THE PRINCIPLES OF MATHEMATICS

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LONDON
GEORGE ALLEN & UNWIN LTD
MUSEUM STREET

the relation affirmed between A and B in the proposition "A differs from B" is the general relation of difference, and is precisely and numerically the same as the relation affirmed between C and D in "C differs from D." And this doctrine must be held, for the same reasons, to be true of all other relations; relations do not have instances, but are strictly the same in all propositions in which they occur.

We may now sum up the main points elicited in our discussion of the verb. The verb, we saw, is a concept which, like the adjective, may occur in a proposition without being one of the terms of the proposition, though it may also be made into a logical subject. One verb, and one only, must occur as verb in every proposition; (but every proposition, by turning its verb into a verbal noun, can be changed into a single logical subject, of a kind which I shall call in future a propositional concept.) Every verb, in the logical sense of the word, may be regarded as a relation; when it occurs as verb, it actually relates, but when it occurs as verbal noun it is the bare relation considered independently of the terms which it relates. Verbs do not, like adjectives, have instances, but are identical in all the cases of their occurrence. Owing to the way in which the verb actually relates the terms of a proposition, every proposition has a unity which renders it distinct from the sum of its constituents. All these points lead to logical problems, which, in a treatise on logic, would deserve to be fully and thoroughly discussed.

Having now given a general sketch of the nature of verbs and adjectives, I shall proceed, in the next two chapters, to discussions arising out of the consideration of adjectives, and in Chapter VII to topics connected with verbs. Broadly speaking, classes are connected with adjectives, while propositional functions involve verbs. It is for this reason that it has been necessary to deal at such length with a subject which might seem, at first sight, to be somewhat remote from the principles of mathematics.

Aristotelian Society, 1900—1901) must not be applied to all concepts. The relation of an instance to its universal, at any rate, must be actually and numerically the same in all cases where it occurs.

CHAPTER V.

DENOTING.

56. THE notion of denoting, like most of the notions of logic, has been obscured hitherto by an undue admixture of psychology. There is a sense in which *we* denote, when we point or describe, or employ words as symbols for concepts; this, however, is not the sense that I wish to discuss. But the fact that description is possible—that we are able, by the employment of concepts, to designate a thing which is not a concept—is due to a logical relation between some concepts and some terms, in virtue of which such concepts inherently and logically *denote* such terms. It is this sense of denoting which is here in question. This notion lies at the bottom (I think) of all theories of substance, of the subject-predicate logic, and of the opposition between things and ideas, discursive thought and immediate perception. These various developments, in the main, appear to me mistaken, while the fundamental fact itself, out of which they have grown, is hardly ever discussed in its logical purity.

A concept *denotes* when, if it occurs in a proposition, the proposition is not *about* the concept, but about a term connected in a certain peculiar way with the concept. If I say "I met a man," the proposition is not about *a man*: this is a concept which does not walk the streets, but lives in the shadowy limbo of the logic-books. What I met was a thing, not a concept, an actual man with a tailor and a bank-account or a public-house and a drunken wife. Again, the proposition "any finite number is odd or even" is plainly true; yet the *concept* "any finite number" is neither odd nor even. It is only particular numbers that are odd or even; there is not, in addition to these, another entity, *any number*, which is either odd or even, and if there were, it is plain that it could not be odd and could not be even. Of the *concept* "any number," almost all the propositions that contain the *phrase* "any number" are false. If we wish to speak of the concept, we have to indicate the fact by italics or inverted commas. People often assert that man is mortal; but what is mortal will die, and yet we should be surprised to find in the "Times" such a notice as the following: "Died at his residence of

Camelot, Gladstone Road, Upper Tooting, on the 18th of June 19—, Man, eldest son of Death and Sin." *Man*, in fact, does not die; hence if "man is mortal" were, as it appears to be, a proposition about *man*, it would be simply false. The fact is, the proposition is about men; and here again, it is not about the concept *men*, but about what this concept denotes. The whole theory of definition, of identity, of classes, of symbolism, and of the variable is wrapped up in the theory of denoting. The notion is a fundamental notion of logic, and, in spite of its difficulties, it is quite essential to be as clear about it as possible.

