta-language the restricted portion of English described above. In addition to this restriction, or perhaps better as part of it, we impose requirements of *effectiveness* as follows: (I) the specification of the primitive symbols shall be effective in the sense that there is a method by which, whenever a symbol is given, it can always be determined effectively whether or not it is one of the primitive symbols; (II) the definition of a well-formed formula

[116]Following Carnap and others, we use the term "language" in such a sense that for any given language there is one fixed notion of a proof in that language. Thus the introduction of one additional axiom or rule of inference, or a change in an axiom or rule of inference, is sufficient to produce a new and different language.

(An alternative, which might be thought to accord better with the everyday use of the word "language," would be to define a "language" as consisting of primitive symbols and a definition of well-formed formula, together with an *interpretation* (see below), and to take the axioms and rules of inference as constituting a "logic" for the language. Instead of speaking of an interpretation as *sound* or *unsound* for a logistic system (see below), we would then speak of a logic as being sound or unsound for a language. Indeed this alternative may have some considerations in its favor. But we reject it here, partly because of reluctance to change a terminology already fairly well established, partly because the alternative terminology leads to a twofold division in each of the subjects of syntax and semantics (§§08, 09)—according as they treat of the object language alone or of the object language together with a logic for it — which, especially in the case of semantics, seems unnatural, and of little use so far as can now be seen.)

[117]Besides these minimum essentials, the primitive basis may also include other notions introduced in order to use them in defining a well-formed formula or in stating the rules of inference. In particular the primitive symbols may be divided in some way into different categories: e.g., they may be classified as *primitive constants*, *variables*, and *improper symbols*, or various categories may be distinguished of primitive constants, of variables, or of improper symbols. The variables and the primitive constants together are usually called *proper symbols*. Rules may be given for distinguishing an occurrence of a variable in a well-formed formula as being a *free occurrence* or a *bound occurrence*, well-formed formulas being then classified as *forms* or *constants* according as they do or do not contain a free occurrence of a variable. Also rules may be given for distinguishing certain of the forms as *propositional forms*, and certain of the constants as *sentences*. In doing all this, the terminology often is so selected that, when the logistic system becomes a language by adoption of one of the intended principal interpretations (see below), the terms *primitive constant, variable, improper symbol, proper symbol, free, bound, form, constant, propositional form, sentence* come to have meanings in accord with the informal semantical explanations of §§02–06.

The *primitive basis* of a formalized language, or interpreted logistic system, is obtained by adding the semantical rules (see below) to the primitive basis of the logistic system.

shall be effective in the sense that there is a method by which, whenever a formula is given, it can always be determined effectively whether or not it is well-formed; (III) the specification of the axioms shall be effective in the sense that there is a method by which, whenever a well-formed formula is given, it can always be determined effectively whether or not it is one of the axioms; (IV) the rules of inference, taken together, shall be effective in the strong sense that there is a method by which, whenever a proposed immediate inference is given of one well-formed formula as conclusion from others as premisses, it can always be determined effectively whether or not this proposed immediate inference is in accordance with the rules of inference.

(From these requirements it follows that the notion of a proof is effective in the sense that there is a method by which, whenever a finite sequence of well-formed formulas is given, it can always be determined effectively whether or not it is a proof. But the notion of a theorem is not necessarily effective in the sense of existence of a method by which, whenever a well-formed formula is given, it can always be determined whether or not it is a theorem—for there may be no certain method by which we can always either find a proof or determine that none exists. This last is a point to which we shall return later.)

As to requirement (I), we suppose that we are able always to determine about two given symbol-occurrences whether or not they are occurrences of the same symbol (thus ruling out by assumption such difficulties as that of illegibility). Therefore, if the number of primitive symbols is finite, the requirement may be satisfied just by giving the complete list of primitive symbols, written out in full. Frequently, however, the number of primitive symbols is infinite. In particular, if there are variables, it is desirable that there should be an infinite number of different variables of each kind because, although in any one well-formed formula the number of different variables is always finite, there is hardly a way to determine a finite upper limit of the number of different variables that may be required for some particular purpose in the actual use of the logistic system. When the number of primitive symbols is infinite, the list cannot be written out in full, but the primitive symbols must rather be fixed in some way by a statement of finite length in the meta-language. And this statement must be such as to conform to (I).

A like remark applies to (III). If the number of axioms is finite, the requirement can be satisfied by writing them out in full. Otherwise the axioms must be specified in some less direct way by means of a statement of finite length in the meta-language, and this must be such as to conform to (III). It may be thought more elegant or otherwise more satisfactory that the number of axioms be finite; but we shall see that it is sometimes convenient to make use of an infinite number of axioms, and no conclusive objections appear to doing so if requirements of effectiveness are obeyed.

We have assumed the reader's understanding of the general notion of effectiveness, and indeed it must be considered as an informally familiar mathematical notion, since it is involved in mathematical problems of a frequently occurring kind, namely, problems to find a method of computation, i.e., a method by which to determine a number, or other thing, effectively.[118] We shall not try to give here a rigorous definition of effectiveness, the informal notion being sufficient to enable us, in cases we shall meet, to distinguish given methods as effective or non-effective.[119]

The requirements of effectiveness are (of course) not meant in the sense that a structure which is analogous to a logistic system except that it fails to satisfy these requirements may not be useful for some purposes or that it is forbidden to consider such—but only that a structure of this kind is unsuitable for use or interpretation as a language. For, however indefinite or imprecisely fixed the common idea of a language may be, it is at least fundamental to it that a language shall serve the purpose of communication. And to the extent that requirements of effectiveness fail, the purpose of communication is defeated.

Consider, in particular, the situation which arises if the definition of well-

[118]A well-known example from topology is the problem (still unsolved even for elementary manifolds of dimensionalities above 2) to find a method of calculating about any two closed simplicial manifolds, given by means of a set of incidence relations, whether or not they are homeomorphic—or, as it is often phrased, the problem to find a complete classification of such manifolds, or to find a complete set of invariants.

As another example, Euclid's algorithm, in the domain of rational integers, or in certain other integral domains, provides an effective method of calculating for any two elements of the domain their greatest common divisor (or highest common factor).

