CHAPTER III

INCOMPLETE SYMBOLS

(1) *Descriptions.* By an "incomplete" symbol we mean a symbol which is not supposed to have any meaning in isolation, but is only defined in certain contexts. In ordinary mathematics, for example, $\frac{d}{dx}$ and $\int_a^b$ are incomplete symbols: something has to be supplied before we have anything significant. Such symbols have what may be called a "definition in use." Thus if we put

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad \text{Df},$$

we define the *use* of ∇^2, but ∇^2 by itself remains without meaning. This distinguishes such symbols from what (in a generalized sense) we may call *proper names*: "Socrates," for example, stands for a certain man, and therefore has a meaning by itself, without the need of any context. If we supply a context, as in "Socrates is mortal," these words express a fact of which Socrates himself is a constituent: there is a certain object, namely Socrates, which does have the property of mortality, and this object is a constituent of the complex fact which we assert when we say "Socrates is mortal." But in other cases, this simple analysis fails us. Suppose we say: "The round square does not exist." It seems plain that this is a true proposition, yet we cannot regard it as denying the existence of a certain object called "the round square." For if there were such an object, it would exist: we cannot first assume that there is a certain object, and then proceed to deny that there is such an object. Whenever the grammatical subject of a proposition can be supposed not to exist without rendering the proposition meaningless, it is plain that the grammatical subject is not a proper name, *i.e.* not a name directly representing some object. Thus in all such cases, the proposition must be capable of being so analysed that what was the grammatical subject shall have disappeared. Thus when we say "the round square does not exist," we may, as a first attempt at such analysis, substitute "it is false that there is an object x which is both round and square." Generally, when "the so-and-so" is said not to exist, we have a proposition of the form[*]

$$\text{“}\sim E\,!\,(\iota x)(\phi x),\text{”}$$

i.e.

$$\sim\{(\exists c):\phi x\,.\equiv_x.\,x=c\},$$

or some equivalent. Here the apparent grammatical subject $(\iota x)(\phi x)$ has completely disappeared; thus in "$\sim E\,!\,(\iota x)(\phi x)$," $(\iota x)(\phi x)$ is an *incomplete* symbol.

[*] Cf. pp. 30, 31.

By an extension of the above argument, it can easily be shown that $(\iota x)(\phi x)$ is *always* an incomplete symbol. Take, for example, the following proposition: "Scott is the author of Waverley." [Here "the author of Waverley" is (ιx) (x wrote Waverley).] This proposition expresses an identity; thus if "the author of Waverley" could be taken as a proper name, and supposed to stand for some object c, the proposition would be "Scott is c." But if c is any one except Scott, this proposition is false; while if c *is* Scott, the proposition is "Scott is Scott," which is trivial, and plainly different from "Scott is the author of Waverley." Generalizing, we see that the proposition

$$a=(\iota x)(\phi x)$$

is one which may be true or may be false, but is never merely trivial, like $a=a$; whereas, if $(\iota x)(\phi x)$ were a proper name, $a=(\iota x)(\phi x)$ would necessarily be either false or the same as the trivial proposition $a=a$. We may express this by saying that $a=(\iota x)(\phi x)$ is not a value of the propositional function $a=y$, from which it follows that $(\iota x)(\phi x)$ is not a value of y. But since y may be anything, it follows that $(\iota x)(\phi x)$ is nothing. Hence, since in use it has meaning, it must be an incomplete symbol.

It might be suggested that "Scott is the author of Waverley" asserts that "Scott" and "the author of Waverley" are two names for the same object. But a little reflection will show that this would be a mistake. For if that were the meaning of "Scott is the author of Waverley," what would be required for its truth would be that Scott should have been *called* the author of Waverley: if he had been so called, the proposition would be true, even if some one else had written Waverley; while if no one called him so, the proposition would be false, even if he had written Waverley. But in fact he was the author of Waverley at a time when no one called him so, and he would not have been the author if every one had called him so but some one else had written Waverley. Thus the proposition "Scott is the author of Waverley" is not a proposition about names, like "Napoleon is Bonaparte"; and this illustrates the sense in which "the author of Waverley" differs from a true proper name.

Thus all phrases (other than propositions) containing the word *the* (in the singular) are incomplete symbols: they have a meaning in use, but not in isolation. For "the author of Waverley" cannot mean the same as "Scott," or "Scott is the author of Waverley" would mean the same as "Scott is Scott," which it plainly does not; nor can "the author of Waverley" mean anything other than "Scott," or "Scott is the author of Waverley" would be false. Hence "the author of Waverley" means nothing.

It follows from the above that we must not attempt to define "$(\iota x)(\phi x)$," but must define the *uses* of this symbol, *i.e.* the propositions in whose symbolic expression it occurs. Now in seeking to define the uses of this symbol, it is important to observe the import of propositions in which it occurs. Take as an illustration: "The author of Waverley was a poet." This implies (1) that Waverley was written, (2) that it was written by one man, and not in collaboration, (3) that the one man who wrote it was a poet. If any one of these fails, the proposition is false. Thus "the author of 'Slawkenburgius on Noses' was a poet" is false, because no such book was ever written; "the author of 'The Maid's Tragedy' was a poet" is false, because this play was written by Beaumont and Fletcher jointly. These two possibilities of falsehood do not arise if we say "Scott was a poet." Thus our interpretation of the uses of $(\iota x)(\phi x)$ must be such as to allow for them. Now taking ϕx to replace "x wrote Waverley," it is plain that any statement apparently about $(\iota x)(\phi x)$ requires (1) $(\exists x) . (\phi x)$ and (2) $\phi x . \phi y . \supset_{x,y} . x = y$; here (1) states that *at least* one object satisfies ϕx, while (2) states that *at most* one object satisfies ϕx. The two together are equivalent to

$$(\exists c) : \phi x . \equiv_x . x = c,$$

which we defined as $\quad E ! (\iota x)(\phi x)$.

Thus "$E ! (\iota x)(\phi x)$" must be part of what is affirmed by any proposition about $(\iota x)(\phi x)$. If our proposition is $f\{(\iota x)(\phi x)\}$, what is further affirmed is fc, if $\phi x . \equiv_x . x = c$. Thus we have

$$f\{(\iota x)(\phi x)\} . = : (\exists c) : \phi x . \equiv_x . x = c : fc \quad \text{Df},$$

i.e. "the x satisfying ϕx satisfies fx" is to mean: "There is an object c such that ϕx is true when, and only when, x is c, and fc is true," or, more exactly: "There is a c such that 'ϕx' is always equivalent to 'x is c,' and fc." In this, "$(\iota x)(\phi x)$" has completely disappeared; thus "$(\iota x)(\phi x)$" is merely symbolic, and does not directly represent an object, as single small Latin letters are assumed to do*.

The proposition "$a = (\iota x)(\phi x)$" is easily shown to be equivalent to "$\phi x . \equiv_x . x = a$." For, by the definition, it is

$$(\exists c) : \phi x . \equiv_x . x = c : a = c,$$

i.e. "there is a c for which $\phi x . \equiv_x . x = c$, and this c is a," which is equivalent to "$\phi x . \equiv_x . x = a$." Thus "Scott is the author of Waverley" is equivalent to:

"'x wrote Waverley' is always equivalent to 'x is Scott,'"

i.e. "x wrote Waverley" is true when x is Scott and false when x is not Scott.