57. The notion of denoting may be obtained by a kind of logical genesis from subject-predicate propositions, upon which it seems more or less dependent. The simplest of propositions are those in which one predicate occurs otherwise than as a term, and there is only one term of which the predicate in question is asserted. Such propositions may be called subject-predicate propositions. Instances are: *A* is, *A* is one, *A* is human. Concepts which are predicates might also be called class-concepts, because they give rise to classes, but we shall find it necessary to distinguish between the words *predicate* and *class-concept*. Propositions of the subject-predicate type always imply and are implied by other propositions of the type which asserts that an individual belongs to a class. Thus the above instances are equivalent to: *A* is an entity, *A* is a unit, *A* is a man. These new propositions are not identical with the previous ones, since they have an entirely different form. To begin with, *is* is now the only concept not used as a term. *A man*, we shall find, is neither a concept nor a term, but a certain kind of combination of certain terms, namely of those which are human. And the relation of Socrates to *a man* is quite different from his relation to humanity; indeed "Socrates is human" must be held, if the above view is correct, to be not, in the most usual sense, a judgment of relation between Socrates and humanity, since this view would make *human* occur as term in "Socrates is human." It is, of course, undeniable that a relation to humanity is implied by "Socrates is human," namely the relation expressed by "Socrates has humanity"; and this relation conversely implies the subject-predicate proposition. But the two propositions can be clearly distinguished, and it is important to the theory of classes that this should be done. Thus we have, in the case of every predicate, three types of propositions which imply one another, namely, "Socrates is human," "Socrates has humanity," and "Socrates is a man." The first contains a term and a predicate, the second two terms and a relation (the second term being identical with the predicate of the first proposition)*, while the third contains a term, a relation, and what I shall call a disjunction (a term which will be explained shortly)†. The class-concept differs little, if at

* Cf. § 49.

† There are two allied propositions expressed by the same words, namely "Socrates is a-man" and "Socrates is-a man." The above remarks apply to the

all, from the predicate, while the class, as opposed to the class-concept, is the sum or conjunction of all the terms which have the given predicate. The relation which occurs in the second type (Socrates has humanity) is characterized completely by the fact that it implies and is implied by a proposition with only one term, in which the other term of the relation has become a predicate. A class is a certain combination of terms, a class-concept is closely akin to a predicate, and the terms whose combination forms the class are determined by the class-concept. Predicates are, in a certain sense, the simplest type of concepts, since they occur in the simplest type of proposition.

58. There is, connected with every predicate, a great variety of closely allied concepts, which, in so far as they are distinct, it is important to distinguish. Starting, for example, with *human*, we have man, men, all men, every man, any man, the human race, of which all except the first are twofold, a denoting concept and an object denoted; we have also, less closely analogous, the notions "a man" and "some man," which again denote objects* other than themselves. This vast apparatus connected with every predicate must be borne in mind, and an endeavour must be made to give an analysis of all the above notions. But for the present, it is the property of denoting, rather than the various denoting concepts, that we are concerned with.

The combination of concepts as such to form new concepts, of greater complexity than their constituents, is a subject upon which writers on logic have said many things. But the combination of terms as such, to form what by analogy may be called complex terms, is a subject upon which logicians, old and new, give us only the scantiest discussion. Nevertheless, the subject is of vital importance to the philosophy of mathematics, since the nature both of number and of the variable turns upon just this point. Six words, of constant occurrence in daily life, are also characteristic of mathematics: these are the words *all*, *every*, *any*, *a*, *some* and *the*. For correctness of reasoning, it is essential that these words should be sharply distinguished one from another; but the subject bristles with difficulties, and is almost wholly neglected by logicians†.