In general, an effective method of calculating, especially if it consists of a sequence of steps with later steps depending on results of earlier ones, is called an *algorithm*. (This is the long established spelling of this word, and should be preserved in spite of any considerations of etymology.)

[119]For a discussion of the question and proposal of a rigorous definition see a paper by the present writer in the *American Journal of Mathematics*, vol. 58 (1936), pp. 345–363, especially §7 thereof. The notion of effectiveness may also be described by saying that an effective method of computation, or algorithm, is one for which it would be possible to build a computing machine. This idea is developed into a rigorous definition by A. M. Turing in the *Proceedings of the London Mathematical Society*, vol. 42 (1936–1937), pp. 230–265 (and vol. 43 (1937), pp. 544–546). See further: S. C. Kleene in the *Mathematische Annalen*, vol. 112 (1936), pp. 727–742; E. L. Post in *The Journal of Symbolic Logic*, vol. 1 (1936), pp. 103–105; A. M. Turing in *The Journal of Symbolic Logic*, vol. 2 (1937), pp. 153–163; Hilbert and Bernays, *Grundlagen der Mathematik*, vol. 2 (1939), Supplement II.

formedness is non-effective. There is then no certain means by which, when an alleged expression of the language is uttered (spoken or written), say as an asserted sentence, the auditor (hearer or reader) may determine whether it is well-formed, and thus whether any actual assertion has been made.[120] Therefore the auditor may fairly demand a proof that the utterance is well-formed, and until such proof is provided may refuse to treat it as constituting an assertion. This proof, which must be added to the original utterance in order to establish its status, ought to be regarded, it seems, as part of the utterance, and the definition of well-formedness ought to be modified to provide this, or its equivalent. When such modification is made, no doubt the non-effectiveness of the definition will disappear; otherwise it would be open to the auditor to make further demand for proof of well-formedness.

Again, consider the situation which arises if the notion of a proof is non-effective. There is then no certain means by which, when a sequence of formulas has been put forward as a proof, the auditor may determine whether it is in fact a proof. Therefore he may fairly demand a proof, in any given case, that the sequence of formulas put forward is a proof; and until this supplementary proof is provided, he may refuse to be convinced that the alleged theorem is proved. This supplementary proof ought to be regarded, it seems, as part of the whole proof of the theorem, and the primitive basis of the logistic system ought to be so modified as to provide this, or its equivalent.[121] Indeed it is essential to the idea of a proof that, to any one who admits the presuppositions on which it is based, a proof carries final

[120]To say that an assertion has been made if there is a meaning evades the issue unless an effective criterion is provided for the presence of meaning. An understanding of the language, however reached, must include effective ability to recognize meaningfulness (in some appropriate sense), and in the purely formal aspect of the language, the logistic system, this appears as an effective criterion of well-formedness.

[121]Perhaps at first sight it will be thought that the proof as so modified might consist of something more than merely a sequence of well-formed formulas. For instance there might be put in at various places indications in the meta-language as to which rule of inference justifies the inclusion of a particular formula as immediately inferred from preceding formulas, or as to which preceding formulas are the premisses of the immediate inference.

But as a matter of fact we consider this inadmissible. For our program is to express proofs (as well as theorems) in a fully formalized object language, and as long as any part of the proof remains in an unformalized meta-language the logical analysis must be held to be incomplete. A statement in the meta-language, e.g., that a particular formula is immediately inferred from particular preceding formulas—if it is not superfluous and therefore simply omissible—must always be replaced in some way by one or more sentences of the object language.

Though we use a meta-language to set up the object language, we require that, once set up, the object language shall be an independent language capable, without continued support and supplementation from the meta-language, of expressing those things for which it was designed.

conviction. And the requirements of effectiveness (I)–(IV) may be thought of as intended just to preserve this essential characteristic of proof.

After setting up the logistic system as described, we still do not have a formalized language until an *interpretation* is provided. This will require a more extensive meta-language than the restricted portion of English used in setting up the logistic system. However, it will proceed not by translations of the well-formed formulas into English phrases but rather by *semantical rules* which, in general, *use* rather than *mention* English phrases (cf. §08), and which shall prescribe for every well-formed formula either how it denotes[122] (so making it a proper name in the sense of §01) or else how it has values[122] (so making it a form in the sense of §02).

In view of our postulation of two truth-values (§04), we impose the requirement that the semantical rules, if they are to be said to provide an interpretation, must be such that the axioms denote truth-values (if they are names) or have always truth-values as values (if they are forms), and the same must hold of the conclusion of any immediate inference if it holds of the premisses. In using the formalized language, only those well-formed formulas shall be capable of being asserted which denote truth-values (if they are names) or have always truth-values as values (if they are forms); and only those shall be capable of being rightly asserted which denote truth (if they are names) or have always the value truth (if they are forms). Since it is intended that proof of a theorem shall justify its assertion, we call an interpretation of a logistic system *sound* if, under it, all the axioms either denote truth or have always the value truth, and if further the same thing holds of the conclusion of any immediate inference if it holds of the premisses. In the contrary case we call the interpretation *unsound*. A formalized language is called sound or unsound according as the interpretation by which it is obtained from a logistic system is sound or unsound. And an unsound interpretation or an unsound language is to be rejected.

[122]Because of the possibility of misunderstanding, we avoid the wordings "what it denotes" and "what values it has."

For example, in one of the logistic systems of Chapter X we may find a well-formed formula which, under a principal interpretation of the system, is interpreted as denoting: the greatest positive integer n such that $1 + n^r$ is prime, r being chosen as the least even positive integer corresponding to which there is such a greatest positive integer n. Thus the semantical rules do in a sense determine what this formula denotes, but the remoteness of this determination is measured by the difficulty of the mathematical problem which must be solved in order to identify in some more familiar manner the positive integer which the formula denotes, or even to say whether or not the formula denotes 1.