Thus although "$(\iota x)(\phi x)$" has no meaning by itself, it may be substituted for y in any propositional function fy, and we get a significant proposition, though not a value of fy.

When $f\{(\iota x)(\phi x)\}$, as above defined, forms part of some other proposition, we shall say that $(\iota x)(\phi x)$ has a *secondary* occurrence. When $(\iota x)(\phi x)$ has a secondary occurrence, a proposition in which it occurs may be true even when $(\iota x)(\phi x)$ does not exist. This applies, *e.g.* to the proposition: "There

* We shall generally write "$f(\iota x)(\phi x)$" rather than "$f\{(\iota x)(\phi x)\}$" in future.

is no such person as the King of France." We may interpret this as

$$\sim \{E ! (\iota x)(\phi x)\},$$

or as

$$\sim \{(\exists c) . c = (\iota x)(\phi x)\},$$

if "ϕx" stands for "x is King of France." In either case, what is asserted is that a proposition p in which $(\iota x)(\phi x)$ occurs is false, and this proposition p is thus part of a larger proposition. The same applies to such a proposition as the following: "If France were a monarchy, the King of France would be of the House of Orleans."

It should be observed that such a proposition as

$$\sim f\{(\iota x)(\phi x)\}$$

is ambiguous; it may deny $f\{(\iota x)(\phi x)\}$, in which case it will be true if $(\iota x)(\phi x)$ does not exist, or it may mean

$$(\exists c) : \phi x . \equiv_x . x = c : \sim fc,$$

in which case it can only be true if $(\iota x)(\phi x)$ exists. In ordinary language, the latter interpretation would usually be adopted. For example, the proposition "the King of France is not bald" would usually be rejected as false, being held to mean "the King of France exists and is not bald," rather than "it is false that the King of France exists and is bald." When $(\iota x)(\phi x)$ exists, the two interpretations of the ambiguity give equivalent results; but when $(\iota x)(\phi x)$ does not exist, one interpretation is true and one is false. It is necessary to be able to distinguish these in our notation; and generally, if we have such propositions as

$$\psi(\iota x)(\phi x) . \supset . p,$$
$$p . \supset . \psi(\iota x)(\phi x),$$
$$\psi(\iota x)(\phi x) . \supset . \chi(\iota x)(\phi x),$$

and so on, we must be able by our notation to distinguish whether the whole or only part of the proposition concerned is to be treated as the "$f(\iota x)(\phi x)$" of our definition. For this purpose, we will put "$[(\iota x)(\phi x)]$" followed by dots at the beginning of the part (or whole) which is to be taken as $f(\iota x)(\phi x)$, the dots being sufficiently numerous to bracket off the $f(\iota x)(\phi x)$; *i.e.* $f(\iota x)(\phi x)$ is to be everything following the dots until we reach an equal number of dots not signifying a logical product, or a greater number signifying a logical product, or the end of the sentence, or the end of a bracket enclosing "$[(\iota x)(\phi x)]$." Thus

$$[(\iota x)(\phi x)] . \psi(\iota x)(\phi x) . \supset . p$$

will mean

$$(\exists c) : \phi x . \equiv_x . x = c : \psi c : \supset . p,$$

but

$$[(\iota x)(\phi x)] : \psi(\iota x)(\phi x) . \supset . p$$

will mean

$$(\exists c) : \phi x . \equiv_x . x = c : \psi c . \supset . p.$$

It is important to distinguish these two, for if $(\iota x)(\phi x)$ does not exist, the first is true and the second false. Again

$$[(\iota x)(\phi x)] . \sim \psi(\iota x)(\phi x)$$

will mean $(\exists c) : \phi x . \equiv_x . x = c : \sim \psi c$,

while $\sim \{[(\imath x)(\phi x)] . \psi (\imath x)(\phi x)\}$

will mean $\sim \{(\exists c) : \phi x . \equiv_x . x = c : \psi c\}$.

Here again, when $(\imath x)(\phi x)$ does not exist, the first is false and the second true.

In order to avoid this ambiguity in propositions containing $(\imath x)(\phi x)$, we amend our definition, or rather our notation, putting

$$[(\imath x)(\phi x)] . f (\imath x)(\phi x) . = : (\exists c) : \phi x . \equiv_x . x = c : fc \quad \text{Df.}$$

By means of this definition, we avoid any doubt as to the portion of our whole asserted proposition which is to be treated as the "$f(\imath x)(\phi x)$" of the definition. This portion will be called the *scope* of $(\imath x)(\phi x)$. Thus in

$$[(\imath x)(\phi x)] . f (\imath x)(\phi x) . \supset . p$$

the scope of $(\imath x)(\phi x)$ is $f(\imath x)(\phi x)$; but in

$$[(\imath x)(\phi x)] : f (\imath x)(\phi x) . \supset . p$$

the scope is $f(\imath x)(\phi x) . \supset . p$;

in $\sim \{[(\imath x)(\phi x)] . f(\imath x)(\phi x)\}$

the scope is $f(\imath x)(\phi x)$; but in

$$[(\imath x)(\phi x)] . \sim f(\imath x)(\phi x)$$

the scope is $\sim f(\imath x)(\phi x)$.

It will be seen that when $(\imath x)(\phi x)$ has the whole of the proposition concerned for its scope, the proposition concerned cannot be true unless $E ! (\imath x)(\phi x)$; but when $(\imath x)(\phi x)$ has only part of the proposition concerned for its scope, it may often be true even when $(\imath x)(\phi x)$ does not exist. It will be seen further that when $E ! (\imath x)(\phi x)$, we may enlarge or diminish the scope of $(\imath x)(\phi x)$ as much as we please without altering the truth-value of any proposition in which it occurs.

If a proposition contains two descriptions, say $(\imath x)(\phi x)$ and $(\imath x)(\psi x)$, we have to distinguish which of them has the larger scope, *i.e.* we have to distinguish

(1) $[(\imath x)(\phi x)] : [(\imath x)(\psi x)] . f\{(\imath x)(\phi x), (\imath x)(\psi x)\}$,

(2) $[(\imath x)(\psi x)] : [(\imath x)(\phi x)] . f\{(\imath x)(\phi x), (\imath x)(\psi x)\}$.

The first of these, eliminating $(\imath x)(\phi x)$, becomes

(3) $(\exists c) : \phi x . \equiv_x : x = c : [(\imath x)(\psi x)] . f\{c, (\imath x)(\psi x)\}$,

which, eliminating $(\imath x)(\psi x)$, becomes

(4) $(\exists c) :. \phi x . \equiv_x . x = c :. (\exists d) : \psi x . \equiv_x . x = d : f(c, d)$,

and the same proposition results if, in (1), we eliminate first $(\imath x)(\psi x)$ and then $(\imath x)(\phi x)$. Similarly (2) becomes, when $(\imath x)(\phi x)$ and $(\imath x)(\psi x)$ are eliminated,

(5) $(\exists d) :. \psi x . \equiv_x . x = d :. (\exists c) : \phi x . \equiv_x . x = c : f(c, d)$.

(4) and (5) are equivalent, so that the truth-value of a proposition containing two descriptions is independent of the question which has the larger scope.