It is plain, to begin with, that a phrase containing one of the above

former; but in future, unless the contrary is indicated by a hyphen or otherwise, the latter will always be in question. The former expresses the identity of Socrates with an ambiguous individual; the latter expresses a relation of Socrates to the class-concept *man*.

* I shall use the word *object* in a wider sense than *term*, to cover both singular and plural, and also cases of ambiguity, such as "a man." The fact that a word can be framed with a wider meaning than *term* raises grave logical problems. Cf. § 47.

† On the indefinite article, some good remarks are made by Meinong, "Abstrahiren und Vergleichen," *Zeitschrift für Psychologie und Physiologie der Sinnesorgane*, Vol. XXIV, p. 63.

six words always denotes. It will be convenient, for the present discussion, to distinguish a class-concept from a predicate: I shall call *human* a predicate, and *man* a class-concept, though the distinction is perhaps only verbal. The characteristic of a class-concept, as distinguished from terms in general, is that "x is a u" is a propositional function when, and only when, u is a class-concept. It must be held that when u is not a class-concept, we do not have a false proposition, but simply no proposition at all, whatever value we may give to x. This enables us to distinguish a class-concept belonging to the null-class, for which all propositions of the above form are false, from a term which is not a class-concept at all, for which there are no propositions of the above form. Also it makes it plain that a class-concept is not a term in the proposition "x is a u," for u has a restricted variability if the formula is to remain a proposition. A denoting phrase, we may now say, consists always of a class-concept preceded by one of the above six words or some synonym of one of them.

59. The question which first meets us in regard to denoting is this: Is there one way of denoting six different kinds of objects, or are the ways of denoting different? And in the latter case, is the object denoted the same in all six cases, or does the object differ as well as the way of denoting it? In order to answer this question, it will be first necessary to explain the differences between the six words in question. Here it will be convenient to omit the word *the* to begin with, since this word is in a different position from the others, and is liable to limitations from which they are exempt.

In cases where the class defined by a class-concept has only a finite number of terms, it is possible to omit the class-concept wholly, and indicate the various objects denoted by enumerating the terms and connecting them by means of *and* or *or* as the case may be. It will help to isolate a part of our problem if we first consider this case, although the lack of subtlety in language renders it difficult to grasp the difference between objects indicated by the same form of words.

Let us begin by considering two terms only, say Brown and Jones. The objects denoted by *all*, *every*, *any*, *a* and *some** are respectively involved in the following five propositions. (1) Brown and Jones are two of Miss Smith's suitors; (2) Brown and Jones are paying court to Miss Smith; (3) if it was Brown or Jones you met, it was a very ardent lover; (4) if it was one of Miss Smith's suitors, it must have been Brown or Jones; (5) Miss Smith will marry Brown or Jones. Although only two forms of words, *Brown and Jones* and *Brown or Jones*, are involved in these propositions, I maintain that five different combinations are involved. The distinctions, some of which are rather subtle, may be

* I intend to distinguish between *a* and *some* in a way not warranted by language; the distinction of *all* and *every* is also a straining of usage. Both are necessary to avoid circumlocution.

brought out by the following considerations. In the first proposition, it is Brown *and* Jones who are two, and this is not true of either separately; nevertheless it is not the whole composed of Brown and Jones which is two, for this is only one. The two are a genuine combination of Brown with Jones, the kind of combination which, as we shall see in the next chapter, is characteristic of classes. In the second proposition, on the contrary, what is asserted is true of Brown and Jones severally; the proposition is equivalent to, though not (I think) identical with, "Brown is paying court to Miss Smith and Jones is paying court to Miss Smith." Thus the combination indicated by *and* is not the same here as in the first case: the first case concerned *all* of them collectively, while the second concerns *all* distributively, *i.e.* each or every one of them. For the sake of distinction, we may call the first a *numerical* conjunction, since it gives rise to number, the second a *propositional* conjunction, since the proposition in which it occurs is equivalent to a conjunction of propositions. (It should be observed that the conjunction of propositions in question is of a wholly different kind from any of the combinations we are considering, being in fact of the kind which is called the logical product. The propositions are combined *quâ* propositions, not *quâ* terms.)