Again in the logistic system F^{1h} of Chapter III (or A^0 of Chapter V) taken with its principal interpretation, there is a well-formed formula which, according to the semantical rules, denotes the truth-value thereof that every even number greater than 2 is the sum of two prime numbers. To say that the semantical rules determine what this formula denotes seems to anticipate the solution of a famous problem, and it may be better to think of the rules as determining indirectly what the formula expresses.

In assigning how (rather than what) a name denotes we are in effect fixing its sense, and in assigning how a form has values we fix the correspondence of sense values of the form (see footnote 27) to concepts of values of its variables. (This statement of the matter will be sufficiently precise for our present purposes, though it remains vague to the extent that we have left the meaning of "sense" uncertain—see footnotes 15, 37.)

It will be seen in particular examples below (such as rules a–g of §10, or rules a–f of §30, or rules α–ζ of §30) that in most of our semantical rules the explicit assertion is that certain well-formed formulas, usually on certain conditions, are to denote certain things or to have certain values. However, as just explained, this explicit assertion is so chosen as to give implicitly also the sense or the sense values. No doubt a fuller treatment of semantics must have additional rules stating the sense or the sense values explicitly, but this would take us into territory still unexplored.

(The requirements, and the definition of soundness, in the foregoing paragraph are based on two truth-values. They are satisfactory for every formalized language which will receive substantial consideration in this book. But they must be modified correspondingly, in case the scheme of two truth-values is modified—cf. the remark in §19.)

The semantical rules must in the first instance be stated in a presupposed and therefore unformalized meta-language, here taken to be ordinary English. Subsequently, for their more exact study, we may formalize the meta-language (using a presupposed meta-meta-language and following the method already described for formalizing the object language) and restate the semantical rules in this formalized language. (This leads to the subject of *semantics* (§09).)

As a condition of rigor, we require that the proof of a theorem (of the object language) shall make no reference to or use of any interpretation, but shall proceed purely by the rules of the logistic system, i.e., shall be a *proof* in the sense defined above for logistic systems. Motivation for this is threefold, three rather different approaches issuing in the same criterion. In the first place this may be considered a more precise formulation of the traditional distinction between form and matter (§00) and of the principle that the validity of an argument depends only on the form—the form of a proof in a logistic system being thought of as something common to its meanings under various interpretations of the logistic system. In the second place this represents the standard mathematical requirement of rigor that a proof must proceed purely from the axioms without use of anything (however supposedly obvious) which is not stated in the axioms; but this requirement is here modified and extended as follows: that a proof must proceed purely from the axioms by the rules of inference, without use of anything not stated in the axioms or any method of inference not validated by the rules. Thirdly there is the motivation that the logistic system is relatively secure and definite, as compared to interpretations which we may wish to adopt, since it is based on a portion of English as meta-language so elementary and restricted that its essential reliability can hardly be doubted if mathematics is to be possible at all.

It is also important that a proof which satisfies our foregoing condition of rigor must then hold under any interpretation of the logistic system, so that there is a resulting economy in proving many things under one process.[123] The extent of the economy is just this, that proofs identical in form but different in matter need not be repeated indefinitely but may be summarized once for all.[124]

Though retaining our freedom to employ any interpretation that may be found useful, we shall indicate, for logistic systems set up in the following chapters, one or more interpretations which we have especially in mind for the system and which shall be called the *principal interpretations*.

The subject of formal logic, when treated by the method of setting up a formalized language, is called *symbolic logic*, or *mathematical logic*, or *logistic*.[125] The method itself we shall call the *logistic method*.

[123]This remark has now long been familiar in connection with the axiomatic method in mathematics (see below).

[124]The summarizing of a proof according to its form may indeed be represented to a certain extent, by the use of variables, within one particular formalized language. But, because of restricted ranges of the variables, such summarizing is less comprehensive in its scope than is obtained by formalizing in a logistic system whose interpretation is left open.

The procedure of formalizing a proof in a logistic system and then employing the formalized proof under various different interpretations of the system may be thought of as a mere device for brevity and convenience of presentation, since it would be possible instead to repeat the proof in full each time it were used with a new interpretation. From this point of view such use of the meta-language may be allowed as being in principle dispensable and therefore not violating the demand (footnote 121) for an independent object language.

(If on the other hand we wish to deal rigorously with the notion of logical form of proofs, this must be in a particular formalized language, namely a formalized meta-language of the language of the proofs. Under the program of §02 each variable of this meta-language will have a fixed range assigned in advance, according, perhaps, with the theory of types. And the notion of form which is dealt with must therefore be correspondingly restricted, it would seem, to proofs of a fixed class, taking no account of sameness of form between proofs of this class and others (in the same or a different language). Presumably our informal references to logical form in the text are to be modified in this way before they can be made rigorous—cf. §09.)

[125]The writer prefers the term "mathematical logic," understood as meaning logic treated by the mathematical method, especially the formal axiomatic or logistic method. But both this term and the term "symbolic logic" are often applied also to logic as treated by a less fully formalized mathematical method, in particular to the "algebra of logic," which had its beginning in the publications of George Boole and Augustus De Morgan in 1847, and received a comprehensive treatment in Ernst Schröder's *Vorlesungen über die Algebra der Logik* (1890–1905). The term "logistic" is more definitely restricted to the method described in this section, and has also the advantage that it is more easily made an adjective. (Sometimes "logistic" has been used with special reference to the school of Russell or to the Frege-Russell doctrine that mathematics is a branch of logic—cf. footnote 545. But we shall follow the more common usage which attaches no such special meaning to this word.)

Familiar in mathematics is the *axiomatic method*, according to which a branch of mathematics begins with a list of *undefined terms* and a list of assumptions, or *postulates* involving these terms, and the theorems are to be derived from the postulates by the methods of formal logic.[126] If the last phrase is left unanalyzed, formal logic being presupposed as already known, we shall say that the development is by the *informal axiomatic method*.[127] And in the opposite case we shall speak of the *formal axiomatic method*.

