PRINCIPIA MATHEMATICA

TO *56

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INTRODUCTION

THE mathematical logic which occupies Part I of the present work has been constructed under the guidance of three different purposes. In the first place, it aims at effecting the greatest possible analysis of the ideas with which it deals and of the processes by which it conducts demonstrations, and at diminishing to the utmost the number of the undefined ideas and undemonstrated propositions (called respectively *primitive* ideas and *primitive* propositions) from which it starts. In the second place, it is framed with a view to the perfectly precise expression, in its symbols, of mathematical propositions: to secure such expression, and to secure it in the simplest and most convenient notation possible, is the chief motive in the choice of topics. In the third place, the system is specially framed to solve the paradoxes which, in recent years, have troubled students of symbolic logic and the theory of aggregates; it is believed that the theory of types, as set forth in what follows, leads both to the avoidance of contradictions, and to the detection of the precise fallacy which has given rise to them.

Of the above three purposes, the first and third often compel us to adopt methods, definitions, and notations which are more complicated or more difficult than they would be if we had the second object alone in view. This applies especially to the theory of descriptive expressions (∗14 and ∗30) and to the theory of classes and relations (∗20 and ∗21). On these two points, and to a lesser degree on others, it has been found necessary to make some sacrifice of lucidity to correctness. The sacrifice is, however, in the main only temporary: in each case, the notation ultimately adopted, though its real meaning is very complicated, has an apparently simple meaning which, except at certain crucial points, can without danger be substituted in thought for the real meaning. It is therefore convenient, in a preliminary explanation of the notation, to treat these apparently simple meanings as primitive ideas, *i.e.* as ideas introduced without definition. When the notation has grown more or less familiar, it is easier to follow the more complicated explanations which we believe to be more correct. In the body of the work, where it is necessary to adhere rigidly to the strict logical order, the easier order of development could not be adopted; it is therefore given in the Introduction. The explanations given in Chapter I of the Introduction are such as place lucidity before correctness; the full explanations are partly supplied in succeeding Chapters of the Introduction, partly given in the body of the work.

The use of a symbolism, other than that of words, in all parts of the book which aim at embodying strictly accurate demonstrative reasoning, has been forced on us by the consistent pursuit of the above three purposes. The reasons for this extension of symbolism beyond the familiar regions of number and allied ideas are many:

(1) The ideas here employed are more abstract than those familiarly considered in language. Accordingly there are no words which are used mainly in the exact consistent senses which are required here. Any use of words would require unnatural limitations to their ordinary meanings, which would be in fact more difficult to remember consistently than are the definitions of entirely new symbols.

(2) The grammatical structure of language is adapted to a wide variety of usages. Thus it possesses no unique simplicity in representing the few simple, though highly abstract, processes and ideas arising in the deductive trains of reasoning employed here. In fact the very abstract simplicity of the ideas of this work defeats language. Language can represent complex ideas more easily. The proposition "a whale is big" represents language at its best, giving terse expression to a complicated fact; while the true analysis of "one is a number" leads, in language, to an intolerable prolixity. Accordingly terseness is gained by using a symbolism especially designed to represent the ideas and processes of deduction which occur in this work.

(3) The adaptation of the rules of the symbolism to the processes of deduction aids the intuition in regions too abstract for the imagination readily to present to the mind the true relation between the ideas employed. For various collocations of symbols become familiar as representing important collocations of ideas; and in turn the possible relations—according to the rules of the symbolism—between these collocations of symbols become familiar, and these further collocations represent still more complicated relations between the abstract ideas. And thus the mind is finally led to construct trains of reasoning in regions of thought in which the imagination would be entirely unable to sustain itself without symbolic help. Ordinary language yields no such help. Its grammatical structure does not represent uniquely the relations between the ideas involved. Thus, "a whale is big" and "one is a number" both look alike, so that the eye gives no help to the imagination.

(4) The terseness of the symbolism enables a whole proposition to be represented to the eyesight as one whole, or at most in two or three parts divided where the natural breaks, represented in the symbolism, occur. This is a humble property, but is in fact very important in connection with the advantages enumerated under the heading (3).

(5) The attainment of the first-mentioned object of this work, namely the complete enumeration of all the ideas and steps in reasoning employed

in mathematics, necessitates both terseness and the presentation of each proposition with the maximum of formality in a form as characteristic of itself as possible.

Further light on the methods and symbolism of this book is thrown by a slight consideration of the limits to their useful employment:

(α) Most mathematical investigation is concerned not with the analysis of the complete process of reasoning, but with the presentation of such an abstract of the proof as is sufficient to convince a properly instructed mind. For such investigations the detailed presentation of the steps in reasoning is of course unnecessary, provided that the detail is carried far enough to guard against error. In this connection it may be remembered that the investigations of Weierstrass and others of the same school have shown that, even in the common topics of mathematical thought, much more detail is necessary than previous generations of mathematicians had anticipated.

(β) In proportion as the imagination works easily in any region of thought, symbolism (except for the express purpose of analysis) becomes only necessary as a convenient shorthand writing to register results obtained without its help. It is a subsidiary object of this work to show that, with the aid of symbolism, deductive reasoning can be extended to regions of thought not usually supposed amenable to mathematical treatment. And until the ideas of such branches of knowledge have become more familiar, the detailed type of reasoning, which is also required for the analysis of the steps, is appropriate to the investigation of the general truths concerning these subjects.

CHAPTER I

PRELIMINARY EXPLANATIONS OF IDEAS AND NOTATIONS

THE notation adopted in the present work is based upon that of Peano, and the following explanations are to some extent modelled on those which he prefixes to his *Formulario Mathematico.* His use of dots as brackets is adopted, and so are many of his symbols.

Variables. The idea of a variable, as it occurs in the present work, is more general than that which is explicitly used in ordinary mathematics. In ordinary mathematics, a variable generally stands for an undetermined number or quantity. In mathematical logic, any symbol whose meaning is not determinate is called a *variable,* and the various determinations of which its meaning is susceptible are called the *values* of the variable. The values may be any set of entities, propositions, functions, classes or relations, according to circumstances. If a statement is made about "Mr A and Mr B," "Mr A" and "Mr B" are variables whose values are confined to men. A variable may either have a conventionally-assigned range of values, or may (in the absence of any indication of the range of values) have as the range of its values all determinations which render the statement in which it occurs significant. Thus when a text-book of logic asserts that "A is A," without any indication as to what A may be, what is meant is that *any* statement of the form "A is A" is true. We may call a variable *restricted* when its values are confined to some only of those of which it is capable; otherwise, we shall call it *unrestricted.* Thus when an unrestricted variable occurs, it represents any object such that the statement concerned can be made significantly (*i.e.* either truly or falsely) concerning that object. For the purposes of logic, the unrestricted variable is more convenient than the restricted variable, and we shall always employ it. We shall find that the unrestricted variable is still subject to limitations imposed by the manner of its occurrence, *i.e.* things which can be said significantly concerning a proposition cannot be said significantly concerning a class or a relation, and so on. But the limitations to which the unrestricted variable is subject do not need to be explicitly indicated, since they are the limits of significance of the statement in which the variable occurs, and are therefore intrinsically determined by this statement. This will be more fully explained later*.

To sum up, the three salient facts connected with the use of the variable are: (1) that a variable is ambiguous in its denotation and accordingly undefined; (2) that a variable preserves a recognizable identity in various occurrences throughout the same context, so that many variables can occur together in the

* Cf. Chapter II of the Introduction.

same context each with its separate identity; and (3) that either the range of possible determinations of two variables may be the same, so that a possible determination of one variable is also a possible determination of the other, or the ranges of two variables may be different, so that, if a possible determination of one variable is given to the other, the resulting complete phrase is meaningless instead of becoming a complete unambiguous proposition (true or false) as would be the case if all variables in it had been given any *suitable* determinations.

The uses of various letters. Variables will be denoted by single letters, and so will certain constants; but a letter which has once been assigned to a constant by a definition must not afterwards be used to denote a variable. The small letters of the ordinary alphabet will all be used for variables, except p and s after $*40$, in which constant meanings are assigned to these two letters. The following capital letters will receive constant meanings: B, C, D, E, F, I and J. Among small Greek letters, we shall give constant meanings to ϵ, ι and (at a later stage) to η, θ and ω. Certain Greek capitals will from time to time be introduced for constants, but Greek capitals will not be used for variables. Of the remaining letters, p, q, r will be called *propositional letters,* and will stand for variable propositions (except that, from $*40$ onwards, p must not be used for a variable); f, g, ϕ, ψ, χ, θ and (until $*33$) F will be called *functional letters,* and will be used for variable functions.

The small Greek letters not already mentioned will be used for variables whose values are classes, and will be referred to simply as *Greek letters.* Ordinary capital letters not already mentioned will be used for variables whose values are relations, and will be referred to simply as *capital letters.* Ordinary small letters other than p, q, r, s, f, g will be used for variables whose values are not known to be functions, classes, or relations; these letters will be referred to simply as *small Latin letters.*

After the early part of the work, variable propositions and variable functions will hardly ever occur. We shall then have three main kinds of variables: variable classes, denoted by small Greek letters; variable relations, denoted by capitals; and variables not given as necessarily classes or relations, which will be denoted by small Latin letters.