The third proposition gives the kind of conjunction by which *any* is defined. There is some difficulty about this notion, which seems half-way between a conjunction and a disjunction. This notion may be further explained as follows. Let a and b be two different propositions, each of which implies a third proposition c. Then the disjunction "a or b" implies c. Now let a and b be propositions assigning the same predicate to two different subjects, then there is a combination of the two subjects to which the given predicate may be assigned so that the resulting proposition is equivalent to the disjunction "a or b." Thus suppose we have "if you met Brown, you met a very ardent lover," and "if you met Jones, you met a very ardent lover." Hence we infer "if you met Brown or if you met Jones, you met a very ardent lover," and we regard this as equivalent to "if you met Brown or Jones, etc." The combination of Brown and Jones here indicated is the same as that indicated by *either* of them. It differs from a disjunction by the fact that it implies and is implied by a statement concerning *both*; but in some more complicated instances, this mutual implication fails. The method of combination is, in fact, different from that indicated by *both*, and is also different from both forms of disjunction. I shall call it the *variable* conjunction. The first form of disjunction is given by (4): this is the form which I shall denote by *a* suitor. Here, although it must have been Brown or Jones, it is not true that it must have been Brown, nor yet that it must have been Jones. Thus the proposition is not equivalent to the disjunction of propositions "it must have been Brown or it must have been Jones." The proposition, in fact, is not capable of statement either as a disjunction or as a conjunction of propositions, except in the very roundabout form: "if it was not Brown, it was Jones, and if it was not Jones, it was Brown," a form which rapidly becomes intolerable when the number of terms is increased beyond two, and becomes theoretically inadmissible when the number of terms is infinite. Thus this form of disjunction denotes a variable term, that is, whichever of the two terms we fix upon, it does not denote this term, and yet it does denote one or other of them. This form accordingly I shall call the *variable* disjunction. Finally, the second form of disjunction is given by (5). This is what I shall call the *constant* disjunction, since here either Brown is denoted, or Jones is denoted, but the alternative is undecided. That is to say, our proposition is now equivalent to a disjunction of propositions, namely "Miss Smith will marry Brown, or she will marry Jones." She will marry *some* one of the two, and the disjunction denotes a particular one of them, though it may denote either particular one. Thus all the five combinations are distinct.

It is to be observed that these five combinations yield neither terms nor concepts, but strictly and only combinations of terms. The first yields many terms, while the others yield something absolutely peculiar, which is neither one nor many. The combinations are combinations of terms, effected without the use of relations. Corresponding to each combination there is, at least if the terms combined form a class, a perfectly definite concept, which *denotes* the various terms of the combination combined in the specified manner. To explain this, let us repeat our distinctions in a case where the terms to be combined are not enumerated, as above, but are defined as the terms of a certain class.

60. When a class-concept a is given, it must be held that the various terms belonging to the class are also given. That is to say, any term being proposed, it can be decided whether or not it belongs to the class. In this way, a collection of terms can be given otherwise than by enumeration. Whether a collection can be given otherwise than by enumeration or by a class-concept, is a question which, for the present, I leave undetermined. But the possibility of giving a collection by a class-concept is highly important, since it enables us to deal with infinite collections, as we shall see in Part V. For the present, I wish to examine the meaning of such phrases as *all* a's, *every* a, *any* a, *an* a, and *some* a. *All* a's, to begin with, denotes a numerical conjunction; it is definite as soon as a is given. The concept *all* a's is a perfectly definite single concept, which denotes the terms of a taken all together. The terms so taken have a number, which may thus be regarded, if we choose, as a property of the class-concept, since it is determinate for any given class-concept. *Every* a, on the contrary, though it still denotes all the a's, denotes them in a different way, *i.e.* severally instead of collectively. *Any* a denotes only one a, but it is wholly irrelevant which it denotes, and what is said will be equally true whichever it may be. Moreover,