The formal axiomatic method thus differs from the logistic method only in the following two ways:

(1) In the logistic system the primitive symbols are given in two categories: the *logical primitive symbols*, thought of as pertaining to the underlying logic, and the *undefined terms*, thought of as pertaining to the particular branch of mathematics. Correspondingly the axioms are divided into two categories: the *logical axioms*, which are well-formed formulas containing only logical primitive symbols, and the *postulates*,[128] which involve also the undefined terms and are thought of as determining the special branch of mathematics. The rules of inference, to accord with the usual conception of

"Logica mathematica" and "logistica" were both used by G. W. v. Leibniz along with "calculus ratiocinator," and many other synonyms, for the calculus of reasoning which he proposed but never developed beyond some brief and inadequate (though significant) fragments. Boole used the expressions "mathematical analysis of logic," "mathematical theory of logic." "Mathematische Logik" was used by Schröder in 1877, "matématičéskaá logika" (Russian) by Platon Poretsky in 1884, "logica matematica" (Italian) by Giuseppe Peano in 1891. "Symbolic logic" seems to have been first used by John Venn (in *The Princeton Review*, 1880), though Boole speaks of "symbolical reasoning." The word "logistic" and its analogues in other languages originally meant the art of calculation or common arithmetic. Its modern use for mathematical logic dates from the International Congress of Philosophy of 1904, where it was proposed independently by Itelson, Lalande, and Couturat. Other terms found in the literature are "logischer Calcul" (Gottfried Ploucquet 1766), "algorithme logique" (G. F. Castillon 1805), "calculus of logic" (Boole 1847), "calculus of inference" (De Morgan 1847), "logique algorithmique" (J. R. L. Delboeuf 1876), "Logikkalkul" (Schröder 1877), "theoretische Logik" (Hilbert and Ackermann 1928). Also "Boole's logical algebra" (C. S. Peirce 1870), "logique algébrique de Boole" (Louis Liard 1877), "algebra of logic" (Alexander Macfarlane 1879, C. S. Peirce 1880).

[126]Accounts of the axiomatic method may of course be found in many mathematical textbooks and other publications. An especially good exposition is in the Introduction to Veblen and Young's *Projective Geometry*, vol. 1 (1910).

[127]This is the method of most mathematical treatises, which proceed axiomatically but are not specifically about logic—in particular of Veblen and Young (preceding footnote).

[128]The words "axiom" and "postulate" have been variously used, either as synonymous or with varying distinctions between them, by the present writer among others. In this book, however, the terminology here set forth will be followed closely.

the axiomatic method, must all be taken as belonging to the underlying logic. And, though they may make reference to particular undefined terms or to classes of primitive symbols which include undefined terms, they must not involve anything which, subjectively, we are unwilling to assign to the underlying logic rather than to the special branch of mathematics.[129]

(2) In the interpretation the semantical rules are given in two categories. Those of the first category fix those general aspects of the interpretation which may be assigned, or which we are willing to assign, to the underlying logic. And the rules of the second category determine the remainder of the interpretation. The consideration of different representations or interpretations of the system of postulates, in the sense of the informal axiomatic method, corresponds here to varying the semantical rules of the second category while those of the first category remain fixed.

08. Syntax. The study of the purely formal part of a formalized language in abstraction from the interpretation, i.e., of the logistic system, is called *syntax*, or, to distinguish it from the narrower sense of "syntax" as concerned with the formation rules alone,[130] *logical syntax*.[131] The meta-language used in order to study the logistic system in this way is called the *syntax language*.[131]

We shall distinguish between *elementary syntax* and *theoretical syntax*.

The elementary syntax of a language is concerned with setting up the logistic system and with the verification of particular well-formed formulas,

[129]Ordinarily, e.g., it would be allowed that the rules of inference should treat differently two undefined terms intended one to denote an individual and one to denote a class of individuals, or two undefined terms intended to denote a class of individuals and a relation between individuals; but not that the rules should treat differently two undefined terms intended both to denote a class of individuals. But no definitive controlling principle can be given.

The subjective and essentially arbitrary character of the distinction between what pertains to the underlying logic and what to the special branch of mathematics is illustrated by the uncertainty which sometimes arises, in treating a branch of mathematics by the informal axiomatic method, as to whether the sign of equality is to be considered as an undefined term (for which it is necessary to state postulates). Again it is illustrated by Zermelo's treatment of axiomatic set theory in his paper of 1908 (cf. Chapter XI) in which, following the informal axiomatic method, he introduces the relation ϵ of membership in a set as an undefined term, though this same relation is usually assigned to the underlying logic when a branch of mathematics is developed by the informal axiomatic method.

[130]Cf. footnote 116.

[131]The terminology is due to Carnap in his *Logische Syntax der Sprache* (1934), translated into English (with some additions) as *The Logical Syntax of Language* (1937). In connection with this book see also reviews of it by Saunders MacLane in the *Bulletin of the American Mathematical Society*, vol. 44 (1938), pp. 171–176, and by S. C. Kleene in *The Journal of Symbolic Logic*, vol. 4 (1939), pp. 82–87.

axioms, immediate inferences, and proofs as being such. The syntax language is the restricted portion of English which was described in the foregoing section, or a correspondingly restricted formalized meta-language, and the requirements of effectiveness, (I)–(IV), must be observed. The demonstration of derived rules and theorem schemata, in the sense of §§12, 33, and their application in particular cases are also considered to belong to elementary syntax, provided that the requirement of effectiveness holds which is explained in §12.

Theoretical syntax, on the other hand, is the general mathematical theory of a logistic system or systems and is concerned with all the consequences of their formal structure (in abstraction from the interpretation). There is no restriction imposed as to what is available in the syntax language, and requirements of effectiveness are or may be abandoned. Indeed the syntax language may be capable of expressing the whole of extant mathematics. But it may also sometimes be desirable to use a weaker syntax language in order to exhibit results as obtained on this weaker basis.

Like any branch of mathematics, theoretical syntax may, and ultimately must, be studied by the axiomatic method. Here the informal and the formal axiomatic method share the important advantage that the particular character of the symbols and formulas of the object language, as marks upon paper, sounds, or the like, is abstracted from, and the pure theory of the structure of the logistic system is developed. But the formal axiomatic method—the syntax language being itself formalized according to the program of §07, by employing a meta-meta-language—has the additional advantage of exhibiting more definitely the basis on which results are obtained, and of clarifying the way and the extent to which certain results may be obtained on a relatively weaker basis.