In addition to this usage of small Greek letters for variable classes, capital letters for variable relations, small Latin letters for variables of type wholly undetermined by the context (these arise from the possibility of "systematic ambiguity," explained later in the explanations of the theory of types), the reader need only remember that all letters represent variables, unless they have been defined as constants in some previous place in the book. In general the structure of the context determines the scope of the variables contained in it; but the special indication of the nature of the variables employed, as here proposed, saves considerable labour of thought.

The fundamental functions of propositions. An aggregation of propositions, considered as wholes not necessarily unambiguously determined, into a single proposition more complex than its constituents, is a function *with propositions as arguments*. The general idea of such an aggregation of propositions, or of variables representing propositions, will not be employed in this work. But there are four special cases which are of fundamental importance, since all the aggregations of subordinate propositions into one complex proposition which occur in the sequel are formed out of them step by step.

They are (1) the Contradictory Function, (2) the Logical Sum, or Disjunctive Function, (3) the Logical Product, or Conjunctive Function, (4) the Implicative Function. These functions in the sense in which they are required in this work are not all independent; and if two of them are taken as primitive undefined ideas, the other two can be defined in terms of them. It is to some extent—though not entirely—arbitrary as to which functions are taken as primitive. Simplicity of primitive ideas and symmetry of treatment seem to be gained by taking the first two functions as primitive ideas.

The Contradictory Function with argument p, where p is any proposition, is the proposition which is the contradictory of p, that is, the proposition asserting that p is not true. This is denoted by $\sim p$. Thus $\sim p$ is the contradictory function with p as argument and means the negation of the proposition p. It will also be referred to as the proposition not-p. Thus $\sim p$ means not-p, which means the negation of p.

The Logical Sum is a propositional function with two arguments p and q, and is the proposition asserting p or q disjunctively, that is, asserting that at least one of the two p and q is true. This is denoted by $p \vee q$. Thus $p \vee q$ is the logical sum with p and q as arguments. It is also called the logical sum of p and q. Accordingly $p \vee q$ means that at least p or q is true, not excluding the case in which both are true.

The Logical Product is a propositional function with two arguments p and q, and is the proposition asserting p and q conjunctively, that is, asserting that both p and q are true. This is denoted by $p . q$, or—in order to make the dots act as brackets in a way to be explained immediately—by $p : q$, or by $p :. q$, or by $p :: q$. Thus $p . q$ is the logical product with p and q as arguments. It is also called the logical product of p and q. Accordingly $p . q$ means that both p and q are true. It is easily seen that this function can be defined in terms of the two preceding functions. For when p and q are both true it must be false that either $\sim p$ or $\sim q$ is true. Hence in this book $p . q$ is merely a shortened form of symbolism for

$$\sim(\sim p \vee \sim q).$$

If any further idea attaches to the proposition "both p and q are true," it is not required here.

The Implicative Function is a propositional function with two arguments p and q, and is the proposition that either not-p or q is true, that is, it is the proposition $\sim p \vee q$. Thus if p is true, $\sim p$ is false, and accordingly the only alternative left by the proposition $\sim p \vee q$ is that q is true. In other words if p and $\sim p \vee q$ are both true, then q is true. In this sense the proposition $\sim p \vee q$ will be quoted as stating that p implies q. The idea contained in this propositional function is so important that it requires a symbolism which with direct simplicity represents the proposition as connecting p and q without the intervention of $\sim p$. But "implies" as used here expresses nothing else than the connection between p and q also expressed by the disjunction "not-p or q." The symbol employed for "p implies q," *i.e.* for "$\sim p \vee q$," is "$p \supset q$." This symbol may also be read "if p, then q." The association of implication with the use of an apparent variable produces an extension called "formal implication." This is explained later: it is an idea derivative from "implication" as here defined. When it is necessary explicitly to discriminate "implication" from "formal implication," it is called "material implication." Thus "material implication" is simply "implication" as here defined. The process of inference, which in common usage is often confused with implication, is explained immediately.

These four functions of propositions are the fundamental constant (*i.e.* definite) propositional functions with *propositions as arguments*, and all other constant propositional functions with propositions as arguments, so far as they are required in the present work, are formed out of them by successive steps. No *variable* propositional functions of this kind occur in this work.

Equivalence. The simplest example of the formation of a more complex function of propositions by the use of these four fundamental forms is furnished by "equivalence." Two propositions p and q are said to be "equivalent" when p implies q and q implies p. This relation between p and q is denoted by "$p \equiv q$." Thus "$p \equiv q$" stands for "$(p \supset q) . (q \supset p)$." It is easily seen that two propositions are equivalent when, and only when, they are both true or are both false. Equivalence rises in the scale of importance when we come to "formal implication" and thus to "formal equivalence." It must not be supposed that two propositions which are equivalent are in any sense identical or even remotely concerned with the same topic. Thus "Newton was a man" and "the sun is hot" are equivalent as being both true, and "Newton was not a man" and "the sun is cold" are equivalent as being both false. But here we have anticipated deductions which follow later from our formal reasoning. Equivalence in its origin is merely mutual implication as stated above.

Truth-values. The "truth-value" of a proposition is *truth* if it is true, and *falsehood* if it is false*. It will be observed that the truth-values of

* This phrase is due to Frege.

$p \vee q$, $p . q$, $p \supset q$, $\sim p$, $p \equiv q$ depend only upon those of p and q, namely the truth-value of "$p \vee q$" is truth if the truth-value of either p or q is truth, and is falsehood otherwise; that of "$p . q$" is truth if that of both p and q is truth, and is falsehood otherwise; that of "$p \supset q$" is truth if either that of p is falsehood or that of q is truth; that of "$\sim p$" is the opposite of that of p; and that of "$p \equiv q$" is truth if p and q have the same truth-value, and is falsehood otherwise. Now the only ways in which propositions will occur in the present work are ways derived from the above by combinations and repetitions. Hence it is easy to see (though it cannot be formally proved except in each particular case) that if a proposition p occurs in any proposition $f(p)$ which we shall ever have occasion to deal with, the truth-value of $f(p)$ will depend, not upon the particular proposition p, but only upon its truth-value; *i.e.* if $p \equiv q$, we shall have $f(p) \equiv f(q)$. Thus whenever two propositions are known to be equivalent, either may be substituted for the other in any formula with which we shall have occasion to deal.

We may call a function $f(p)$ a "truth-function" when its argument p is a proposition, and the truth-value of $f(p)$ depends only upon the truth-value of p. Such functions are by no means the only common functions of propositions. For example, "A believes p" is a function of p which will vary its truth-value for different arguments having the same truth-value: A may believe one true proposition without believing another, and may believe one false proposition without believing another. Such functions are not excluded from our consideration, and are included in the scope of any general propositions we may make about functions; but the particular functions of propositions which we shall have occasion to construct or to consider explicitly are all truth-functions. This fact is closely connected with a characteristic of mathematics, namely, that mathematics is always concerned with extensions rather than intensions. The connection, if not now obvious, will become more so when we have considered the theory of classes and relations.

Assertion-sign. The sign "$\vdash$," called the "assertion-sign," means that what follows is asserted. It is required for distinguishing a complete proposition, which we assert, from any subordinate propositions contained in it but not asserted. In ordinary written language a sentence contained between full stops denotes an asserted proposition, and if it is false the book is in error. The sign "$\vdash$" prefixed to a proposition serves this same purpose in our symbolism. For example, if "$\vdash (p \supset p)$" occurs, it is to be taken as a complete assertion convicting the authors of error unless the proposition "$p \supset p$" is true (as it is). Also a proposition stated in symbols without this sign "$\vdash$" prefixed is not asserted, and is merely put forward for consideration, or as a subordinate part of an asserted proposition.

Inference. The process of inference is as follows: a proposition "p" is asserted, and a proposition "p implies q" is asserted, and then as a sequel

the proposition "q" is asserted. The trust in inference is the belief that if the two former assertions are not in error, the final assertion is not in error. Accordingly whenever, in symbols, where p and q have of course special determinations,

"$\vdash p$" and "$\vdash (p \supset q)$"

have occurred, then "$\vdash q$" will occur if it is desired to put it on record. The process of the inference cannot be reduced to symbols. Its sole record is the occurrence of "$\vdash q$." It is of course convenient, even at the risk of repetition, to write "$\vdash p$" and "$\vdash (p \supset q)$" in close juxtaposition before proceeding to "$\vdash q$" as the result of an inference. When this is to be done, for the sake of drawing attention to the inference which is being made, we shall write instead

"$\vdash p \supset \vdash q$,"

which is to be considered as a mere abbreviation of the threefold statement

"$\vdash p$" and "$\vdash (p \supset q)$" and "$\vdash q$."