any a denotes a variable a, that is, whatever particular a we may fasten upon, it is certain that *any* a does not denote that one; and yet of that one any proposition is true which is true of any a. *An* a denotes a variable disjunction: that is to say, a proposition which holds of *an* a may be false concerning each particular a, so that it is not reducible to a disjunction of propositions. For example, a point lies between any point and any other point; but it would not be true of any one particular point that it lay between any point and any other point, since there would be many pairs of points between which it did not lie. This brings us finally to *some* a, the constant disjunction. This denotes just one term of the class a, but the term it denotes may be any term of the class. Thus "some moment does not follow any moment" would mean that there was a first moment in time, while "a moment precedes any moment" means the exact opposite, namely, that every moment has predecessors.

61. In the case of a class a which has a finite number of terms—say $a_1, a_2, a_3, \ldots a_n$, we can illustrate these various notions as follows:

(1) *All* a's denotes a_1 and a_2 and ... and a_n.

(2) *Every* a denotes a_1 and denotes a_2 and ... and denotes a_n.

(3) *Any* a denotes a_1 or a_2 or ... or a_n, where *or* has the meaning that it is irrelevant which we take.

(4) *An* a denotes a_1 or a_2 or ... or a_n, where *or* has the meaning that no one in particular must be taken, just as in *all* a's we must not take any one in particular.

(5) *Some* a denotes a_1 or denotes a_2 or ... or denotes a_n, where it is not irrelevant which is taken, but on the contrary some one particular a must be taken.

As the nature and properties of the various ways of combining terms are of vital importance to the principles of mathematics, it may be well to illustrate their properties by the following important examples.

(a) Let a be a class, and b a class of classes. We then obtain in all six possible relations of a to b from various combinations of *any*, *a* and *some*. *All* and *every* do not, in this case, introduce anything new. The six cases are as follows.

(1) Any a belongs to any class belonging to b, in other words, the class a is wholly contained in the common part or logical product of the various classes belonging to b.

(2) Any a belongs to a b, *i.e.* the class a is contained in any class which contains all the b's, or, is contained in the logical sum of all the b's.

(3) Any a belongs to some b, *i.e.* there is a class belonging to b, in which the class a is contained. The difference between this case and the second arises from the fact that here there is one b to which every a belongs, whereas before it was only decided that every a belonged to a b, and different a's might belong to different b's.

(4) An *a* belongs to any *b*, *i.e.* whatever *b* we take, it has a part in common with *a*.

(5) An *a* belongs to a *b*, *i.e.* there is a *b* which has a part in common with *a*. This is equivalent to "some (or an) *a* belongs to some *b*."

(6) Some *a* belongs to any *b*, *i.e.* there is an *a* which belongs to the common part of all the *b*'s, or *a* and all the *b*'s have a common part. These are all the cases that arise here.

(β) It is instructive, as showing the generality of the type of relations here considered, to compare the above case with the following. Let *a*, *b* be two series of real numbers; then six precisely analogous cases arise.

(1) Any *a* is less than any *b*, or, the series *a* is contained among numbers less than every *b*.

(2) Any *a* is less than a *b*, or, whatever *a* we take, there is a *b* which is greater, or, the series *a* is contained among numbers less than a (variable) term of the series *b*. It does not follow that some term of the series *b* is greater than all the *a*'s.

(3) Any *a* is less than some *b*, or, there is a term of *b* which is greater than all the *a*'s. This case is not to be confounded with (2).

(4) An *a* is less than any *b*, *i.e.* whatever *b* we take, there is an *a* which is less than it.

(5) An *a* is less than a *b*, *i.e.* it is possible to find an *a* and a *b* such that the *a* is less than the *b*. This merely denies that any *a* is greater than any *b*.