In this book we shall be concerned with the task of formalizing an object language, and theoretical syntax will be treated informally, presupposing in any connection such general knowledge of mathematics as is necessary for the work at hand. Thus we do not apply even the informal axiomatic method to our treatment of syntax. But the reader must always understand that syntactical discussions are carried out in a syntax language whose formalization is ultimately contemplated, and distinctions based upon such formalization may be relevant to the discussion.

In such informal development of syntax, we shall think of the syntax language as being a different language from the object language. But the possibility is important that a sufficiently adequate object language may be capable of expressing its own syntax, so that in this case the ultimate formalization of the syntax language may if desired consist in identifying it with the object language.[132]

We shall distinguish between theorems *of* the object language and theorems of the syntax language (which often are theorems *about* the object language) by calling the latter *syntactical theorems*. Though we demonstrate syntactical theorems informally, it is contemplated that the ultimate formalization of the syntax language shall make them theorems in the sense of §07, i.e., theorems of the syntax language in the same sense as that in which we speak of theorems of the object language.

We shall require, as belonging to the syntax language: first, names of the various symbols and formulas of the object language; and secondly, variables which have these symbols and formulas as their values. The former will be called *syntactical constants*, and the latter, *syntactical variables*.[133]

As syntactical variables we shall use the following: as variables whose range is the primitive symbols of the object language, bold Greek small letters ($\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, etc.); as variables whose range is the primitive constants and variables of the object language—see footnote 117—bold roman small letters (**a**, **b**, **c**, etc.); as variables whose range is the formulas of the object language, bold Greek capitals ($\boldsymbol{\Gamma}$, $\boldsymbol{\Delta}$, etc.); and as variables whose range is the well-formed formulas of the object language, bold roman capitals (**A**, **B**, **C**, etc.). Wherever these bold letters are used in the following chapters, the reader must bear in mind that they are not part of the symbolic apparatus of the object language but that they belong to the syntax language and serve the purpose of talking about the object language. In use of the object language as an independent language, bold letters do not appear (just as English words never appear in the pure text of a Latin author though they do appear in a Latin grammar written in English).

As a preliminary to explaining the device to which we resort for syntactical constants, it is desirable first to consider the situation in ordinary

[132]Cf. footnote 109. In particular the developments of Chapter VIII show that the logistic system of Chapter VII is capable of expressing its own syntax if given a suitable interpretation different from the principal interpretation of Chapter VII, namely, an interpretation in which the symbols and formulas of the logistic system itself are counted among the individuals, as well as all finite sequences of such formulas, and the functional constant *S* is given an appropriate (quite complicated) interpretation, details of which may be made out by following the scheme of Gödel numbers that is set forth in Chapter VIII.

[133]Given the apparatus of syntactical variables, we could actually avoid the use of syntactical constants by resorting to appropriate circumlocutions in cases where syntactical constants would otherwise seem to be demanded. Indeed the example of the preceding footnote illustrates this, as will become clear in connection with the cited chapters. But it is more natural and convenient, especially in an informal treatment of syntax, to allow free use of syntactical constants.

English, with no formalized object language specially in question. We must take into account the fact that English is not a formalized language and the consequent uncertainty as to what are its formation rules, rules of inference, and semantical rules, the contents of ordinary English grammars and dictionaries providing only some incomplete and rather vague approximations to such rules. But, with such reservations as this remark implies, we go on to consider the use of English in making syntactical statements about the English language itself.

Frequently found in practice is the use of English words *autonymously* (to adopt a terminology due to Carnap), i.e., as names of those same words.[134] Examples are such statements as "The second letter of man is a vowel," "Man is monosyllabic," "Man is a noun with an irregular plural." Of course it is equivocal to use the same word, man, both as a proper name of the English word which is spelled by the thirteenth, first, fourteenth letters of the alphabet in that order, and as a common name (see footnote 6) of featherless plantigrade biped mammals[135]—but an equivocacy which, like many others in the natural languages, is often both convenient and harmless. Whenever there would otherwise be real doubt of the meaning, it may be removed by the use of added words in the sentence, or by the use of quotation marks, or of italics, as in: "The word man is monosyllabic"; "'Man' is monosyllabic"; "*Man* is monosyllabic."

Following the convenient and natural phraseology of Quine, we may distinguish between *use* and *mention* of a word or symbol. In "Man is a rational animal" the word "man" is used but not mentioned. In "The English translation of the French word *homme* has three letters" the word "man" is mentioned but not used. In "Man is a monosyllable" the word "man" is both mentioned and used, though used in an anomalous manner, namely autonymously.

Frege introduced the device of systematically indicating autonymy by quotation marks, and in his later publications (though not in the *Begriffsschrift*) words and symbols used autonymously are enclosed in single quotation marks in all cases. This has the effect that a word enclosed in single

[134]In the terminology of the Scholastics, use of a word as a name of itself, i.e., to denote itself as a word, was called *suppositio materialis*. Opposed to this as *suppositio formalis* was the use of a noun in its proper or ordinary meaning. This terminology is sometimes still convenient.

The various further distinctions of *suppositiones* are too cumbrous, and too uncertain, to be usable. All of them, like that between *suppositio materialis* and *formalis*, refer to peculiarities and irregularities of meaning which are found in many natural languages but which have to be eliminated in setting up a formalized language.

[135]To follow a definition found in *The Century Dictionary*.

quotation marks is to be treated as a different word from that without the quotation marks—as if the quotation marks were two additional letters in the spelling of the word—and equivocacy is thus removed by providing two different words to correspond to the different meanings. Many recent writers follow Frege in this systematic use of quotation marks, some using double quotation marks in this way, and others following Frege in using single quotation marks for the purpose, in order to reserve double quotation marks for their regular use as punctuation. As the reader has long since observed, Frege's systematic use of quotation marks is not adopted in this book.[136] But we may employ quotation marks or other devices from time to time, especially in cases in which there might otherwise be real doubt of the meaning.