Thus "$\vdash p \supset \vdash q$" may be read "$p$, therefore q," being in fact the same abbreviation, essentially, as this is; for "p, therefore q" does not explicitly state, what is part of its meaning, that p implies q. An inference is the dropping of a true premiss; it is the dissolution of an implication.

The use of dots. Dots on the line of the symbols have two uses, one to bracket off propositions, the other to indicate the logical product of two propositions. Dots immediately preceded or followed by "$\vee$" or "$\supset$" or "$\equiv$" or "$\vdash$," or by "(x)," "(x, y)," "(x, y, z)"... or "$(\exists x)$," "$(\exists x, y)$," "$(\exists x, y, z)$"... or "$[(\iota x)(\phi x)]$" or "$[R\text{‘}y]$" or analogous expressions, serve to bracket off a proposition; dots occurring otherwise serve to mark a logical product. The general principle is that a larger number of dots indicates an outside bracket, a smaller number indicates an inside bracket. The exact rule as to the scope of the bracket indicated by dots is arrived at by dividing the occurrences of dots into three groups which we will name I, II, and III. Group I consists of dots adjoining a sign of implication ($\supset$) or of equivalence ($\equiv$) or of disjunction ($\vee$) or of equality by definition ($=$ Df). Group II consists of dots following brackets indicative of an apparent variable, such as (x) or (x, y) or $(\exists x)$ or $(\exists x, y)$ or $[(\iota x)(\phi x)]$ or analogous expressions*. Group III consists of dots which stand between propositions in order to indicate a logical product. Group I is of greater force than Group II, and Group II than Group III. The scope of the bracket indicated by any collection of dots extends backwards or forwards beyond any *smaller* number of dots, or any *equal* number from a group of less force, until we reach either the end of the asserted proposition or a *greater* number of dots or an *equal* number belonging to a group of equal or superior force. Dots indicating a logical product have a scope which works both backwards and forwards; other dots only work away from the

* The meaning of these expressions will be explained later, and examples of the use of dots in connection with them will be given on pp. 16, 17.

adjacent sign of disjunction, implication, or equivalence, or forward from the adjacent symbol of one of the other kinds enumerated in Group II.

Some examples will serve to illustrate the use of dots.

"$p \vee q . \supset . q \vee p$" means the proposition "'p or q' implies 'q or p.'" When we *assert* this proposition, instead of merely considering it, we write

$$\text{"}\vdash : p \vee q . \supset . q \vee p,\text{"}$$

where the two dots after the assertion-sign show that what is asserted is the whole of what follows the assertion-sign, since there are not as many as two dots anywhere else. If we had written "$p : \vee : q . \supset . q \vee p$," that would mean the proposition "either p is true, or q implies 'q or p.'" If we wished to assert this, we should have to put three dots after the assertion-sign. If we had written "$p \vee q . \supset . q : \vee : p$," that would mean the proposition "either 'p or q' implies q, or p is true." The forms "$p . \vee . q . \supset . q \vee p$" and "$p \vee q . \supset . q . \vee . p$" have no meaning.

"$p \supset q . \supset : q \supset r . \supset . p \supset r$" will mean "if p implies q, then if q implies r, p implies r." If we wish to assert this (which is true) we write

$$\text{"}\vdash :. p \supset q . \supset : q \supset r . \supset . p \supset r.\text{"}$$

Again "$p \supset q . \supset . q \supset r : \supset . p \supset r$" will mean "if '$p$ implies q' implies 'q implies r,' then p implies r." This is in general untrue. (Observe that "$p \supset q$" is sometimes most conveniently read as "p implies q," and sometimes as "if p, then q.") "$p \supset q . q \supset r . \supset . p \supset r$" will mean "if p implies q, and q implies r, then p implies r." In this formula, the first dot indicates a logical product; hence the scope of the second dot extends backwards to the beginning of the proposition. "$p \supset q : q \supset r . \supset . p \supset r$" will mean "$p$ implies q; and if q implies r, then p implies r." (This is not true in general.) Here the two dots indicate a logical product; since two dots do not occur anywhere else, the scope of these two dots extends backwards to the beginning of the proposition, and forwards to the end.

"$p \vee q . \supset :. p . \vee . q \supset r : \supset . p \vee r$" will mean "if either p or q is true, then if either p or 'q implies r' is true, it follows that either p or r is true." If this is to be asserted, we must put four dots after the assertion-sign, thus:

$$\text{"}\vdash :: p \vee q . \supset :. p . \vee . q \supset r : \supset . p \vee r.\text{"}$$

(This proposition is proved in the body of the work; it is ∗2·75.) If we wish to assert (what is equivalent to the above) the proposition: "if either p or q is true, and either p or 'q implies r' is true, then either p or r is true," we write

$$\text{"}\vdash :. p \vee q : p . \vee . q \supset r : \supset . p \vee r.\text{"}$$

Here the first pair of dots indicates a logical product, while the second pair does not. Thus the scope of the second pair of dots passes over the first pair, and back until we reach the three dots after the assertion-sign.

Other uses of dots follow the same principles, and will be explained as they are introduced. In reading a proposition, the dots should be noticed

first, as they show its structure. In a proposition containing several signs of implication or equivalence, the one with the greatest number of dots before or after it is the *principal* one: everything that goes before this one is stated by the proposition to imply or be equivalent to everything that comes after it.

Definitions. A definition is a declaration that a certain newly-introduced symbol or combination of symbols is to mean the same as a certain other combination of symbols of which the meaning is already known. Or, if the defining combination of symbols is one which only acquires meaning when combined in a suitable manner with other symbols*, what is meant is that any combination of symbols in which the newly-defined symbol or combination of symbols occurs is to have that meaning (if any) which results from substituting the defining combination of symbols for the newly-defined symbol or combination of symbols wherever the latter occurs. We will give the names of *definiendum* and *definiens* respectively to what is defined and to that which it is defined as meaning. We express a definition by putting the *definiendum* to the left and the *definiens* to the right, with the sign "=" between, and the letters "Df" to the right of the *definiens*. It is to be understood that the sign "=" and the letters "Df" are to be regarded as together forming one symbol. The sign "=" without the letters "Df" will have a different meaning, to be explained shortly.

An example of a definition is

$$p \supset q . = . \sim p \vee q \quad \text{Df.}$$

It is to be observed that a definition is, strictly speaking, no part of the subject in which it occurs. For a definition is concerned wholly with the symbols, not with what they symbolise. Moreover it is not true or false, being the expression of a volition, not of a proposition. (For this reason, definitions are not preceded by the assertion-sign.) Theoretically, it is unnecessary ever to give a definition: we might always use the *definiens* instead, and thus wholly dispense with the *definiendum*. Thus although we employ definitions and do not define "definition," yet "definition" does not appear among our primitive ideas, because the definitions are no part of our subject, but are, strictly speaking, mere typographical conveniences. Practically, of course, if we introduced no definitions, our formulae would very soon become so lengthy as to be unmanageable; but theoretically, all definitions are superfluous.

In spite of the fact that definitions are theoretically superfluous, it is nevertheless true that they often convey more important information than is contained in the propositions in which they are used. This arises from two causes. First, a definition usually implies that the *definiens* is worthy of careful consideration. Hence the collection of definitions embodies our choice

* This case will be fully considered in Chapter III of the Introduction. It need not further concern us at present.

of subjects and our judgment as to what is most important. Secondly, when what is defined is (as often occurs) something already familiar, such as cardinal or ordinal numbers, the definition contains an analysis of a common idea, and may therefore express a notable advance. Cantor's definition of the continuum illustrates this: his definition amounts to the statement that what he is defining is the object which has the properties commonly associated with the word "continuum," though what precisely constitutes these properties had not before been known. In such cases, a definition is a "making definite": it gives definiteness to an idea which had previously been more or less vague.

For these reasons, it will be found, in what follows, that the definitions are what is most important, and what most deserves the reader's prolonged attention.

Some important remarks must be made respecting the variables occurring in the *definiens* and the *definiendum*. But these will be deferred till the notion of an "apparent variable" has been introduced, when the subject can be considered as a whole.

Summary of preceding statements. There are, in the above, three primitive ideas which are not "defined" but only descriptively explained. Their primitiveness is only relative to our exposition of logical connection and is not absolute; though of course such an exposition gains in importance according to the simplicity of its primitive ideas. These ideas are symbolised by "$\sim p$" and "$p \vee q$," and by "$\vdash$" prefixed to a proposition.