(6) Some *a* is less than any *b*, *i.e.* there is an *a* which is less than all the *b*'s. This was not implied in (4), where the *a* was variable, whereas here it is constant.

In this case, actual mathematics have compelled the distinction between the variable and the constant disjunction. But in other cases, where mathematics have not obtained sway, the distinction has been neglected; and the mathematicians, as was natural, have not investigated the logical nature of the disjunctive notions which they employed.

(γ) I shall give one other instance, as it brings in the difference between *any* and *every*, which has not been relevant in the previous cases. Let *a* and *b* be two classes of classes; then twenty different relations between them arise from different combinations of the terms of their terms. The following technical terms will be useful. If *a* be a class of classes, its logical sum consists of all terms belonging to any *a*, *i.e.* all terms such that there is an *a* to which they belong, while its logical product consists of all terms belonging to every *a*, *i.e.* to the common part of all the *a*'s. We have then the following cases.

(1) Any term of any *a* belongs to every *b*, *i.e.* the logical sum of *a* is contained in the logical product of *b*.

(2) Any term of any *a* belongs to a *b*, *i.e.* the logical sum of *a* is contained in the logical sum of *b*.

(3) Any term of any *a* belongs to some *b*, *i.e.* there is a *b* which contains the logical sum of *a*.

(4) Any term of some (or an) *a* belongs to every *b*, *i.e.* there is an *a* which is contained in the product of *b*.

(5) Any term of some (or an) *a* belongs to a *b*, *i.e.* there is an *a* which is contained in the sum of *b*.

(6) Any term of some (or an) *a* belongs to some *b*, *i.e.* there is a *b* which contains one class belonging to *a*.

(7) A term of any *a* belongs to any *b*, *i.e.* any class of *a* and any class of *b* have a common part.

(8) A term of any *a* belongs to a *b*, *i.e.* any class of *a* has a part in common with the logical sum of *b*.

(9) A term of any *a* belongs to some *b*, *i.e.* there is a *b* with which any *a* has a part in common.

(10) A term of an *a* belongs to every *b*, *i.e.* the logical sum of *a* and the logical product of *b* have a common part.

(11) A term of an *a* belongs to any *b*, *i.e.* given any *b*, an *a* can be found with which it has a common part.

(12) A term of an *a* belongs to a *b*, *i.e.* the logical sums of *a* and of *b* have a common part.

(13) Any term of every *a* belongs to every *b*, *i.e.* the logical product of *a* is contained in the logical product of *b*.

(14) Any term of every *a* belongs to a *b*, *i.e.* the logical product of *a* is contained in the logical sum of *b*.

(15) Any term of every *a* belongs to some *b*, *i.e.* there is a term of *b* in which the logical product of *a* is contained.

(16) A (or some) term of every *a* belongs to every *b*, *i.e.* the logical products of *a* and of *b* have a common part.

(17) A (or some) term of every *a* belongs to a *b*, *i.e.* the logical product of *a* and the logical sum of *b* have a common part.

(18) Some term of any *a* belongs to every *b*, *i.e.* any *a* has a part in common with the logical product of *b*.

(19) A term of some *a* belongs to any *b*, *i.e.* there is some term of *a* with which any *b* has a common part.

(20) A term of every *a* belongs to any *b*, *i.e.* any *b* has a part in common with the logical product of *a*.

The above examples show that, although it may often happen that there is a mutual implication (which has not always been stated) of corresponding propositions concerning *some* and *a*, or concerning *any* and *every*, yet in other cases there is no such mutual implication. Thus the five notions discussed in the present chapter are genuinely distinct, and to confound them may lead to perfectly definite fallacies.