To return to the question of syntactical constants for use in developing the syntax of a formalized object language, we find that there is in this case

[136]Besides being rather awkward in practice, such systematic use of quotation marks is open to some unfortunate abuses and misunderstandings. One of these is the misuse of quotation marks as if they denoted a function from things (of some category) to names of such things, or as if such a function might be employed at all without some more definite account of it. Related to this is the temptation to use in the role of a syntactical variable the expression obtained by enclosing a variable of an object language in quotation marks, though such an expression, correctly used, is not a variable of any kind, and not a form but a constant.

Also not uncommon is the false impression that trivial or self-evident propositions are expressed in such statements as the following: ' 'Snow is white' is true if and only if snow is white' (Tarski's example); ' 'Snow is white' means that snow is white'; ' 'Cape Town' is the [or a] name of Cape Town.'

This last misunderstanding may arise also in connection with autonymy. A useful method of combatting it is that of translation into another language (cf. a remark by C. H. Langford in *The Journal of Symbolic Logic*, vol. 2 (1937), p. 53). For example, the proposition that 'Cape Town' is the name of Cape Town would be conveyed thus to an Italian (whom we may suppose to have no knowledge of English): ' 'Cape Town' è il nome di Città del Capo.' Assuming, as we may, that the Italian words have exactly the same sense as the English words of which we use them as translations—in particular that 'Città del Capo' has the same sense as 'Cape Town' and that ' 'Cape Town' ' has the same sense in Italian as in English—we see that the Italian sentence and its English translation must express the very same proposition, which can no more be a triviality when conveyed in one language than it can in another.

The foregoing example may be clarified by recalling the remark of footnote 8 that the name relation is properly a ternary relation, and may be reduced to a binary relation only by fixing the language in a particular context. Thus we have the more explicit English sentences: ' 'Cape Town' is the English name of Cape Town'; ' 'Città del Capo' is the Italian name of Cape Town.' The Italian translations are: ' 'Cape Town' è il nome inglese di Città del Capo'; ' 'Città del Capo' è il nome italiano di Città del Capo.' Of the two propositions in question, the first one has a false appearance of obviousness when expressed in English, the illusion being dispelled on translation into Italian; the second one contrariwise does not seem obvious or trivial when expressed in English, but on translation into Italian acquires the appearance of being so.

(In the three preceding paragraphs of this footnote, we have followed Frege's systematic use of single quotation marks, and the paragraphs are to be read with that understanding. As explained, we do not follow this usage elsewhere.)

nothing equivocal in using the symbols and formulas of the object language autonymously in the syntax language, provided that care is taken that no formula of the object language is also a formula of the syntax language in any other wise than as an autonym. Therefore we adopt the following practice:

The primitive symbols of the object language will be used in the syntax language as names of themselves, and juxtaposition will be used for juxtaposition.[137]

This is the ordinary usage in mathematical writing, and has the advantage of being self-explanatory. Though we employ it only informally, it is also readily adapted to incorporation in a formalized syntax language[138] (and in fact more so than the convention of quotation marks).

As a precaution against equivocation, we shall hereafter avoid the practice—which might otherwise sometimes be convenient—of borrowing formulas of the object language for use in the syntax language (or other meta-language) with the same meaning that they have in the object language. Thus in all cases where a single symbol or a formula of the object language is found as a constituent in an English sentence, it is to be understood in accordance with the italicized rule above, i.e., autonymously.

Since we shall later often introduce conventions for abbreviating well-formed formulas of an object language, some additional explanations will be necessary concerning the use of syntactical variables and syntactical constants (and concerning autonymy) in connection with such abbreviations. These will be indicated in §11, where such abbreviations first appear. But, as explained in that section, the abbreviations themselves and therefore any special usages in connection with them are dispensable in principle, however necessary practically. In theoretical discussions of syntax and in particular in formalizing the syntax language, the matter of abbreviations of well-formed formulas may be ignored.

[137]I.e., juxtaposition will be used in the syntax language as a binary connective having the operation of juxtaposition as its associated function. Technically, some added notation is needed to show association, or some convention about the matter, such as that of association to the left (as in §11). But in practice, because of the associativity of juxtaposition, there is no difficulty in this respect.

[138]This is, of course, on the assumption that the syntax language is a different language from the object language.

If on the contrary a formalized language is to contain names of its own formulas, then a name of a formula must ordinarily not be that formula. E.g., a variable of a language must not be, in that same language, also a name of itself; for a proper name of a variable is no variable but a constant (as already remarked, in another connection, in footnote 136).

09. Semantics. Let us imagine the users of a formalized language, say a written language, engaged in writing down well-formed formulas of the language, and in assembling sequences of formulas which constitute chains of immediate inferences or, in particular, proofs. And let us imagine an observer of this activity who not only does not understand the language but refuses to believe that it is a language, i.e., that the formulas have meanings. He recognizes, let us say, the syntactical criteria by which formulas are accepted as well-formed, and those by which sequences of well-formed formulas are accepted as immediate inferences or as proofs; but he supposes that the activity is merely a game—analogous to a game of chess or, better, to a chess problem or a game of solitaire at cards—the point of the game being to discover unexpected theorems or ingenious chains of inferences, and to solve puzzles as to whether and how some given formula can be proved or can be inferred from other given formulas.[139]

To this observer the symbols have only such meaning as is given to them by the rules of the game—only such meaning as belongs, for example, to the various pieces at chess. A formula is for him like a position on a chess-board, significant only as a step in the game, which leads in accordance with the rules to various other steps.

All those things about the language which can be said to and understood by such an observer while he continues to regard the use of the language as merely a game constitute the (theoretical) syntax of the language. But those things which are intelligible only through an understanding that the well-formed formulas have meaning in the proper sense, e.g., that certain of them express propositions or that they denote or have values in certain ways, belong to the semantics of the language.

Thus the study of the interpretation of the language as an interpretation is called *semantics*.[140] The name is applied especially when the treatment is

[139]A comparison of the rules of arithmetic to those of a game of chess was made by J. Thomae (1898) and figures in the controversy between Thomae and Frege (1903–1908). The same comparison was used by Hermann Weyl (1924) in order to describe Hilbert's program of *metamathematics* or syntax of a mathematical object language.