Three definitions have been introduced:

$$p \,.\, q \,.=.\, \sim(\sim p \vee \sim q) \quad \text{Df,}$$

$$p \supset q \,.=.\, \sim p \vee q \quad \text{Df,}$$

$$p \equiv q \,.=.\, p \supset q \,.\, q \supset p \quad \text{Df.}$$

Primitive propositions. Some propositions must be assumed without proof, since all inference proceeds from propositions previously asserted. These, as far as they concern the functions of propositions mentioned above, will be found stated in $*1$, where the formal and continuous exposition of the subject commences. Such propositions will be called "primitive propositions." These, like the primitive ideas, are to some extent a matter of arbitrary choice; though, as in the previous case, a logical system grows in importance according as the primitive propositions are few and simple. It will be found that owing to the weakness of the imagination in dealing with simple abstract ideas no very great stress can be laid upon their obviousness. They are obvious to the instructed mind, but then so are many propositions which cannot be quite true, as being disproved by their contradictory consequences. The proof of a logical system is its adequacy and its coherence. That is: (1) the system must embrace among its deductions all those propositions which we believe to be true and capable of deduction from logical premisses alone, though possibly they may

require some slight limitation in the form of an increased stringency of enunciation; and (2) the system must lead to no contradictions, namely in pursuing our inferences we must never be led to assert both p and not-p, *i.e.* both "$\vdash . p$" and "$\vdash . \sim p$" cannot legitimately appear.

The following are the primitive propositions employed in the calculus of propositions. The letters "Pp" stand for "primitive proposition."

(1) Anything implied by a true premiss is true Pp.

This is the rule which justifies inference.

(2) $\vdash : p \vee p \,.\supset.\, p$ Pp,

i.e. if p or p is true, then p is true.

(3) $\vdash : q \,.\supset.\, p \vee q$ Pp,

i.e. if q is true, then p or q is true.

(4) $\vdash : p \vee q \,.\supset.\, q \vee p$ Pp,

i.e. if p or q is true, then q or p is true.

(5) $\vdash : p \vee (q \vee r) \,.\supset.\, q \vee (p \vee r)$ Pp,

i.e. if either p is true or "q or r" is true, then either q is true or "p or r" is true.

(6) $\vdash :. q \supset r \,.\supset:\, p \vee q \,.\supset.\, p \vee r$ Pp,

i.e. if q implies r, then "p or q" implies "p or r."

(7) Besides the above primitive propositions, we require a primitive proposition called "the axiom of identification of real variables." When we have separately asserted two different functions of x, where x is undetermined, it is often important to know whether we can identify the x in one assertion with the x in the other. This will be the case—so our axiom allows us to infer—if both assertions present x as the argument to some one function, that is to say, if ϕx is a constituent in both assertions (whatever propositional function ϕ may be), or, more generally, if $\phi(x, y, z, \ldots)$ is a constituent in one assertion, and $\phi(x, u, v, \ldots)$ is a constituent in the other. This axiom introduces notions which have not yet been explained; for a fuller account, see the remarks accompanying $*3{\cdot}03$, $*1{\cdot}7$, $*1{\cdot}71$, and $*1{\cdot}72$ (which is the statement of this axiom) in the body of the work, as well as the explanation of propositional functions and ambiguous assertion to be given shortly.

Some simple propositions. In addition to the primitive propositions we have already mentioned, the following are among the most important of the elementary properties of propositions appearing among the deductions.

The law of excluded middle:

$$\vdash . p \vee \sim p.$$

This is $*2{\cdot}11$ below. We shall indicate in brackets the numbers given to the following propositions in the body of the work.

The law of contradiction ($*3{\cdot}24$):

$$\vdash . \sim(p \,.\sim p).$$

The law of double negation (*4·13):

$$\vdash . p \equiv \sim(\sim p).$$

The principle of *transposition, i.e.* "if p implies q, then not-q implies not-p," and vice versa: this principle has various forms, namely

$$(*4{\cdot}1) \quad \vdash : p \supset q . \equiv . \sim q \supset \sim p,$$

$$(*4{\cdot}11) \quad \vdash : p \equiv q . \equiv . \sim p \equiv \sim q,$$

$$(*4{\cdot}14) \quad \vdash :. p . q . \supset . r : \equiv : p . \sim r . \supset . \sim q,$$

as well as others which are variants of these.

The law of tautology, in the two forms:

$$(*4{\cdot}24) \quad \vdash : p . \equiv . p . p,$$

$$(*4{\cdot}25) \quad \vdash : p . \equiv . p \vee p,$$

i.e. "p is true" is equivalent to "p is true and p is true," as well as to "p is true or p is true." From a formal point of view, it is through the law of tautology and its consequences that the algebra of logic is chiefly distinguished from ordinary algebra.

The law of absorption:

$$(*4{\cdot}71) \quad \vdash :. p \supset q . \equiv : p . \equiv . p . q,$$

i.e. "p implies q" is equivalent to "p is equivalent to $p . q$." This is called the law of absorption because it shows that the factor q in the product is absorbed by the factor p, if p implies q. This principle enables us to replace an implication $(p \supset q)$ by an equivalence $(p . \equiv . p . q)$ whenever it is convenient to do so.

An analogous and very important principle is the following:

$$(*4{\cdot}73) \quad \vdash :. q . \supset : p . \equiv . p . q.$$

Logical addition and multiplication of propositions obey the associative and commutative laws, and the distributive law in two forms, namely

$$(*4{\cdot}4) \quad \vdash :. p . q \vee r . \equiv : p . q . \vee . p . r,$$

$$(*4{\cdot}41) \quad \vdash :. p . \vee . q . r : \equiv : p \vee q . p \vee r.$$

The second of these distinguishes the relations of logical addition and multiplication from those of arithmetical addition and multiplication.

Propositional functions. Let ϕx be a statement containing a variable x and such that it becomes a proposition when x is given any fixed determined meaning. Then ϕx is called a "propositional function"; it is not a proposition, since owing to the ambiguity of x it really makes no assertion at all. Thus "x is hurt" really makes no assertion at all, till we have settled who x is. Yet owing to the individuality retained by the ambiguous variable x, it is an ambiguous example from the collection of propositions arrived at by giving all possible determinations to x in "x is hurt" which yield a proposition, true or false. Also if "x is hurt" and "y is hurt" occur *in the same context*, where y is another variable, then according to the determinations given to x and y, they can be settled to be (possibly) the same proposition or (possibly) different propositions. But apart from some determination given to x and y, they retain in that context their ambiguous differentiation. Thus "x is hurt" is an ambiguous "value" of a propositional function. When we wish to speak of the propositional function corresponding to "x is hurt," we shall write "$\hat{x}$ is hurt." Thus "$\hat{x}$ is hurt" is the propositional function and "x is hurt" is an ambiguous value of that function. Accordingly though "x is hurt" and "y is hurt" *occurring in the same context* can be distinguished, "$\hat{x}$ is hurt" and "$\hat{y}$ is hurt" convey no distinction of meaning at all. More generally, ϕx is an ambiguous value of the propositional function $\phi\hat{x}$, and when a definite signification a is substituted for x, ϕa is an unambiguous value of $\phi\hat{x}$.

Propositional functions are the fundamental kind from which the more usual kinds of function, such as "$\sin x$" or "$\log x$" or "the father of x," are derived. These derivative functions are considered later, and are called "descriptive functions." The functions of propositions considered above are a particular case of propositional functions.

The range of values and total variation. Thus corresponding to any propositional function $\phi\hat{x}$, there is a range, or collection, of values, consisting of all the propositions (true or false) which can be obtained by giving every possible determination to x in ϕx. A value of x for which ϕx is true will be said to "satisfy" $\phi\hat{x}$. Now in respect to the truth or falsehood of propositions of this range three important cases must be noted and symbolised. These cases are given by three propositions of which one at least must be true. Either (1) all propositions of the range are true, or (2) some propositions of the range are true, or (3) no proposition of the range is true. The statement (1) is symbolised by "$(x) . \phi x$," and (2) is symbolised by "$(\exists x) . \phi x$." No definition is given of these two symbols, which accordingly embody two new primitive ideas in our system. The symbol "$(x) . \phi x$" may be read "ϕx always," or "ϕx is always true," or "ϕx is true for all possible values of x." The symbol "$(\exists x) . \phi x$" may be read "there exists an x for which ϕx is true," or "there exists an x satisfying $\phi\hat{x}$," and thus conforms to the natural form of the expression of thought.

Proposition (3) can be expressed in terms of the fundamental ideas now on hand. In order to do this, note that "$\sim \phi x$" stands for the contradictory of ϕx. Accordingly $\sim \phi\hat{x}$ is another propositional function such that each value of $\phi\hat{x}$ contradicts a value of $\sim \phi\hat{x}$, and vice versa. Hence "$(x) . \sim \phi x$" symbolises the proposition that every value of $\phi\hat{x}$ is untrue. This is number (3) as stated above.

It is an obvious error, though one easy to commit, to assume that cases (1) and (3) are each other's contradictories. The symbolism exposes this fallacy at once, for (1) is $(x) . \phi x$, and (3) is $(x) . \sim \phi x$, while the contradictory of (1) is $\sim \{(x) . \phi x\}$. For the sake of brevity of symbolism a definition is made, namely

$$\sim (x) . \phi x . = . \sim \{(x) . \phi x\} \quad \text{Df.}$$

Definitions of which the object is to gain some trivial advantage in brevity by a slight adjustment of symbols will be said to be of "merely symbolic import," in contradistinction to those definitions which invite consideration of an important idea.

The proposition $(x) . \phi x$ is called the "total variation" of the function $\phi \hat{x}$.