It will be found that, in most cases in which descriptions occur, their scope is, in practice, the smallest proposition enclosed in dots or other brackets in which they are contained. Thus for example

$$[(\imath x)(\phi x)] . \psi (\imath x)(\phi x) . \supset . [(\imath x)(\phi x)] . \chi (\imath x)(\phi x)$$

will occur much more frequently than

$$[(\imath x)(\phi x)] : \psi (\imath x)(\phi x) . \supset . \chi (\imath x)(\phi x).$$

For this reason it is convenient to decide that, when the scope of an occurrence of $(\imath x)(\phi x)$ is the smallest proposition, enclosed in dots or other brackets, in which the occurrence in question is contained, the scope need not be indicated by "$[(\imath x)(\phi x)]$." Thus *e.g.*

$$p . \supset . a = (\imath x)(\phi x)$$

will mean $p . \supset . [(\imath x)(\phi x)] . a = (\imath x)(\phi x)$;

and $p . \supset . (\exists a) . a = (\imath x)(\phi x)$

will mean $p . \supset . (\exists a) . [(\imath x)(\phi x)] . a = (\imath x)(\phi x)$;

and $p . \supset . a \neq (\imath x)(\phi x)$

will mean $p . \supset . [(\imath x)(\phi x)] . \sim \{a = (\imath x)(\phi x)\}$;

but $p . \supset . \sim \{a = (\imath x)(\phi x)\}$

will mean $p . \supset . \sim \{[(\imath x)(\phi x)] . a = (\imath x)(\phi x)\}$.

This convention enables us, in the vast majority of cases that actually occur, to dispense with the explicit indication of the scope of a descriptive symbol; and it will be found that the convention agrees very closely with the tacit conventions of ordinary language on this subject. Thus for example, if "$(\imath x)(\phi x)$" is "the so-and-so," "$a \neq (\imath x)(\phi x)$" is to be read "$a$ is not the so-and-so," which would ordinarily be regarded as implying that "the so-and-so" exists; but "$\sim \{a = (\imath x)(\phi x)\}$" is to be read "it is not true that a is the so-and-so," which would generally be allowed to hold if "the so-and-so" does not exist. Ordinary language is, of course, rather loose and fluctuating in its implications on this matter; but subject to the requirement of definiteness, our convention seems to keep as near to ordinary language as possible.

In the case when the smallest proposition enclosed in dots or other brackets contains two or more descriptions, we shall assume, in the absence of any indication to the contrary, that one which typographically occurs earlier has a larger scope than one which typographically occurs later. Thus

$$(\imath x)(\phi x) = (\imath x)(\psi x)$$

will mean $(\exists c) : \phi x . \equiv_x . x = c : [(\imath x)(\psi x)] . c = (\imath x)(\psi x)$,

while $(\imath x)(\psi x) = (\imath x)(\phi x)$

will mean $(\exists d) : \psi x . \equiv_x . x = d : [(\imath x)(\phi x)] . (\imath x)(\phi x) = d$.

These two propositions are easily shown to be equivalent.

(2) *Classes.* The symbols for classes, like those for descriptions, are, in our system, incomplete symbols: their *uses* are defined, but they themselves are not assumed to mean anything at all. That is to say, the uses of such

symbols are so defined that, when the *definiens* is substituted for the *definiendum*, there no longer remains any symbol which could be supposed to represent a class. Thus classes, so far as we introduce them, are merely symbolic or linguistic conveniences, not genuine objects as their members are if they are individuals.

It is an old dispute whether formal logic should concern itself mainly with intensions or with extensions. In general, logicians whose training was mainly philosophical have decided for intensions, while those whose training was mainly mathematical have decided for extensions. The facts seem to be that, while mathematical logic requires extensions, philosophical logic refuses to supply anything except intensions. Our theory of classes recognizes and reconciles these two apparently opposite facts, by showing that an extension (which is the same as a class) is an incomplete symbol, whose use always acquires its meaning through a reference to intension.

In the case of descriptions, it was possible to *prove* that they are incomplete symbols. In the case of classes, we do not know of any equally definite proof, though arguments of more or less cogency can be elicited from the ancient problem of the One and the Many*. It is not necessary for our purposes, however, to assert dogmatically that there are no such things as classes. It is only necessary for us to show that the incomplete symbols which we introduce as representatives of classes yield all the propositions for the sake of which classes might be thought essential. When this has been shown, the mere principle of economy of primitive ideas leads to the non-introduction of classes except as incomplete symbols.

To explain the theory of classes, it is necessary first to explain the distinction between *extensional* and *intensional* functions. This is effected by the following definitions:

The *truth-value* of a proposition is truth if it is true, and falsehood if it is false. (This expression is due to Frege.)

Two propositions are said to be *equivalent* when they have the same truth-value, *i.e.* when they are both true or both false.

Two propositional functions are said to be *formally equivalent* when they are equivalent with every possible argument, *i.e.* when any argument which satisfies the one satisfies the other, and vice versa. Thus "$\hat{x}$ is a man" is formally equivalent to "$\hat{x}$ is a featherless biped"; "$\hat{x}$ is an even prime" is formally equivalent to "$\hat{x}$ is identical with 2."

A function of a function is called *extensional* when its truth-value with any argument is the same as with any formally equivalent argument. That is to

* Briefly, these arguments reduce to the following: If there is such an object as a class, it must be in some sense *one* object. Yet it is only of classes that *many* can be predicated. Hence, if we admit classes as objects, we must suppose that the same object can be both one and many, which seems impossible.

say, $f(\phi\hat{z})$ is an extensional function of $\phi\hat{z}$ if, provided $\psi\hat{z}$ is formally equivalent to $\phi\hat{z}$, $f(\phi\hat{z})$ is equivalent to $f(\psi\hat{z})$. Here the apparent variables ϕ and ψ are necessarily of the type from which arguments can significantly be supplied to f. We find no need to use as apparent variables any functions of non-predicative types; accordingly in the sequel all extensional functions considered are in fact functions of predicative functions*.

A function of a function is called *intensional* when it is not extensional.

The nature and importance of the distinction between intensional and extensional functions will be made clearer by some illustrations. The proposition "'x is a man' always implies 'x is a mortal'" is an extensional function of the function "$\hat{x}$ is a man," because we may substitute, for "x is a man," "x is a featherless biped," or any other statement which applies to the same objects to which "x is a man" applies, and to no others. But the proposition "A believes that 'x is a man' always implies 'x is a mortal'" is an intensional function of "$\hat{x}$ is a man," because A may never have considered the question whether featherless bipeds are mortal, or may believe wrongly that there are featherless bipeds which are not mortal. Thus even if "x is a featherless biped" is formally equivalent to "x is a man," it by no means follows that a person who believes that all men are mortal must believe that all featherless bipeds are mortal, since he may have never thought about featherless bipeds, or have supposed that featherless bipeds were not always men. Again the proposition "the number of arguments that satisfy the function $\phi ! \hat{z}$ is n" is an extensional function of $\phi ! \hat{z}$, because its truth or falsehood is unchanged if we substitute for $\phi ! \hat{z}$ any other function which is true whenever $\phi ! \hat{z}$ is true, and false whenever $\phi ! \hat{z}$ is false. But the proposition "A asserts that the number of arguments satisfying $\phi ! \hat{z}$ is n" is an intensional function of $\phi ! \hat{z}$, since, if A asserts this concerning $\phi ! \hat{z}$, he certainly cannot assert it concerning all predicative functions that are equivalent to $\phi ! \hat{z}$, because life is too short. Again, consider the proposition "two white men claim to have reached the North Pole." This proposition states "two arguments satisfy the function '$\hat{x}$ is a white man who claims to have reached the North Pole.'" The truth or falsehood of this proposition is unaffected if we substitute for "$\hat{x}$ is a white man who claims to have reached the North Pole" any other statement which holds of the same arguments, and of no others. Hence it is an extensional function. But the proposition "it is a strange coincidence that two white men should claim to have reached the North Pole," which states "it is a strange coincidence that two arguments should satisfy the function '$\hat{x}$ is a white man who claims to have reached the North Pole,'" is not equivalent to "it is a strange coincidence that two arguments should satisfy the function '$\hat{x}$ is Dr Cook or Commander Peary.'" Thus "it is a strange coincidence that $\phi ! \hat{x}$ should be satisfied by two arguments" is an intensional function of $\phi ! \hat{x}$.