62. It appears from the above discussion that, whether there are different ways of denoting or not, the objects denoted by *all men*, *every man*, etc. are certainly distinct. It seems therefore legitimate to say that the whole difference lies in the objects, and that denoting itself is the same in all cases. There are, however, many difficult problems connected with the subject, especially as regards the nature of the objects denoted. *All men*, which I shall identify with the class of men, seems to be an unambiguous object, although grammatically it is plural. But in the other cases the question is not so simple: we may doubt whether an ambiguous object is unambiguously denoted, or a definite object ambiguously denoted. Consider again the proposition "I met a man." It is quite certain, and is implied by this proposition, that what I met was an unambiguous perfectly definite man: in the technical language which is here adopted, the proposition is expressed by "I met some man." But the actual man whom I met forms no part of the proposition in question, and is not specially denoted by *some man*. Thus the concrete event which happened is not asserted in the proposition. What is asserted is merely that some one of a class of concrete events took place. The whole human race is involved in my assertion: if any man who ever existed or will exist had not existed or been going to exist, the purport of my proposition would have been different. Or, to put the same point in more intensional language, if I substitute for *man* any of the other class-concepts applicable to the individual whom I had the honour to meet, my proposition is changed, although the individual in question is just as much denoted as before. What this proves is, that *some man* must not be regarded as actually denoting Smith and actually denoting Brown, and so on: the whole procession of human beings throughout the ages is always relevant to every proposition in which *some man* occurs, and what is denoted is essentially not each separate man, but a kind of combination of all men. This is more evident in the case of *every*, *any*, and *a*. There is, then, a definite something, different in each of the five cases, which must, in a sense, be an object, but is characterized as a set of terms combined in a certain way, which something is denoted by *all men*, *every man*, *any man*, *a man* or *some man*; and it is with this very paradoxical object that propositions are concerned in which the corresponding concept is used as denoting.

63. It remains to discuss the notion of *the*. This notion has been symbolically emphasized by Peano, with very great advantage to his calculus; but here it is to be discussed philosophically. The use of identity and the theory of definition are dependent upon this notion, which has thus the very highest philosophical importance.

The word *the*, in the singular, is correctly employed only in relation to a class-concept of which there is only one instance. We speak of *the* King, *the* Prime Minister, and so on (understanding *at the present time*); and in such cases there is a method of denoting one single definite term by means of a concept, which is not given us by any of our other five words. It is owing to this notion that mathematics can give definitions of terms which are not concepts—a possibility which illustrates the difference between mathematical and philosophical definition. Every term is the only instance of *some* class-concept, and thus every term, theoretically, is capable of definition, provided we have not adopted a system in which the said term is one of our indefinables. It is a curious paradox, puzzling to the symbolic mind, that definitions, theoretically, are nothing but statements of symbolic abbreviations, irrelevant to the reasoning and inserted only for practical convenience, while yet, in the development of a subject, they always require a very large amount of thought, and often embody some of the greatest achievements of analysis. This fact seems to be explained by the theory of denoting. An object may be present to the mind, without our knowing any concept of which the said object is *the* instance; and the discovery of such a concept is not a mere improvement in notation. The reason why this appears to be the case is that, as soon as the definition is found, it becomes wholly unnecessary to the reasoning to remember the actual object defined, since only concepts are relevant to our deductions. In the moment of discovery, the definition is seen to be *true*, because the object to be defined was already in our thoughts; but as part of our reasoning it is not true, but merely symbolic, since what the reasoning requires is not that it should deal with *that* object, but merely that it should deal with the object denoted by the definition.

In most actual definitions of mathematics, what is defined is a *class* of entities, and the notion of *the* does not then explicitly appear. But even in this case, what is really defined is *the* class satisfying certain conditions; for a class, as we shall see in the next chapter, is always a term or conjunction of terms and never a concept. Thus the notion of *the* is always relevant in definitions; and we may observe generally that the adequacy of concepts to deal with things is wholly dependent upon the unambiguous denoting of a single term which this notion gives.