For reasons which will be explained in Chapter II, we do not take negation as a primitive idea when propositions of the forms $(x) . \phi x$ and $(\exists x) . \phi x$ are concerned, but we *define* the negation of $(x) . \phi x$, *i.e.* of "ϕx is always true," as being "ϕx is sometimes false," *i.e.* "$(\exists x) . \sim \phi x$," and similarly we *define* the negation of $(\exists x) . \phi x$ as being $(x) . \sim \phi x$. Thus we put

$$\sim \{(x) . \phi x\} . = . (\exists x) . \sim \phi x \quad \text{Df,}$$

$$\sim \{(\exists x) . \phi x\} . = . (x) . \sim \phi x \quad \text{Df.}$$

In like manner we define a disjunction in which one of the propositions is of the form "$(x) . \phi x$" or "$(\exists x) . \phi x$" in terms of a disjunction of propositions not of this form, putting

$$(x) . \phi x . \vee . p : = . (x) . \phi x \vee p \quad \text{Df,}$$

i.e. "either ϕx is always true, or p is true" is to mean "'ϕx or p' is always true," with similar definitions in other cases. This subject is resumed in Chapter II, and in ∗9 in the body of the work.

Apparent variables. The symbol "$(x) . \phi x$" denotes one definite proposition, and there is no distinction in meaning between "$(x) . \phi x$" and "$(y) . \phi y$" when they occur in the same context. Thus the "x" in "$(x) . \phi x$" is not an ambiguous constituent of any expression in which "$(x) . \phi x$" occurs; and such an expression does not cease to convey a determinate meaning by reason of the ambiguity of the x in the "ϕx." The symbol "$(x) . \phi x$" has some analogy to the symbol

$$\text{“} \int_a^b \phi(x)\, dx \text{”}$$

for definite integration, since in neither case is the expression a function of x.

The range of x in "$(x) . \phi x$" or "$(\exists x) . \phi x$" extends over the complete field of the values of x for which "ϕx" has meaning, and accordingly the meaning of "$(x) . \phi x$" or "$(\exists x) . \phi x$" involves the supposition that such a field is determinate. The x which occurs in "$(x) . \phi x$" or "$(\exists x) . \phi x$" is called (following Peano) an "apparent variable." It follows from the meaning of "$(\exists x) . \phi x$" that the x in this expression is also an apparent variable. A proposition in which x occurs as an apparent variable is not a function of x. Thus *e.g.* "$(x) . x = x$" will mean "everything is equal to itself." This is an absolute constant, not a function of a variable x. This is why the x is called an *apparent* variable in such cases.

Besides the "*range*" of x in "$(x) . \phi x$" or "$(\exists x) . \phi x$," which is the field of the values that x may have, we shall speak of the "*scope*" of x, meaning

the function of which all values or some value are being affirmed. If we are asserting all values (or some value) of "ϕx," "ϕx" is the scope of x; if we are asserting all values (or some value) of "$\phi x \supset p$," "$\phi x \supset p$" is the scope of x; if we are asserting all values (or some value) of "$\phi x \supset \psi x$," "$\phi x \supset \psi x$" will be the scope of x, and so on. The scope of x is indicated by the number of dots after the "(x)" or "$(\exists x)$"; that is to say, the scope extends forwards until we reach an equal number of dots not indicating a logical product, or a greater number indicating a logical product, or the end of the asserted proposition in which the "(x)" or "$(\exists x)$" occurs, whichever of these happens first*. Thus *e.g.*

$$\text{“}(x) : \phi x . \supset . \psi x\text{”}$$

will mean "ϕx always implies ψx," but

$$\text{“}(x) . \phi x . \supset . \psi x\text{”}$$

will mean "if ϕx is always true, then ψx is true for the argument x."

Note that in the proposition

$$(x) . \phi x . \supset . \psi x$$

the two x's have no connection with each other. Since only one dot follows the x in brackets, the scope of the first x is limited to the "ϕx" immediately following the x in brackets. It usually conduces to clearness to write

$$(x) . \phi x . \supset . \psi y$$

rather than $\quad (x) . \phi x . \supset . \psi x,$

since the use of different letters emphasises the absence of connection between the two variables; but there is no logical necessity to use different letters, and it is *sometimes* convenient to use the same letter.

Ambiguous assertion and the real variable. Any value "ϕx" of the function $\phi \hat{x}$ can be asserted. Such an assertion of an ambiguous member of the values of $\phi \hat{x}$ is symbolised by

$$\text{“} \vdash . \phi x \text{.”}$$

Ambiguous assertion of this kind is a primitive idea, which cannot be defined in terms of the assertion of propositions. This primitive idea is the one which embodies the use of the variable. Apart from ambiguous assertion, the consideration of "ϕx," which is an ambiguous member of the values of $\phi \hat{x}$, would be of little consequence. When we are considering or asserting "ϕx," the variable x is called a "real variable." Take, for example, the law of excluded middle in the form which it has in traditional formal logic:

"*a* is either *b* or not *b*."

Here a and b are real variables: as they vary, different propositions are expressed, though all of them are true. While a and b are undetermined, as in the above enunciation, no one definite proposition is asserted, but what is asserted is *any* value of the propositional function in question. This can only

* This agrees with the rules for the occurrences of dots of the type of Group II as explained above, pp. 9 and 10.

be legitimately asserted if, whatever value may be chosen, that value is true, *i.e.* if all the values are true. Thus the above form of the law of excluded middle is equivalent to

$$\text{“}(a, b) \,.\, a \text{ is either } b \text{ or not } b\text{,”}$$

i.e. to "it is always true that a is either b or not b." But these two, though equivalent, are not identical, and we shall find it necessary to keep them distinguished.

When we assert something containing a real variable, as in *e.g.*

$$\text{“}\vdash . \, x = x\text{,”}$$

we are asserting *any* value of a propositional function. When we assert something containing an apparent variable, as in

$$\text{“}\vdash . \, (x) \,.\, x = x\text{”}$$

or

$$\text{“}\vdash . \, (\exists x) \,.\, x = x\text{,”}$$

we are asserting, in the first case *all* values, in the second case *some* value (undetermined), of the propositional function in question. It is plain that we can only legitimately assert "*any* value" if *all* values are true; for otherwise, since the value of the variable remains to be determined, it might be so determined as to give a false proposition. Thus in the above instance, since we have

$$\vdash . \, x = x$$

we may infer $\vdash . \, (x) \,.\, x = x.$

And generally, given an assertion containing a real variable x, we may transform the real variable into an apparent one by placing the x in brackets at the beginning, followed by as many dots as there are after the assertion-sign.

When we assert something containing a real variable, we cannot strictly be said to be asserting a *proposition*, for we only obtain a definite proposition by assigning a value to the variable, and then our assertion only applies to one definite case, so that it has not at all the same force as before. When what we assert contains a real variable, we are asserting a wholly undetermined one of all the propositions that result from giving various values to the variable. It will be convenient to speak of such assertions as *asserting a propositional function*. The ordinary formulae of mathematics contain such assertions; for example

$$\text{“}\sin^2 x + \cos^2 x = 1\text{”}$$

does not assert this or that particular case of the formula, nor does it assert that the formula holds for *all* possible values of x, though it is equivalent to this latter assertion; it simply asserts that the formula holds, leaving x wholly undetermined; and it is able to do this legitimately, because, however x may be determined, a true proposition results.

Although an assertion containing a real variable does not, in strictness, assert a proposition, yet it will be spoken of as asserting a proposition except when the nature of the ambiguous assertion involved is under discussion.

Definition and real variables. When the *definiens* contains one or more real variables, the *definiendum* must also contain them. For in this case we have a function of the real variables, and the *definiendum* must have the same meaning as the *definiens* for all values of these variables, which requires that the symbol which is the *definiendum* should contain the letters representing the real variables. This rule is not always observed by mathematicians, and its infringement has sometimes caused important confusions of thought, notably in geometry and the philosophy of space.

In the definitions given above of "$p \,.\, q$" and "$p \supset q$" and "$p \equiv q$," p and q are real variables, and therefore appear on both sides of the definition. In the definition of "$\sim\{(x) \,.\, \phi x\}$" only the function considered, namely $\phi\hat{z}$, is a real variable; thus so far as concerns the rule in question, x need not appear on the left. But when a real variable is a function, it is necessary to indicate how the argument is to be supplied, and therefore there are objections to omitting an apparent variable where (as in the case before us) this is the argument to the function which is the real variable. This appears more plainly if, instead of a general function $\phi\hat{x}$, we take some particular function, say "$\hat{x} = a$," and consider the definition of $\sim\{(x) \,.\, x = a\}$. Our definition gives

$$\sim\{(x) \,.\, x = a\} \,.\!=.\, (\exists x) \,.\, \sim(x = a) \quad \text{Df.}$$

But if we had adopted a notation in which the ambiguous value "$x = a$," containing the apparent variable x, did not occur in the *definiendum*, we should have had to construct a notation employing the function itself, namely "$\hat{x} = a$." This does not involve an apparent variable, but would be clumsy in practice. In fact we have found it convenient and possible—except in the explanatory portions—to keep the explicit use of symbols of the type "$\phi\hat{x}$," either as constants [*e.g.* $\hat{x} = a$] or as real variables, almost entirely out of this work.