[140]The name (or its analogue in Polish) was introduced by Tarski in a paper in *Przegląd Filozoficzny*, vol. 39 (1936), pp. 50–57, translated into German as "Grundlegung der wissenschaftlichen Semantik" in *Actes du Congrès International de Philosophie Scientifique* (1936). Other important publications in the field of semantics are: Tarski's *Pojęcie Prawdy w Językach Nauk Dedukcyjnych* (1933), afterwards translated into German (and an important appendix added) as "Der Wahrheitsbegriff in den formalisierten Sprachen" in *Studia Philosophica*, vol. 1 (1936) pp. 261–405; and Carnap's *Introduction to Semantics* (1942). Concerning Carnap's book see a review by the present writer in *The Philosophical Review*, vol. 52 (1943), pp. 298–304.

The word *semantics* has various other meanings, most of them older than that in question here. Care must be taken to avoid confusion on this account. But in this book the word will have always the one meaning, intended to be the same (or substantially

in a formalized meta-language. But in this book we shall not go beyond some unformalized semantical discussion, in ordinary English.

Theorems of the semantical meta-language will be called *semantical theorems*, and both semantical and syntactical theorems will be called *metatheorems*, in order to distinguish them from theorems of the object language.

As appears from the work of Tarski, there is a sense in which semantics can be reduced to syntax. Tarski has emphasized especially the possibility of finding, for a given formalized language, a purely syntactical property of the well-formed formulas which coincides in extension with the semantical property of being a true sentence. And in Tarski's *Wahrheitsbegriff*[141] the problem of finding such a syntactical property is solved for various particular formalized languages.[142] But like methods apply to the two semantical concepts of denoting and having values, so that syntactical concepts may be found which coincide with them in extension.[143] Therefore, if names expressing

so) as that in which it is used by Tarski, C. W. Morris (*Foundations of the Theory of Signs*, 1938), Carnap, G. D. W. Berry (*Harvard University, Summaries of Theses 1942*, pp. 330–334).

[141]Cited in the preceding footnote.

[142]Tarski solves also, for various particular formalized languages, the problem of finding a syntactical relation which coincides in extension with the semantical relation of satisfying a propositional form.

In a paper published in *Monatshefte für Mathematik und Physik*, vol. 42, no. 1 (1935), therefore later than Tarski's *Pojęcie Prawdy* but earlier than the German translation and its appendix, Carnap also solves both problems (of finding syntactical equivalents of being a true sentence and of satisfying a propositional form) for a particular formalized language and in fact for a stronger language than any for which this had previously been done by Tarski. Carnap's procedure can be simplified in the light of Tarski's appendix or as suggested by Kleene in his review cited in footnote 131.

On the theory of meaning which we are here adopting, the semantical concepts of being a true sentence and of satisfying a propositional form are reducible to those of denoting and having values, and these results of Tarski and Carnap are therefore implicit in the statement of the following footnote.

[143]More explicitly, this may be done as follows. In §07, in discussing the semantical rules of a formalized language, we thought of the concepts of denoting and of having values as being known in advance, and we used the semantical rules for the purpose of giving meaning to the previously uninterpreted logistic system. But instead of this it would be possible to give no meaning in advance to the words "denote" and "have values" as they occur in the semantical rules, and then to regard the semantical rules, taken together, as constituting definitions of "denote" and "have values" (in the same way that the formation rules of a logistic system constitute a definition of "well-formed"). The concepts expressed by "denote" and "have values" as thus defined belong to theoretical syntax, nothing semantical having been used in their definition. But they coincide in extension with the semantical concepts of denoting and having values, as applied to the particular formalized language.

The situation may be clarified by recalling that a particular logistic system may be expected to have many sound interpretations, leading to many different assignments of denotations and values to its well-formed formulas. These assignments of denotations and values to the well-formed formulas may be made as abstract correspondences, so that their treatment belongs to theoretical syntax. Semantics begins when we decide the meaning of the well-formed formulas by fixing a particular interpretation of the system. The distinction between semantics and syntax is found in the different signifthese two concepts are the only specifically semantical (non-syntactical) primitive symbols of a semantical meta-language, it is possible to transform the semantical meta-language into a syntax language by a change of interpretation which consists only in altering the sense of those names without changing their denotations.

However, a sound syntax language capable of expressing such syntactical equivalents of the semantical concepts of denoting and having values—or even only a syntactical equivalent of the semantical property of truth—must ordinarily be stronger than the object language (assumed sound), in the sense that there will be theorems of the syntax language of which no translation (i.e., sentence expressing the same proposition) is a theorem of the object language. Else there will be simple elementary propositions about the semantical concepts such that the sentences expressing the corresponding propositions about the syntactical equivalents of the semantical concepts are not theorems of the syntax language.[144]

For various particular formalized languages this was proved (in effect) by Tarski in his *Wahrheitsbegriff*. And Tarski's methods[145] are such that they can be applied to obtain the same result in many other cases—in particular in the case of each of the object languages studied in this book, when a formalized syntax language of it is set up in a straightforward manner. No doubt Tarski's result is capable of precise formulation and proof as a result about a very general class of languages, but we shall not attempt this.

The significance of Tarski's result should be noticed as it affects the question of the use of a formalized language as semantical meta-language of itself. A sound and sufficiently adequate language may indeed be capable

icance given to one particular interpretation and to its assignment of denotations and values to the well-formed formulas; but within the domain of formal logic, including pure syntax and pure semantics, nothing can be said about this different significance except to postulate it as different.

Many similar situations are familiar in mathematics. For instance, the distinction between plane Euclidean metric geometry and plane projective geometry may be found in the different significance given to one particular straight line and one particular elliptic involution on it. And it seems not unjustified to say that the sense in which semantics can be reduced to syntax is like that in which Euclidean metric geometry can be reduced to projective geometry.

All this suggests that, in order to maintain the distinction of semantics from syntax, "denote" and "have values" should be introduced as undefined terms and treated by the axiomatic method. Our use of semantical rules is intended as a step towards this. And in fact Tarski's *Wahrheitsbegriff* already contains the proposal of an axiomatic theory of truth as an alternative to that of finding a syntactical equivalent of the concept of truth.