* Cf. p. 53.

The above instances illustrate the fact that the functions of functions with which mathematics is specially concerned are extensional, and that intensional functions of functions only occur where non-mathematical ideas are introduced, such as what somebody believes or affirms, or the emotions aroused by some fact. Hence it is natural, in a mathematical logic, to lay special stress on *extensional* functions of functions.

When two functions are formally equivalent, we may say that they *have the same extension*. In this definition, we are in close agreement with usage. We do not assume that there is such a thing as an extension: we merely define the whole phrase "having the same extension." We may now say that an extensional function of a function is one whose truth or falsehood depends only upon the extension of its argument. In such a case, it is convenient to regard the statement concerned as being about the extension. Since extensional functions are many and important, it is natural to regard the extension as an object, called a *class*, which is supposed to be the subject of all the equivalent statements about various formally equivalent functions. Thus *e.g.* if we say "there were twelve Apostles," it is natural to regard this statement as attributing the property of being twelve to a certain collection of men, namely those who were Apostles, rather than as attributing the property of being satisfied by twelve arguments to the function "$\hat{x}$ was an Apostle." This view is encouraged by the feeling that there is something which is identical in the case of two functions which "have the same extension." And if we take such simple problems as "how many combinations can be made of n things?" it seems at first sight necessary that each "combination" should be a single object which can be counted as one. This, however, is certainly not necessary technically, and we see no reason to suppose that it is true philosophically. The technical procedure by which the apparent difficulty is overcome is as follows.

We have seen that an extensional function of a function may be regarded as a function of the class determined by the argument-function, but that an intensional function cannot be so regarded. In order to obviate the necessity of giving different treatment to intensional and extensional functions of functions, we construct an extensional function derived from any function of a predicative function $\psi ! \hat{z}$, and having the property of being equivalent to the function from which it is derived, provided this function is extensional, as well as the property of being significant (by the help of the systematic ambiguity of equivalence) with any argument $\phi\hat{z}$ whose arguments are of the same type as those of $\psi ! \hat{z}$. The derived function, written "$f\{\hat{z}(\phi z)\}$," is defined as follows: Given a function $f(\psi ! \hat{z})$, our derived function is to be "there is a predicative function which is formally equivalent to $\phi\hat{z}$ and satisfies f." If $\phi\hat{z}$ is a predicative function, our derived function will be true whenever $f(\phi\hat{z})$ is true. If $f(\phi\hat{z})$ is an extensional function, and $\phi\hat{z}$ is a predicative function, our derived function will not be true unless $f(\phi\hat{z})$ is true; thus in this case, our derived function is equivalent to $f(\phi\hat{z})$. If $f(\phi\hat{z})$ is not an extensional function, and if $\phi\hat{z}$ is a predicative function, our derived function may sometimes be true when the original function is false. But in any case the derived function is always extensional.

In order that the derived function should be significant for any function $\phi\hat{z}$, of whatever order, provided it takes arguments of the right type, it is necessary and sufficient that $f(\psi ! \hat{z})$ should be significant, where $\psi ! \hat{z}$ is any *predicative* function. The reason of this is that we only require, concerning an argument $\phi\hat{z}$, the hypothesis that it is formally equivalent to some predicative function $\psi ! \hat{z}$, and formal equivalence has the same kind of systematic ambiguity as to type that belongs to truth and falsehood, and can therefore hold between functions of any two different orders, provided the functions take arguments of the same type. Thus by means of our derived function we have not merely provided extensional functions everywhere in place of intensional functions, but we have *practically* removed the necessity for considering differences of type among functions whose arguments are of the same type. This effects the same kind of simplification in our hierarchy as would result from never considering any but predicative functions.

If $f(\psi ! \hat{z})$ can be built up by means of the primitive ideas of disjunction, negation, $(x) . \phi x$, and $(\exists x) . \phi x$, as is the case with all the functions of functions that explicitly occur in the present work, it will be found that, in virtue of the systematic ambiguity of the above primitive ideas, any function $\phi\hat{z}$ whose arguments are of the same type as those of $\psi ! \hat{z}$ can significantly be substituted for $\psi ! \hat{z}$ in f without any other symbolic change. Thus in such a case what is symbolically, though not really, the same function f can receive as arguments functions of various different types. If, with a given argument $\phi\hat{z}$, the function $f(\phi\hat{z})$, so interpreted, is equivalent to $f(\psi ! \hat{z})$ whenever $\psi ! \hat{z}$ is formally equivalent to $\phi\hat{z}$, then $f\{\hat{z}(\phi z)\}$ is equivalent to $f(\phi\hat{z})$ provided there is any predicative function formally equivalent to $\phi\hat{z}$. At this point, we make use of the axiom of reducibility, according to which there always is a predicative function formally equivalent to $\phi\hat{z}$.

As was explained above, it is convenient to regard an extensional function of a function as having for its argument not the function, but the class determined by the function. Now we have seen that our derived function is always extensional. Hence if our original function was $f(\psi ! \hat{z})$, we write the derived function $f\{\hat{z}(\phi z)\}$, where "$\hat{z}(\phi z)$" may be read "the class of arguments which satisfy $\phi\hat{z}$," or more simply "the class determined by $\phi\hat{z}$." Thus "$f\{\hat{z}(\phi z)\}$" will mean: "There is a predicative function $\psi ! \hat{z}$ which is formally equivalent to $\phi\hat{z}$ and is such that $f(\psi ! \hat{z})$ is true." This is in reality a function of $\phi\hat{z}$, but we treat it symbolically as if it had an argument $\hat{z}(\phi z)$. By the help of the axiom of reducibility, we find that the usual properties of classes result. For example, two formally equivalent functions determine the same class, and conversely, two functions which determine the same class are formally equivalent. Also to say that x is a member of $\hat{z}(\phi z)$, *i.e.* of the class determined by $\phi\hat{z}$, is true when ϕx is true, and false when ϕx is false. Thus all the mathematical purposes for which classes might seem to be required are fulfilled by the purely symbolic objects $\hat{z}(\phi z)$, provided we assume the axiom of reducibility.