Propositions connecting real and apparent variables. The most important propositions connecting real and apparent variables are the following:

(1) "When a propositional function can be asserted, so can the proposition that all values of the function are true." More briefly, if less exactly, "what holds of any, however chosen, holds of all." This translates itself into the rule that when a real variable occurs in an assertion, we may turn it into an apparent variable by putting the letter representing it in brackets immediately after the assertion-sign.

(2) "What holds of all, holds of any," *i.e.*

$$\vdash : (x) \,.\, \phi x \,.\supset.\, \phi y.$$

This states "if ϕx is always true, then ϕy is true."

(3) "If ϕy is true, then ϕx is sometimes true," *i.e.*

$$\vdash : \phi y \,.\supset.\, (\exists x) \,.\, \phi x.$$

An asserted proposition of the form "$(\exists x) . \phi x$" expresses an "existence-theorem," namely "there exists an x for which ϕx is true." The above proposition gives what is in practice the only way of proving existence-theorems: we always have to find some particular y for which ϕy holds, and thence to infer "$(\exists x) . \phi x$." If we were to assume what is called the multiplicative axiom, or the equivalent axiom enunciated by Zermelo, that would, in an important class of cases, give an existence-theorem where no particular instance of its truth can be found.

In virtue of "$\vdash : (x) . \phi x . \supset . \phi y$" and "$\vdash : \phi y . \supset . (\exists x) . \phi x$," we have "$\vdash : (x) . \phi x . \supset . (\exists x) . \phi x$," *i.e.* "what is always true is sometimes true." This would not be the case if nothing existed; thus our assumptions contain the assumption that there is something. This is involved in the principle that what holds of all, holds of any; for this would not be true if there were no "any."

(4) "If ϕx is always true, and ψx is always true, then '$\phi x . \psi x$' is always true," *i.e.*

$$\vdash :. (x) . \phi x : (x) . \psi x : \supset . (x) . \phi x . \psi x.$$

(This requires that ϕ and ψ should be functions which take arguments of the same *type*. We shall explain this requirement at a later stage.) The converse also holds; *i.e.* we have

$$\vdash :. (x) . \phi x . \psi x . \supset : (x) . \phi x : (x) . \psi x.$$

It is to some extent optional which of the propositions connecting real and apparent variables are taken as primitive propositions. The primitive propositions assumed, on this subject, in the body of the work (∗9), are the following:

(1) $\vdash : \phi x . \supset . (\exists z) . \phi z.$

(2) $\vdash : \phi x \vee \phi y . \supset . (\exists z) . \phi z,$

i.e. if either ϕx is true, or ϕy is true, then $(\exists z) . \phi z$ is true. (On the necessity for this primitive proposition, see remarks on ∗9·11 in the body of the work.)

(3) If we can assert ϕy, where y is a real variable, then we can assert $(x) . \phi x$; *i.e.* what holds of any, however chosen, holds of all.

Formal implication and formal equivalence. When an implication, say $\phi x . \supset . \psi x$, is said to hold always, *i.e.* when $(x) : \phi x . \supset . \psi x$, we shall say that ϕx *formally implies* ψx; and propositions of the form "$(x) : \phi x . \supset . \psi x$" will be said to state *formal implications*. In the usual instances of implication, such as "'Socrates is a man' implies 'Socrates is mortal,'" we have a proposition of the form "$\phi x . \supset . \psi x$" in a case in which "$(x) : \phi x . \supset . \psi x$" is true. In such a case, we feel the implication as a particular case of a formal implication. Thus it has come about that implications which are not particular cases of formal implications have not been regarded as implications at all. There is also a practical ground for the neglect of such implications, for, speaking

generally, they can only be *known* when it is already known either that their hypothesis is false or that their conclusion is true; and in neither of these cases do they serve to make us know the conclusion, since in the first case the conclusion need not be true, and in the second it is known already. Thus such implications do not serve the purpose for which implications are chiefly useful, namely that of making us know, by deduction, conclusions of which we were previously ignorant. *Formal* implications, on the contrary, do serve this purpose, owing to the psychological fact that we often know "$(x) : \phi x . \supset . \psi x$" and ϕy, in cases where ψy (which follows from these premisses) cannot easily be known directly.

These reasons, though they do not warrant the complete neglect of implications that are not instances of formal implications, are reasons which make formal implication very important. A formal implication states that, for all possible values of x, if the hypothesis ϕx is true, the conclusion ψx is true. Since "$\phi x . \supset . \psi x$" will always be true when ϕx is false, it is only the values of x that make ϕx true that are *important* in a formal implication; what is effectively stated is that, for all these values, ψx is true. Thus propositions of the form "all α is β," "no α is β" state formal implications, since the first (as appears by what has just been said) states

$$(x) : x \text{ is an } \alpha . \supset . x \text{ is a } \beta,$$

while the second states

$$(x) : x \text{ is an } \alpha . \supset . x \text{ is not a } \beta.$$

And any formal implication "$(x) : \phi x . \supset . \psi x$" may be interpreted as: "All values of x which satisfy* ϕx satisfy ψx," while the formal implication "$(x) : \phi x . \supset . \sim\psi x$" may be interpreted as: "No values of x which satisfy ϕx satisfy ψx."

We have similarly for "some α is β" the formula

$$(\exists x) . x \text{ is an } \alpha . x \text{ is a } \beta,$$

and for "some α is not β" the formula

$$(\exists x) . x \text{ is an } \alpha . x \text{ is not a } \beta.$$

Two functions ϕx, ψx are called *formally equivalent* when each always implies the other, *i.e.* when

$$(x) : \phi x . \equiv . \psi x,$$

and a proposition of this form is called a *formal equivalence*. In virtue of what was said about truth-values, if ϕx and ψx are formally equivalent, either may replace the other in any truth-function. Hence for all the purposes of mathematics or of the present work, $\phi \hat{z}$ may replace $\psi \hat{z}$ or vice versa in any proposition with which we shall be concerned. Now to say that ϕx and ψx are formally equivalent is the same thing as to say that $\phi \hat{z}$ and $\psi \hat{z}$ have the same *extension*, *i.e.* that any value of x which satisfies either satisfies the other.

* A value of x is said to *satisfy* ϕx or $\phi \hat{x}$ when ϕx is true for that value of x.

Thus whenever a constant function occurs in our work, the truth-value of the proposition in which it occurs depends only upon the extension of the function. A proposition containing a function $\phi\hat{z}$ and having this property (*i.e.* that its truth-value depends only upon the extension of $\phi\hat{z}$) will be called an *extensional* function of $\phi\hat{z}$. Thus the functions of functions with which we shall be specially concerned will all be extensional functions of functions.

What has just been said explains the connection (noted above) between the fact that the functions of propositions with which mathematics is specially concerned are all truth-functions and the fact that mathematics is concerned with extensions rather than intensions.

Convenient abbreviation. The following definitions give alternative and often more convenient notations:

$$\phi x . \supset_x . \psi x : = : (x) : \phi x . \supset . \psi x \quad \text{Df,}$$
$$\phi x . \equiv_x . \psi x : = : (x) : \phi x . \equiv . \psi x \quad \text{Df.}$$

This notation "$\phi x . \supset_x . \psi x$" is due to Peano, who, however, has no notation for the general idea "$(x) . \phi x$." It may be noticed as an exercise in the use of dots as brackets that we might have written

$$\phi x \supset_x \psi x . = . (x) . \phi x \supset \psi x \quad \text{Df,}$$
$$\phi x \equiv_x \psi x . = . (x) . \phi x \equiv \psi x \quad \text{Df.}$$

In practice however, when $\phi\hat{x}$ and $\psi\hat{x}$ are special functions, it is not possible to employ fewer dots than in the first form, and often more are required.

The following definitions give abbreviated notations for functions of two or more variables:

$$(x, y) . \phi(x, y) . = : (x) : (y) . \phi(x, y) \quad \text{Df,}$$

and so on for any number of variables;

$$\phi(x, y) . \supset_{x,y} . \psi(x, y) : = : (x, y) : \phi(x, y) . \supset . \psi(x, y) \quad \text{Df,}$$

and so on for any number of variables.

Identity. The propositional function "x is identical with y" is expressed by

$$x = y.$$

This will be defined (cf. ∗13·01), but, owing to certain difficult points involved in the definition, we shall here omit it (cf. Chapter II). We have, of course,

$$\vdash . x = x \quad \text{(the law of identity),}$$
$$\vdash : x = y . \equiv . y = x,$$
$$\vdash : x = y . y = z . \supset . x = z.$$

The first of these expresses the *reflexive* property of identity: a relation is called *reflexive* when it holds between a term and itself, either universally, or whenever it holds between that term and some term. The second of the above propositions expresses that identity is a *symmetrical* relation: a relation is called *symmetrical* if, whenever it holds between x and y, it also holds between y and x. The third proposition expresses that identity is a *transitive* relation: a relation is called *transitive* if, whenever it holds between x and y and between y and z, it holds also between x and z.