[144]A more precise statement of this will be found in Chapter VIII, as it applies to the special case of the logistic system of Chapter VII when interpreted, in the manner indicated in footnote 132, so as to be capable of expressing its own syntax.

[145]Related to those used by Kurt Gödel in the proof of his incompleteness theorems, set forth in Chapter VIII.

of expressing its own syntax (cf. footnote 132) and its own semantics, in the sense of containing sentences which express at least a very comprehensive class of the propositions of its syntax and its semantics. But among these sentences, if certain very general conditions are satisfied, there will always be true sentences of a very elementary semantical character which are not theorems—sentences to the effect, roughly speaking, that such and such a particular sentence is true if and only if ____, the blank being filled by that particular sentence.[146] Hence, on the assumption that the language satisfies ordinary conditions of adequacy in other respects, not all the semantical rules (in the sense of §07), when written as sentences of the language, are theorems.

On account of this situation, the distinction between object language and meta-language, which first arises in formalizing the object language, remains of importance even after the task of formalization is complete for both the object language and the meta-language.

In concluding this Introduction, let us observe that much of what we have been saying has been concerned with the relation between linguistic expressions and their meaning, and therefore belongs to semantics. However, our interest has been less in the semantics of this or that particular language than in general features common to the semantics of many languages. And very general semantical principles, imposed as a demand upon any language that we wish to consider at all, have been put forward in some cases, notably assumptions (1), (2), (3) of §01 and assumption (4) of §02.[147]

We have not, however, attempted to formalize this semantical discussion, or even to put the material into such preliminary order as would constitute a first step toward formalization. Our purpose has been introductory and explanatory, and it is hoped that ideas to which the reader has thus been informally introduced will be held subject to revision or more precise formulation as the development continues.

From time to time in the following chapters we shall interrupt the rigorous treatment of a logistic system in order to make an informal semantical aside. Though in studying a logistic system we shall wish to hold its interpretation open, such semantical explanations about a system may serve in

[146]A more careful statement is given by Tarski.

By the results of Gödel referred to in the preceding footnote (or alternatively by Tarski's reduction of semantics to syntax), true syntactical sentences which are not theorems must also be expected. But these are of not quite so elementary a character. And the fundamental syntactical rules described in §07 may nevertheless all be theorems when written as sentences of the language.

[147]And assumption (5) of footnote 30.

particular to show a motivation for consideration of it by indicating its principal interpretations (cf. §07). *Except in this Introduction, semantical passages will be distinguished from others by being printed in smaller type, the small type serving as a warning that the material is not part of the formal logistic development and must not be used as such.*

As we have already indicated, it is contemplated that semantics itself should ultimately be studied by the logistic method.

But if semantical passages in this Introduction and in later chapters are to be rewritten in a formalized semantical language, certain refinements become necessary. Thus if the semantical language is to be a functional calculus of order ω in the sense of Chapter VI, or a language like that of Chapter X, then various semantical terms, such as the term "denote" introduced in §01, must give way to a multiplicity of terms of different types,[148] and statements which we have made using these terms must be replaced by axiom schemata[149] or theorem schemata[149] with typical ambiguity.[149] Or if the semantical language should conform to some alternative to the theory of types, changes of a different character would be required. In particular, following the Zermelo set theory (Chapter XI), we would have to weaken substantially the assumption made in §03 that every singulary form has an associated function, and explanations regarding the notation λ would have to be modified in some way in consequence.

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[148]All the expressions of the language—formulas, or well-formed formulas—may be treated as values of (syntactical) variables of one type. But terms "denote" of different types are nevertheless necessary, because in "____ denotes ____," after filling the first blank with a syntactical variable or syntactical constant, we may still fill the second blank with a variable or constant of any type.

Analogously, various other terms that we have used have to be replaced each by a multiplicity of terms of different types. This applies in particular to "thing," and the consequent weakening is especially striking in the case of footnote 9—which must become a schema with typical ambiguity.

See also the remark in the last paragraph of footnote 87.

[149]The terminology is explained in §§27, 30, 33, and Chapter VI. (The typical ambiguity required here is ambiguity with respect to *type* in the sense described in footnote 578, and is therefore not the same as the typical ambiguity mentioned in footnote 585, which is ambiguity rather with respect to *level*.)

I. The Propositional Calculus

The name *propositional calculus*[150] is given to any one of various logistic systems—which, however, are all equivalent to one another in a sense which will be made clear later. When we are engaged in developing a particular one of these systems, or when (as often happens) it is unnecessary for the purpose in hand to distinguish among the different systems, we speak of *the* propositional calculus. Otherwise the various logistic systems are distinguished as various *formulations of* the propositional calculus.

The importance of the propositional calculus in one or another of its formulations arises from its frequent occurrence as a part of more extensive logistic systems which are considered in this book or have been considered elsewhere, the variables of the propositional calculus (propositional variables) being replaceable by sentences of the more extensive system. Because of its greater simplicity, in many ways than other logistic systems which we consider, the propositional calculus also serves the purposes of introduction and illustration, many of the things which we do in connection with it being afterwards extended, with greater or less modification, to other systems.

In this chapter we develop in detail a particular formulation of the propositional calculus, the logistic system P_1. Some other formulations will be considered in the next chapter.

10. The primitive basis of P_1.[150] The primitive symbols of P_1 are three improper symbols

$$[\quad \supset \quad]$$

(of which the first and third are called *brackets*) and one primitive constant

$$f$$

and an infinite list of variables

$$p \quad q \quad r \quad s \quad p_1 \quad q_1 \quad r_1 \quad s_1 \quad p_2 \quad q_2 \quad \ldots$$

(the order here indicated being called the *alphabetic order* of the variables). The variables and the primitive constant are called *proper symbols*.[151]

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[150]Historical questions in connection with the propositional calculus will be treated briefly in the concluding section of Chapter II.

[151]Regarding the terminology, see explanations in §07 and in footnote 117.