In virtue of the axiom of reducibility, if $\phi\hat{z}$ is any function, there is a formally equivalent predicative function $\psi ! \hat{z}$; then the class $\hat{z}(\phi z)$ is identical with the class $\hat{z}(\psi ! z)$, so that every class can be defined by a *predicative* function. Hence the totality of the *classes* to which a given term can be significantly said to belong or not to belong is a legitimate totality, although the totality of *functions* which a given term can be significantly said to satisfy or not to satisfy is not a legitimate totality. The classes to which a given term a belongs or does not belong are the classes defined by a-functions; they are also the classes defined by *predicative* a-functions. Let us call them a-classes. Then "a-classes" form a legitimate totality, derived from that of predicative a-functions. Hence many kinds of general statements become possible which would otherwise involve vicious-circle paradoxes. These general statements are none of them such as lead to contradictions, and many of them such as it is very hard to suppose illegitimate. The fact that they are rendered possible by the axiom of reducibility, and that they would otherwise be excluded by the vicious-circle principle, is to be regarded as an argument in favour of the axiom of reducibility.

The above definition of "the class defined by the function $\phi\hat{z}$," or rather, of any proposition in which this phrase occurs, is, in symbols, as follows:

$$f\{\hat{z}(\phi z)\} . = : (\exists \psi) : \phi x . \equiv_x . \psi ! x : f\{\psi ! \hat{z}\} \quad \text{Df.}$$

In order to recommend this definition, we shall enumerate five requisites which a definition of classes must satisfy, and we shall then show that the above definition satisfies these five requisites.

We require of classes, if they are to serve the purposes for which they are commonly employed, that they shall have certain properties, which may be enumerated as follows. (1) Every propositional function must determine a class, which may be regarded as the collection of all the arguments satisfying the function in question. This principle must hold when the function is satisfied by an infinite number of arguments as well as when it is satisfied by a finite number. It must hold also when no arguments satisfy the function; *i.e.* the "null-class" must be just as good a class as any other. (2) Two propositional functions which are formally equivalent, *i.e.* such that any argument which satisfies either satisfies the other, must determine the same class; that is to say, a class must be something wholly determined by its membership, so that *e.g.* the class "featherless bipeds" is identical with the class "men," and

the class "even primes" is identical with the class "numbers identical with 2." (3) Conversely, two propositional functions which determine the same class must be formally equivalent; in other words, when the class is given, the membership is determinate: two different sets of objects cannot yield the same class. (4) In the same sense in which there are classes (whatever this sense may be), or in some closely analogous sense, there must also be classes of classes. Thus for example "the combinations of n things m at a time," where the n things form a given class, is a class of classes; each combination of m things is a class, and each such class is a member of the specified set of combinations, which set is therefore a class whose members are classes. Again, the class of unit classes, or of couples, is absolutely indispensable; the former is the number 1, the latter the number 2. Thus without classes of classes, arithmetic becomes impossible. (5) It must under all circumstances be meaningless to suppose a class identical with one of its own members. For if such a supposition had any meaning "$\alpha \epsilon \alpha$" would be a significant propositional function*, and so would "$\alpha \sim \epsilon \alpha$." Hence, by (1) and (4), there would be a class of all classes satisfying the function "$\alpha \sim \epsilon \alpha$." If we call this class κ, we shall have

$$\alpha \epsilon \kappa . \equiv_\alpha . \alpha \sim \epsilon \alpha.$$

Since, by our hypothesis, "$\kappa \epsilon \kappa$" is supposed significant, the above equivalence, which holds with all possible values of α, holds with the value κ, *i.e.*

$$\kappa \epsilon \kappa . \equiv . \kappa \sim \epsilon \kappa.$$

But this is a contradiction†. Hence "$\alpha \epsilon \alpha$" and "$\alpha \sim \epsilon \alpha$" must always be meaningless. In general, there is nothing surprising about this conclusion, but it has two consequences which deserve special notice. In the first place, a class consisting of only one member must not be identical with that one member, *i.e.* we must not have $\iota ' x = x$. For we have $x \epsilon \iota ' x$, and therefore, if $x = \iota ' x$, we have $\iota ' x \epsilon \iota ' x$, which, we saw, must be meaningless. It follows that "$x = \iota ' x$" must be absolutely meaningless, not simply false. In the second place, it might appear as if the class of all classes were a class, *i.e.* as if (writing "Cls" for "class") "Cls ϵ Cls" were a true proposition. But this combination of symbols must be meaningless; unless, indeed, an ambiguity exists in the meaning of "Cls," so that, in "Cls ϵ Cls," the first "Cls" can be supposed to have a different meaning from the second.

As regards the above requisites, it is plain, to begin with, that, in accordance with our definition, every propositional function $\phi\hat{z}$ determines a class $\hat{z}(\phi z)$. Assuming the axiom of reducibility, there must always be true propositions about $\hat{z}(\phi z)$, *i.e.* true propositions of the form $f\{\hat{z}(\phi z)\}$. For suppose $\phi\hat{z}$ is formally equivalent to $\psi ! \hat{z}$, and suppose $\psi ! \hat{z}$ satisfies some function f. Then

* As explained in Chapter I (p. 25), "$x \epsilon a$" means "x is a member of the class a," or, more shortly, "x is an a." The definition of this expression in terms of our theory of classes will be given shortly.

† This is the second of the contradictions discussed at the end of Chapter II.

$\hat{z}(\phi z)$ also satisfies f. Hence, given any function $\phi\hat{z}$, there are true propositions of the form $f\{\hat{z}(\phi z)\}$, *i.e.* true propositions in which "the class determined by $\phi\hat{z}$" is grammatically the subject. This shows that our definition fulfils the first of our five requisites.

The second and third requisites together demand that the classes $\hat{z}(\phi z)$ and $\hat{z}(\psi z)$ should be identical when, and only when, their defining functions are formally equivalent, *i.e.* that we should have

$$\hat{z}(\phi z)=\hat{z}(\psi z) . \equiv : \phi x . \equiv_x . \psi x.$$

Here the meaning of "$\hat{z}(\phi z)=\hat{z}(\psi z)$" is to be derived, by means of a two-fold application of the definition of $f\{\hat{z}(\phi z)\}$, from the definition of

$$\text{“}\chi!\hat{z}=\theta!\hat{z},\text{”}$$

which is $\chi!\hat{z}=\theta!\hat{z} . = : (f) : f!\chi!\hat{z} . \supset . f!\theta!\hat{z}$ Df

by the general definition of identity.