We shall find that no new definition of the sign of equality is required in mathematics: all mathematical equations in which the sign of equality is used in the ordinary way express some identity, and thus use the sign of equality in the above sense.

If x and y are identical, either can replace the other in any proposition without altering the truth-value of the proposition; thus we have

$$\vdash : x = y . \supset . \phi x \equiv \phi y.$$

This is a fundamental property of identity, from which the remaining properties mostly follow.

It might be thought that identity would not have much importance, since it can only hold between x and y if x and y are different symbols for the same object. This view, however, does not apply to what we shall call "descriptive phrases," *i.e.* "the so-and-so." It is in regard to such phrases that identity is important, as we shall shortly explain. A proposition such as "Scott was the author of Waverley" expresses an identity in which there is a descriptive phrase (namely "the author of Waverley"); this illustrates how, in such cases, the assertion of identity may be important. It is essentially the same case when the newspapers say "the identity of the criminal has not transpired." In such a case, the criminal is known by a descriptive phrase, namely "the man who did the deed," and we wish to find an x of whom it is true that "x = the man who did the deed." When such an x has been found, the identity of the criminal has transpired.

Classes and relations. A *class* (which is the same as a *manifold* or *aggregate*) is all the objects satisfying some propositional function. If α is the class composed of the objects satisfying $\phi\hat{x}$, we shall say that α is the class *determined* by $\phi\hat{x}$. Every propositional function thus determines a class, though if the propositional function is one which is always false, the class will be *null*, *i.e.* will have no members. The class determined by the function $\phi\hat{x}$ will be represented by $\hat{z}(\phi z)$*. Thus for example if ϕx is an equation, $\hat{z}(\phi z)$ will be the class of its roots; if ϕx is "x has two legs and no feathers," $\hat{z}(\phi z)$ will be the class of men; if ϕx is "$0 < x < 1$," $\hat{z}(\phi z)$ will be the class of proper fractions, and so on.

It is obvious that the same class of objects will have many determining functions. When it is not necessary to specify a determining function of a class, the class may be conveniently represented by a single Greek letter. Thus Greek letters, other than those to which some constant meaning is assigned, will be exclusively used for classes.

* Any other letter may be used instead of z.

There are two kinds of difficulties which arise in formal logic; one kind arises in connection with classes and relations and the other in connection with descriptive functions. The point of the difficulty for classes and relations, so far as it concerns classes, is that a class cannot be an object suitable as an argument to any of its determining functions. If α represents a class and $\phi\hat{x}$ one of its determining functions [so that $\alpha = \hat{z}(\phi z)$], it is not sufficient that $\phi\alpha$ be a false proposition, it must be nonsense. Thus a certain classification of what appear to be objects into things of essentially different types seems to be rendered necessary. This whole question is discussed in Chapter II, on the theory of types, and the formal treatment in the systematic exposition, which forms the main body of this work, is guided by this discussion. The part of the systematic exposition which is specially concerned with the theory of classes is ∗20, and in this Introduction it is discussed in Chapter III. It is sufficient to note here that, in the complete treatment of ∗20, we have avoided the decision as to whether a class of things has in any sense an existence as one object. A decision of this question in either way is indifferent to our logic, though perhaps, if we had regarded some solution which held classes and relations to be in some real sense objects as both true and likely to be universally received, we might have simplified one or two definitions and a few preliminary propositions. Our symbols, such as "$\hat{x}(\phi x)$" and α and others, which represent classes and relations, are merely defined in their use, just as ∇^2, standing for

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2},$$

has no meaning apart from a suitable function of x, y, z on which to operate. The result of our definitions is that the way in which we use classes corresponds in general to their use in ordinary thought and speech; and whatever may be the ultimate interpretation of the one is also the interpretation of the other. Thus in fact our classification of types in Chapter II really performs the single, though essential, service of justifying us in refraining from entering on trains of reasoning which lead to contradictory conclusions. The justification is that what seem to be propositions are really nonsense.

The definitions which occur in the theory of classes, by which the idea of a class (at least in use) is based on the other ideas assumed as primitive, cannot be understood without a fuller discussion than can be given now (cf. Chapter II of this Introduction and also ∗20). Accordingly, in this preliminary survey, we proceed to state the more important simple propositions which result from those definitions, leaving the reader to employ in his mind the ordinary unanalysed idea of a class of things. Our symbols in their usage conform to the ordinary usage of this idea in language. It is to be noticed that in the systematic exposition our treatment of classes and relations requires no new primitive ideas and only two new primitive propositions, namely the two forms of the "Axiom of Reducibility" (cf. next Chapter) for one and two variables respectively.

The propositional function "x is a member of the class α" will be expressed, following Peano, by the notation

$$x \in \alpha.$$

Here ϵ is chosen as the initial of the word ἐστί. "$x \in \alpha$" may be read "x is an α." Thus "$x \in$ man" will mean "x is a man," and so on. For typographical convenience we shall put

$$x \sim \in \alpha \,.=.\, \sim(x \in \alpha) \quad \text{Df},$$

$$x, y \in \alpha \,.=.\, x \in \alpha \,.\, y \in \alpha \quad \text{Df}.$$

For "class" we shall write "Cls"; thus "$\alpha \in \mathrm{Cls}$" means "α is a class."

We have

$$\vdash : x \in \hat{z}(\phi z) \,.\equiv.\, \phi x,$$

i.e. "'x is a member of the class determined by $\phi\hat{z}$' is equivalent to 'x satisfies $\phi\hat{z}$,' or to 'ϕx is true.'"

A class is wholly determinate when its membership is known, that is, there cannot be two different classes having the same membership. Thus if ϕx, ψx are formally equivalent functions, they determine the same class; for in that case, if x is a member of the class determined by $\phi\hat{x}$, and therefore satisfies ϕx, it also satisfies ψx, and is therefore a member of the class determined by $\psi\hat{x}$. Thus we have

$$\vdash :. \hat{z}(\phi z) = \hat{z}(\psi z) \,.\equiv:\, \phi x \,.\equiv_x.\, \psi x.$$

The following propositions are obvious and important:

$$\vdash :. \alpha = \hat{z}(\phi z) \,.\equiv:\, x \in \alpha \,.\equiv_x.\, \phi x,$$

i.e. α is identical with the class determined by $\phi\hat{z}$ when, and only when, "x is an α" is formally equivalent to ϕx;

$$\vdash :. \alpha = \beta \,.\equiv:\, x \in \alpha \,.\equiv_x.\, x \in \beta,$$

i.e. two classes α and β are identical when, and only when, they have the same membership;

$$\vdash .\, \hat{x}(x \in \alpha) = \alpha,$$

i.e. the class whose determining function is "x is an α" is α, in other words, α is the class of objects which are members of α;

$$\vdash .\, \hat{z}(\phi z) \in \mathrm{Cls},$$

i.e. the class determined by the function $\phi\hat{z}$ is a class.

It will be seen that, according to the above, any function of one variable can be replaced by an equivalent function of the form "$x \in \alpha$." Hence any extensional function of functions which holds when its argument is a function of the form "$\hat{z} \in \alpha$," whatever possible value α may have, will hold also when its argument is any function $\phi\hat{z}$. Thus variation of classes can replace variation of functions of one variable in all the propositions of the sort with which we are concerned.

In an exactly analogous manner we introduce dual or dyadic relations, *i.e.* relations between two terms. Such relations will be called simply "relations"; relations between more than two terms will be distinguished as *multiple* relations, or (when the number of their terms is specified) as triple, quadruple,... relations, or as triadic, tetradic,... relations. Such relations will not concern us until we come to Geometry. For the present, the only relations we are concerned with are *dual* relations.

Relations, like classes, are to be taken in *extension*, *i.e.* if R and S are relations which hold between the same pairs of terms, R and S are to be identical. We may regard a relation, in the sense in which it is required for our purposes, as a class of couples; *i.e.* the couple (x, y) is to be one of the class of couples constituting the relation R if x has the relation R to y*. This view of relations as classes of couples will not, however, be introduced into our symbolic treatment, and is only mentioned in order to show that it is possible so to understand the meaning of the word *relation* that a relation shall be determined by its extension.

Any function $\phi(x, y)$ determines a relation R between x and y. If we regard a relation as a class of couples, the relation determined by $\phi(x, y)$ is the class of couples (x, y) for which $\phi(x, y)$ is true. The relation determined by the function $\phi(x, y)$ will be denoted by

$$\hat{x}\hat{y}\phi(x, y).$$

We shall use a capital letter for a relation when it is not necessary to specify the determining function. Thus whenever a capital letter occurs, it is to be understood that it stands for a relation.

The propositional function "x has the relation R to y" will be expressed by the notation

$$xRy.$$

This notation is designed to keep as near as possible to common language, which, when it has to express a relation, generally mentions it between its terms, as in "x loves y," "x equals y," "x is greater than y," and so on. For "relation" we shall write "Rel"; thus "$R \epsilon \mathrm{Rel}$" means "R is a relation."