64. The connection of denoting with the nature of identity is important, and helps, I think, to solve some rather serious problems. The question whether identity is or is not a relation, and even whether there is such a concept at all, is not easy to answer. For, it may be said, identity cannot be a relation, since, where it is truly asserted, we have only one term, whereas two terms are required for a relation. And indeed identity, an objector may urge, cannot be anything at all: two terms plainly are not identical, and one term cannot be, for what is it identical with? Nevertheless identity must be something. We might attempt to remove identity from terms to relations, and say that two terms are identical in some respect when they have a given relation to a given term. But then we shall have to hold either that there is strict identity between the two cases of the given relation, or that the two cases have identity in the sense of having a given relation to a given term; but the latter view leads to an endless process of the illegitimate kind. Thus identity must be admitted, and the difficulty as to the two terms of a relation must be met by a sheer denial that two different terms are necessary. There must always be a referent and a relatum, but these need not be distinct; and where identity is affirmed, they are not so*.

But the question arises: Why is it ever worth while to affirm identity? This question is answered by the theory of denoting. If we say "Edward VII is the King," we assert an identity; the reason why this assertion is worth making is, that in the one case the actual term occurs, while in the other a denoting concept takes its place. (For purposes of discussion, I ignore the fact that Edwards form a class, and that seventh Edwards form a class having only one term. Edward VII is practically, though not formally, a proper name.) Often two denoting concepts occur, and the term itself is not mentioned, as in the proposition "the present Pope is the last survivor of his generation." When a term is given, the assertion of its identity with itself, though true, is perfectly futile, and is never made outside the logic-books; but where denoting concepts are introduced, identity is at once seen to be significant. In this case, of course, there is involved, though not asserted, a relation of the denoting concept to the term, or of the two denoting concepts to each other. But the *is* which occurs in such propositions does not itself state this further relation, but states pure identity†.

65. To sum up. When a class-concept, preceded by one of the six words *all*, *every*, *any*, *a*, *some*, *the*, occurs in a proposition, the proposition is, as a rule, not *about* the concept formed of the two words together, but about an object quite different from this, in general not a concept at all, but a term or complex of terms. This may be seen by the fact that propositions in which such concepts occur are in general false concerning the concepts themselves. At the same time, it is possible to consider and make propositions about the concepts themselves, but these are not the natural propositions to make in employing the concepts. "Any number is odd or even" is a perfectly natural proposition, whereas "*Any number* is a variable conjunction" is a proposition only to be made in a logical discussion. In such cases, we say that the concept in question *denotes*. We decided that denoting is a perfectly

* On relations of terms to themselves, *v. inf.* Chap. IX, § 95.

† The word *is* is terribly ambiguous, and great care is necessary in order not to confound its various meanings. We have (1) the sense in which it asserts Being, as in "*A* is"; (2) the sense of identity; (3) the sense of predication, in "*A* is human"; (4) the sense of "*A* is a-man" (cf. p. 54, note), which is very like identity. In addition to these there are less common uses, as "to be good is to be happy," where a relation of assertions is meant, that relation, in fact, which, where it exists, gives rise to formal implication. Doubtless there are further meanings which have not occurred to me. On the meanings of *is*, cf. De Morgan, *Formal Logic*, pp. 49, 50.

definite relation, the same in all six cases, and that it is the nature of the denoted object and the denoting concept which distinguishes the cases. We discussed at some length the nature and the differences of the denoted objects in the five cases in which these objects are combinations of terms. In a full discussion, it would be necessary also to discuss the denoting concepts: the actual meanings of these concepts, as opposed to the nature of the objects they denote, have not been discussed above. But I do not know that there would be anything further to say on this topic. Finally, we discussed *the*, and showed that this notion is essential to what mathematics calls definition, as well as to the possibility of uniquely determining a term by means of concepts; the actual use of identity, though not its meaning, was also found to depend upon this way of denoting a single term. From this point we can advance to the discussion of classes, thereby continuing the development of the topics connected with adjectives.