In interpreting "$\hat{z}(\phi z)=\hat{z}(\psi z)$," we will adopt the convention which we adopted in regard to $(\iota x)(\phi x)$ and $(\iota x)(\psi x)$, namely that the incomplete symbol which occurs first is to have the larger scope. Thus $\hat{z}(\phi z)=\hat{z}(\psi z)$ becomes, by our definition,

$$(\exists\chi) : \phi x . \equiv_x . \chi!x : \chi!\hat{z}=\hat{z}(\psi z),$$

which, by eliminating $\hat{z}(\psi z)$, becomes

$$(\exists\chi) :. \phi x . \equiv_x . \chi!x :. (\exists\theta) : \psi x . \equiv_x . \theta!x : \chi!\hat{z}=\theta!\hat{z},$$

which is equivalent to

$$(\exists\chi,\theta) : \phi x . \equiv_x . \chi!x : \psi x . \equiv_x . \theta!x : \chi!\hat{z}=\theta!\hat{z},$$

which, again, is equivalent to

$$(\exists\chi) : \phi x . \equiv_x . \chi!x : \psi x . \equiv_x . \chi!x,$$

which, in virtue of the axiom of reducibility, is equivalent to

$$\phi x . \equiv_x . \psi x.$$

Thus our definition of the use of $\hat{z}(\phi z)$ is such as to satisfy the conditions (2) and (3) which we laid down for classes, *i.e.* we have

$$\vdash :. \hat{z}(\phi z)=\hat{z}(\psi z) . \equiv : \phi x . \equiv_x . \psi x.$$

Before considering classes of classes, it will be well to define membership of a class, *i.e.* to define the symbol "$x\in\hat{z}(\phi z)$," which may be read "x is a member of the class determined by $\phi\hat{z}$." Since this is a function of the form $f\{\hat{z}(\phi z)\}$, it must be derived, by means of our general definition of such functions, from the corresponding function $f\{\psi!\hat{z}\}$. We therefore put

$$x\in\psi!\hat{z} . = . \psi!x \quad \text{Df.}$$

This definition is only needed in order to give a meaning to "$x\in\hat{z}(\phi z)$"; the meaning it gives is, in virtue of the definition of $f\{\hat{z}(\phi z)\}$,

$$(\exists\psi) : \phi y . \equiv_y . \psi!y : \psi!x.$$

It thus appears that "$x\in\hat{z}(\phi z)$" implies ϕx, since it implies $\psi!x$, and $\psi!x$ is equivalent to ϕx; also, in virtue of the axiom of reducibility, ϕx implies "$x\in\hat{z}(\phi z)$," since there is a predicative function ψ formally equivalent to ϕ,

and x must satisfy ψ, since x (*ex hypothesi*) satisfies ϕ. Thus in virtue of the axiom of reducibility we have

$$\vdash : x\in\hat{z}(\phi z) . \equiv . \phi x,$$

i.e. x is a member of the class $\hat{z}(\phi z)$ when, and only when, x satisfies the function ϕ which defines the class.

We have next to consider how to interpret a class of classes. As we have defined $f\{\hat{z}(\phi z)\}$, we shall naturally regard a class of classes as consisting of those values of $\hat{z}(\phi z)$ which satisfy $f\{\hat{z}(\phi z)\}$. Let us write α for $\hat{z}(\phi z)$; then we may write $\hat{\alpha}(f\alpha)$ for the class of values of α which satisfy $f\alpha$*. We shall apply the same definition, and put

$$F\{\hat{\alpha}(f\alpha)\} . = : (\exists g) : f\beta . \equiv_\beta . g!\beta : F\{g!\hat{\alpha}\} \quad \text{Df,}$$

where "β" stands for any expression of the form $\hat{z}(\psi!z)$.

Let us take "$\gamma\in\hat{\alpha}(f\alpha)$" as an instance of $F\{\hat{\alpha}(f\alpha)\}$. Then

$$\vdash :. \gamma\in\hat{\alpha}(f\alpha) . \equiv : (\exists g) : f\beta . \equiv_\beta . g!\beta : \gamma\in g!\hat{\alpha}.$$

Just as we put $x\in\psi!\hat{z} . = . \psi!x$ Df,

so we put $\gamma\in g!\hat{\alpha} . = . g!\gamma$ Df.

Thus we find

$$\vdash :. \gamma\in\hat{\alpha}(f\alpha) . \equiv : (\exists g) : f\beta . \equiv_\beta . g!\beta : g!\gamma.$$

If we now extend the axiom of reducibility so as to apply to functions of functions, *i.e.* if we assume

$$(\exists g) : f(\psi!\hat{z}) . \equiv_\psi . g!(\psi!\hat{z}),$$

we easily deduce

$$\vdash : (\exists g) : f\{\hat{z}(\psi!z)\} . \equiv_\psi . g!\{\hat{z}(\psi!z)\},$$

i.e. $\vdash : (\exists g) : f\beta . \equiv_\beta . g!\beta.$

Thus $\vdash : \gamma\in\hat{\alpha}(f\alpha) . \equiv . f\gamma.$

Thus every function which can take classes as arguments, *i.e.* every function of functions, determines a class of classes, whose members are those classes which satisfy the determining function. Thus the theory of classes of classes offers no difficulty.

We have next to consider our fifth requisite, namely that "$\hat{z}(\phi z)\in\hat{z}(\phi z)$" is to be meaningless. Applying our definition of $f\{\hat{z}(\phi z)\}$, we find that if this collection of symbols had a meaning, it would mean

$$(\exists\psi) : \phi x . \equiv_x . \psi!x : \psi!\hat{z}\in\psi!\hat{z},$$

i.e. in virtue of the definition

$$x\in\psi!\hat{z} . = . \psi!x \quad \text{Df,}$$

it would mean $(\exists\psi) : \phi x . \equiv_x . \psi!x : \psi!(\psi!\hat{z})$.

But here the symbol "$\psi!(\psi!\hat{z})$" occurs, which assigns a function as argument to itself. Such a symbol is always meaningless, for the reasons explained at the beginning of Chapter II (pp. 38—41). Hence "$\hat{z}(\phi z)\in\hat{z}(\phi z)$" is meaningless, and our fifth and last requisite is fulfilled.

* The use of a single letter, such as α or β, to represent a variable class, will be further explained shortly.

As in the case of $f(\imath x)(\phi x)$, so in that of $f\{\hat{z}(\phi z)\}$, there is an ambiguity as to the scope of $\hat{z}(\phi z)$ if it occurs in a proposition which itself is part of a larger proposition. But in the case of classes, since we always have the axiom of reducibility, namely

$$(\exists\psi) : \phi x . \equiv_x . \psi ! x,$$

which takes the place of $E!(\imath x)(\phi x)$, it follows that the truth-value of any proposition in which $\hat{z}(\phi z)$ occurs is the same whatever scope we may give to $\hat{z}(\phi z)$, provided the proposition is an extensional function of whatever functions it may contain. Hence we may adopt the convention that the scope is to be always the smallest proposition enclosed in dots or brackets in which $\hat{z}(\phi z)$ occurs. If at any time a larger scope is required, we may indicate it by "$[\hat{z}(\phi z)]$" followed by dots, in the same way as we did for $[(\imath x)(\phi x)]$.

Similarly when two class symbols occur, *e.g.* in a proposition of the form $f\{\hat{z}(\phi z), \hat{z}(\psi z)\}$, we need not remember rules for the scopes of the two symbols, since all choices give equivalent results, as it is easy to prove. For the preliminary propositions a rule is desirable, so we can decide that the class symbol which occurs first in the order of writing is to have the larger scope.

The representation of a class by a single letter α can now be understood. For the denotation of α is ambiguous, in so far as it is undecided as to which of the symbols $\hat{z}(\phi z)$, $\hat{z}(\psi z)$, $\hat{z}(\chi z)$, etc. it is to stand for, where $\phi\hat{z}$, $\psi\hat{z}$, $\chi\hat{z}$, etc. are the various determining functions of the class. According to the choice made, different propositions result. But all the resulting propositions are equivalent by virtue of the easily proved proposition:

$$\text{“}\vdash : \phi x \equiv_x \psi x . \supset . f\{\hat{z}(\phi z)\} \equiv f\{\hat{z}(\psi z)\}.\text{”}$$

Hence unless we wish to discuss the determining function itself, so that the notion of a class is really not properly present, the ambiguity in the denotation of α is entirely immaterial, though, as we shall see immediately, we are led to limit ourselves to predicative determining functions. Thus "$f(\alpha)$," where α is a variable class, is really "$f\{\hat{z}(\phi z)\}$," where ϕ is a variable function, that is, it is

$$\text{“}(\exists\psi) . \phi x \equiv_x \psi ! x . f\{\psi ! \hat{z}\},\text{”}$$

where ϕ is a variable function. But here a difficulty arises which is removed by a limitation to our practice and by the axiom of reducibility. For the determining functions $\phi\hat{z}$, $\psi\hat{z}$, etc. will be of different types, though the axiom of reducibility secures that some are predicative functions. Then, in interpreting α as a variable in terms of the variation of any determining function, we shall be led into errors unless we confine ourselves to predicative determining functions. These errors especially arise in the transition to total variation (cf. pp. 15, 16). Accordingly

$$f\alpha = . (\exists\psi) . \phi ! x \equiv_x \psi ! x . f\{\psi ! \hat{z}\} \quad \text{Df.}$$

It is the peculiarity of a definition of the use of a single letter [viz. α] for a variable incomplete symbol that it, though in a sense a real variable, occurs only in the *definiendum*, while "ϕ," though a real variable, occurs only in the *definiens*.

Thus "$f\hat{\alpha}$" stands for

$$\text{“}(\exists\psi) . \hat{\phi} ! x \equiv_x \psi ! x . f\{\psi ! \hat{z}\},\text{”}$$

and "$(\alpha) . f\alpha$" stands for

$$\text{“}(\phi) : (\exists\psi) . \phi ! x \equiv_x \psi ! x . f\{\psi ! \hat{z}\}.\text{”}$$

Accordingly, in mathematical reasoning, we can dismiss the whole apparatus of functions and think only of classes as "quasi-things," capable of immediate representation by a single name. The advantages are two-fold: (1) classes are determined by their membership, so that to one set of members there is one class, (2) the "type" of a class is entirely defined by the type of its members.

Also a predicative function of a class can be defined thus

$$f!\alpha = . (\exists\psi) . \phi ! x \equiv_x \psi ! x . f!\{\psi ! \hat{z}\} \quad \text{Df.}$$

Thus a predicative function of a class is always a predicative function of any predicative determining function of the class, though the converse does not hold.

(3) *Relations.* With regard to relations, we have a theory strictly analogous to that which we have just explained as regards classes. Relations in extension, like classes, are incomplete symbols. We require a division of functions of two variables into predicative and non-predicative functions, again for reasons which have been explained in Chapter II. We use the notation "$\phi!(x, y)$" for a *predicative* function of x and y.

We use "$\phi!(\hat{x}, \hat{y})$" for the function as opposed to its values; and we use "$\hat{x}\hat{y}\phi(x, y)$" for the relation (in extension) determined by $\phi(x, y)$. We put

$$f\{\hat{x}\hat{y}\phi(x, y)\} . = : (\exists\psi) : \phi(x, y) . \equiv_{x,y} . \psi!(x, y) : f\{\psi!(\hat{x}, \hat{y})\} \quad \text{Df.}$$

Thus even when $f\{\psi!(\hat{x}, \hat{y})\}$ is not an extensional function of ψ, $f\{\hat{x}\hat{y}\phi(x, y)\}$ *is* an extensional function of ϕ. Hence, just as in the case of classes, we deduce

$$\vdash :. \hat{x}\hat{y}\phi(x, y) = \hat{x}\hat{y}\psi(x, y) . \equiv : \phi(x, y) . \equiv_{x,y} . \psi(x, y),$$

i.e. a relation is determined by its extension, and vice versa.

On the analogy of the definition of "$x \epsilon \psi ! \hat{z}$," we put

$$x\{\psi!(\hat{x}, \hat{y})\} y . = . \psi!(x, y) \quad \text{Df*.}$$

This definition, like that of "$x \epsilon \psi ! \hat{z}$," is not introduced for its own sake, but in order to give a meaning to

$$x\{\hat{x}\hat{y}\phi(x, y)\} y.$$

This meaning, in virtue of our definitions, is

$$(\exists\psi) : \phi(x, y) . \equiv_{x,y} . \psi!(x, y) : x\{\psi!(\hat{x}, \hat{y})\} y,$$

i.e.

$$(\exists\psi) : \phi(x, y) . \equiv_{x,y} . \psi!(x, y) : \psi!(x, y),$$

and this, in virtue of the axiom of reducibility

$$\text{“}(\exists\psi) : \phi(x, y) . \equiv_{x,y} . \psi!(x, y),\text{”}$$

is equivalent to $\phi(x, y)$.

Thus we have always

$$\vdash : x\{\hat{x}\hat{y}\phi(x, y)\} y . \equiv . \phi(x, y).$$

* This definition raises certain questions as to the two senses of a relation, which are dealt with in *21.

Whenever the determining function of a relation is not relevant, we may replace $\hat{x}\hat{y}\,\phi(x,y)$ by a single capital letter. In virtue of the propositions given above,

$$\vdash :. R = S . \equiv : xRy . \equiv_{x,y} . xSy,$$

$$\vdash :. R = \hat{x}\hat{y}\,\phi(x,y) . \equiv : xRy . \equiv_{x,y} . \phi(x,y),$$

and $$\vdash . R = \hat{x}\hat{y}\,(xRy).$$

Classes of relations, and relations of relations, can be dealt with as classes of classes were dealt with above.

Just as a class must not be capable of being or not being a member of itself, so a relation must neither be nor not be referent or relatum with respect to itself. This turns out to be equivalent to the assertion that $\phi\,!\,(\hat{x},\hat{y})$ cannot significantly be either of the arguments x or y in $\phi\,!\,(x,y)$. This principle, again, results from the limitation to the possible arguments to a function explained at the beginning of Chapter II.

We may sum up this whole discussion on incomplete symbols as follows.

The use of the symbol "$(\imath x)(\phi x)$" as if in "$f(\imath x)(\phi x)$" it *directly* represented an argument to the function $f\hat{z}$ is rendered possible by the theorems

$$\vdash :. E\,!\,(\imath x)(\phi x) . \supset : (x) . fx . \supset . f(\imath x)(\phi x),$$

$$\vdash : (\imath x)(\phi x) = (\imath x)(\psi x) . \supset . f(\imath x)(\phi x) \equiv f(\imath x)(\psi x),$$

$$\vdash : E\,!\,(\imath x)(\phi x) . \supset . (\imath x)(\phi x) = (\imath x)(\phi x),$$

$$\vdash : (\imath x)(\phi x) = (\imath x)(\psi x) . \equiv . (\imath x)(\psi x) = (\imath x)(\phi x),$$

$$\vdash : (\imath x)(\phi x) = (\imath x)(\psi x) . (\imath x)(\psi x) = (\imath x)(\chi x) . \supset . (\imath x)(\phi x) = (\imath x)(\chi x).$$

The use of the symbol "$\hat{x}(\phi x)$" (or of a single letter, such as α, to represent such a symbol) as if, in "$f\{\hat{x}(\phi x)\}$," it *directly* represented an argument α to a function $f\hat{\alpha}$, is rendered possible by the theorems

$$\vdash : (\alpha) . f\alpha . \supset . f\{\hat{x}(\phi x)\},$$

$$\vdash : \hat{x}(\phi x) = \hat{x}(\psi x) . \supset . f\{\hat{x}(\phi x)\} \equiv f\{\hat{x}(\psi x)\},$$

$$\vdash . \hat{x}(\phi x) = \hat{x}(\phi x),$$

$$\vdash : \hat{x}(\phi x) = \hat{x}(\psi x) . \equiv . \hat{x}(\psi x) = \hat{x}(\phi x),$$

$$\vdash : \hat{x}(\phi x) = \hat{x}(\psi x) . \hat{x}(\psi x) = \hat{x}(\chi x) . \supset . \hat{x}(\phi x) = \hat{x}(\chi x).$$

Throughout these propositions the types must be supposed to be properly adjusted, where ambiguity is possible.

The use of the symbol "$\hat{x}\hat{y}\,\{\phi(x,y)\}$" (or of a single letter, such as R, to represent such a symbol) as if, in "$f\{\hat{x}\hat{y}\,\phi(x,y)\}$," it *directly* represented an argument R to a function $f\hat{R}$, is rendered possible by the theorems

$$\vdash : (R) . fR . \supset . f\{\hat{x}\hat{y}\,\phi(x,y)\},$$

$$\vdash : \hat{x}\hat{y}\,\phi(x,y) = \hat{x}\hat{y}\,\psi(x,y) . \supset . f\{\hat{x}\hat{y}\,\phi(x,y)\} \equiv f\{\hat{x}\hat{y}\,\psi(x,y)\},$$

$$\vdash . \hat{x}\hat{y}\,\phi(x,y) = \hat{x}\hat{y}\,\phi(x,y),$$

$$\vdash : \hat{x}\hat{y}\,\phi(x,y) = \hat{x}\hat{y}\,\psi(x,y) . \equiv . \hat{x}\hat{y}\,\psi(x,y) = \hat{x}\hat{y}\,\phi(x,y),$$

$$\vdash : \hat{x}\hat{y}\,\phi(x,y) = \hat{x}\hat{y}\,\psi(x,y) . \hat{x}\hat{y}\,\psi(x,y) = \hat{x}\hat{y}\,\chi(x,y) . \supset . \hat{x}\hat{y}\,\phi(x,y) = \hat{x}\hat{y}\,\chi(x,y).$$

Throughout these propositions the types must be supposed to be properly adjusted where ambiguity is possible.

It follows from these three groups of theorems that these incomplete symbols are obedient to the same formal rules of identity as symbols which directly represent objects, so long as we only consider the *equivalence* of the resulting variable (or constant) values of propositional functions and not their identity. This consideration of the *identity* of propositions never enters into our formal reasoning.

Similarly the *limitations* to the use of these symbols can be summed up as follows. In the case of $(\imath x)(\phi x)$, the chief way in which its incompleteness is relevant is that we do not always have

$$(x) . fx . \supset . f(\imath x)(\phi x),$$

i.e. a function which is always true may nevertheless not be true of $(\imath x)(\phi x)$. This is possible because $f(\imath x)(\phi x)$ is not a value of $f\hat{x}$, so that even when all values of $f\hat{x}$ are true, $f(\imath x)(\phi x)$ may not be true. This happens when $(\imath x)(\phi x)$ does not exist. Thus for example we have $(x) . x = x$, but we do not have

the round square = the round square.

The inference $$(x) . fx . \supset . f(\imath x)(\phi x)$$

is only valid when $E\,!\,(\imath x)(\phi x)$. As soon as we know $E\,!\,(\imath x)(\phi x)$, the fact that $(\imath x)(\phi x)$ is an incomplete symbol becomes irrelevant so long as we confine ourselves to truth-functions* of whatever proposition is its scope. But even when $E\,!\,(\imath x)(\phi x)$, the incompleteness of $(\imath x)(\phi x)$ may be relevant when we pass outside truth-functions. For example, George IV wished to know whether Scott was the author of Waverley, *i.e.* he wished to know whether a proposition of the form "$c = (\imath x)(\phi x)$" was true. But there was no proposition of the form "$c = y$" concerning which he wished to know if it was true.

In regard to classes, the relevance of their incompleteness is somewhat different. It may be illustrated by the fact that we may have

$$\hat{z}(\phi z) = \psi\,!\,\hat{z} . \hat{z}(\phi z) = \chi\,!\,\hat{z}$$

without having $$\psi\,!\,\hat{z} = \chi\,!\,\hat{z}.$$

For, by a direct application of the definitions, we find that

$$\vdash : \hat{z}(\phi z) = \psi\,!\,\hat{z} . \equiv . \phi x \equiv_x \psi\,!\,x.$$

Thus we shall have

$$\vdash : \phi x \equiv_x \psi\,!\,x . \phi x \equiv_x \chi\,!\,x . \supset . \hat{z}(\phi z) = \psi\,!\,\hat{z} . \hat{z}(\phi z) = \chi\,!\,\hat{z},$$

but we shall not necessarily have $\psi\,!\,\hat{z} = \chi\,!\,\hat{z}$ under these circumstances, for two functions may well be formally equivalent without being identical; for example,

$$x = \text{Scott} . \equiv_x . x = \text{the author of Waverley},$$

but the function "$\hat{z}$ = the author of Waverley" has the property that George IV wished to know whether its value with the argument "Scott" was true, whereas

* Cf. p. 8.

the function "$\hat{z} = \text{Scott}$" has no such property, and therefore the two functions are not identical. Hence there is a propositional function, namely

$$x = y \,.\, x = z \,.\, \supset \,.\, y = z,$$

which holds without any exception, and yet does not hold when for x we substitute a class, and for y and z we substitute functions. This is only possible because a class is an incomplete symbol, and therefore "$\hat{z}(\phi z) = \psi ! \hat{z}$" is not a value of "$x = y$."

It will be observed that "$\theta ! \hat{z} = \psi ! \hat{z}$" is not an extensional function of $\psi ! \hat{z}$. Thus the scope of $\hat{z}(\phi z)$ is relevant in interpreting the product

$$\hat{z}(\phi z) = \psi ! \hat{z} \,.\, \hat{z}(\phi z) = \chi ! \hat{z}.$$

If we take the whole of the product as the scope of $\hat{z}(\phi z)$, the product is equivalent to

$$(\exists \theta) : \phi x \equiv_x \theta ! x \,.\, \theta ! \hat{z} = \psi ! \hat{z} \,.\, \theta ! \hat{z} = \chi ! \hat{z},$$

and this *does* imply $\psi ! \hat{z} = \chi ! \hat{z}$.

We may say generally that the fact that $\hat{z}(\phi z)$ is an incomplete symbol is not relevant so long as we confine ourselves to extensional functions of functions, but is apt to become relevant for other functions of functions.

PART I

MATHEMATICAL LOGIC