Owing to our taking relations in extension, we shall have

$$\vdash :. \hat{x}\hat{y}\phi(x, y) = \hat{x}\hat{y}\psi(x, y) . \equiv : \phi(x, y) . \equiv_{x, y} . \psi(x, y),$$

i.e. two functions of two variables determine the same relation when, and only when, the two functions are formally equivalent.

We have $\quad \vdash . z\{\hat{x}\hat{y}\phi(x, y)\}w . \equiv . \phi(z, w),$

* Such a couple has a *sense*, *i.e.* the couple (x, y) is different from the couple (y, x), unless $x = y$. We shall call it a "couple with sense," to distinguish it from the class consisting of x and y. It may also be called an *ordered* couple.

i.e. "z has to w the relation determined by the function $\phi(x, y)$" is equivalent to $\phi(z, w)$;

$$\vdash :. R = \hat{x}\hat{y}\phi(x, y) . \equiv : xRy . \equiv_{x, y} . \phi(x, y),$$
$$\vdash :. R = S . \equiv : xRy . \equiv_{x, y} . xSy,$$
$$\vdash . \hat{x}\hat{y}(xRy) = R,$$
$$\vdash . \{\hat{x}\hat{y}\phi(x, y)\} \epsilon \mathrm{Rel}.$$

These propositions are analogous to those previously given for classes. It results from them that any function of two variables is formally equivalent to some function of the form xRy; hence, in extensional functions of two variables, variation of relations can replace variation of functions of two variables.

Both classes and relations have properties analogous to most of those of propositions that result from negation and the logical sum. The *logical product* of two classes α and β is their common part, *i.e.* the class of terms which are members of both. This is represented by $\alpha \cap \beta$. Thus we put

$$\alpha \cap \beta = \hat{x}(x \epsilon \alpha . x \epsilon \beta) \quad \mathrm{Df.}$$

This gives us $\quad \vdash : x \epsilon \alpha \cap \beta . \equiv . x \epsilon \alpha . x \epsilon \beta,$

i.e. "x is a member of the logical product of α and β" is equivalent to the logical product of "x is a member of α" and "x is a member of β."

Similarly the *logical sum* of two classes α and β is the class of terms which are members of either; we denote it by $\alpha \cup \beta$. The definition is

$$\alpha \cup \beta = \hat{x}(x \epsilon \alpha . \vee . x \epsilon \beta) \quad \mathrm{Df,}$$

and the connection with the logical sum of propositions is given by

$$\vdash :. x \epsilon \alpha \cup \beta . \equiv : x \epsilon \alpha . \vee . x \epsilon \beta.$$

The *negation* of a class α consists of those terms x for which "$x \epsilon \alpha$" can be *significantly and truly* denied. We shall find that there are terms of other types for which "$x \epsilon \alpha$" is neither true nor false, but nonsense. These terms are not members of the negation of α.

Thus the *negation* of a class α is the class of terms of suitable type which are not members of it, *i.e.* the class $\hat{x}(x \sim\epsilon \alpha)$. We call this class "$-\alpha$" (read "not-$\alpha$"); thus the definition is

$$-\alpha = \hat{x}(x \sim\epsilon \alpha) \quad \mathrm{Df,}$$

and the connection with the negation of propositions is given by

$$\vdash : x \epsilon -\alpha . \equiv . x \sim\epsilon \alpha.$$

In place of implication we have the relation of *inclusion*. A class α is said to be included or contained in a class β if all members of α are members of β, *i.e.* if $x \epsilon \alpha . \supset_x . x \epsilon \beta$. We write "$\alpha \subset \beta$" for "$\alpha$ is contained in β." Thus we put

$$\alpha \subset \beta . = : x \epsilon \alpha . \supset_x . x \epsilon \beta \quad \mathrm{Df.}$$

Most of the formulae concerning $p . q$, $p \vee q$, $\sim p$, $p \supset q$ remain true if we substitute $\alpha \cap \beta$, $\alpha \cup \beta$, $-\alpha$, $\alpha \subset \beta$. In place of equivalence, we substitute identity; for "$p \equiv q$" was defined as "$p \supset q . q \supset p$," but "$\alpha \subset \beta . \beta \subset \alpha$" gives "$x \epsilon \alpha . \equiv_x . x \epsilon \beta$," whence $\alpha = \beta$.

The following are some propositions concerning classes which are analogues of propositions previously given concerning propositions:

$$\vdash . \alpha \cap \beta = -(-\alpha \cup -\beta),$$

i.e. the common part of α and β is the negation of "not-α or not-β";

$$\vdash . x \epsilon (\alpha \cup -\alpha),$$

i.e. "x is a member of α or not-α";

$$\vdash . x \sim\epsilon (\alpha \cap -\alpha),$$

i.e. "x is not a member of both α and not-α";

$$\vdash . \alpha = -(-\alpha),$$

$$\vdash : \alpha \subset \beta . \equiv . -\beta \subset -\alpha,$$

$$\vdash : \alpha = \beta . \equiv . -\alpha = -\beta,$$

$$\vdash : \alpha = \alpha \cap \alpha,$$

$$\vdash : \alpha = \alpha \cup \alpha.$$

The two last are the two forms of the law of tautology.

The law of absorption holds in the form

$$\vdash : \alpha \subset \beta . \equiv . \alpha = \alpha \cap \beta.$$

Thus for example "all Cretans are liars" is equivalent to "Cretans are identical with lying Cretans."

Just as we have $\quad \vdash : p \supset q . q \supset r . \supset . p \supset r,$

so we have $\quad \vdash : \alpha \subset \beta . \beta \subset \gamma . \supset . \alpha \subset \gamma.$

This expresses the ordinary syllogism in Barbara (with the premisses interchanged); for "$\alpha \subset \beta$" means the same as "all α's are β's," so that the above proposition states: "If all α's are β's, and all β's are γ's, then all α's are γ's." (It should be observed that syllogisms are traditionally expressed with "therefore," as if they asserted both premisses and conclusion. This is, of course, merely a slipshod way of speaking, since what is really asserted is only the connection of premisses with conclusion.)

The syllogism in Barbara when the minor premiss has an individual subject is

$$\vdash : x \epsilon \beta . \beta \subset \gamma . \supset . x \epsilon \gamma,$$

e.g. "if Socrates is a man, and all men are mortals, then Socrates is a mortal." This, as was pointed out by Peano, is not a particular case of "$\alpha \subset \beta . \beta \subset \gamma . \supset . \alpha \subset \gamma$," since "$x \epsilon \beta$" is not a particular case of "$\alpha \subset \beta$." This point is important, since traditional logic is here mistaken. The nature and magnitude of its mistake will become clearer at a later stage.

For relations, we have precisely analogous definitions and propositions. We put

$$R \dot{\cap} S = \hat{x}\hat{y}(xRy . xSy) \quad \text{Df},$$

which leads to $\quad \vdash : x (R \dot{\cap} S) y . \equiv . xRy . xSy.$

Similarly $\quad R \dot{\cup} S = \hat{x}\hat{y}(xRy . \vee . xSy) \quad \text{Df},$

$$\dot{-} R = \hat{x}\hat{y}\{\sim(xRy)\} \quad \text{Df},$$

$$R \mathrel{\dot{\subset}} S . = : xRy . \supset_{x,y} . xSy \quad \text{Df}.$$

Generally, when we require analogous but different symbols for relations and for classes, we shall choose for relations the symbol obtained by adding a dot, in some convenient position, to the corresponding symbol for classes. (The dot must not be put on the line, since that would cause confusion with the use of dots as brackets.) But such symbols require and receive a special definition in each case.

A class is said to *exist* when it has at least one member: "α exists" is denoted by "$\exists ! \alpha$." Thus we put

$$\exists ! \alpha . = . (\exists x) . x \epsilon \alpha \quad \text{Df}.$$

The class which has no members is called the "null-class," and is denoted by "Λ." Any propositional function which is always false determines the null-class. One such function is known to us already, namely "x is not identical with x," which we denote by "$x \neq x$." Thus we may use this function for defining Λ, and put

$$\Lambda = \hat{x}(x \neq x) \quad \text{Df}.$$

The class determined by a function which is always true is called the *universal class*, and is represented by V; thus

$$\mathrm{V} = \hat{x}(x = x) \quad \text{Df}.$$

Thus Λ is the negation of V. We have

$$\vdash . (x) . x \epsilon \mathrm{V},$$

i.e. "'x is a member of V' is always true"; and

$$\vdash . (x) . x \sim\epsilon \Lambda,$$

i.e. "'x is a member of Λ' is always false." Also

$$\vdash : \alpha = \Lambda . \equiv . \sim\exists ! \alpha,$$

i.e. "α is the null-class" is equivalent to "α does not exist."

For relations we use similar notations. We put

$$\dot{\exists} ! R . = . (\exists x, y) . xRy,$$

i.e. "$\dot{\exists} ! R$" means that there is at least one couple x, y between which the relation R holds. $\dot{\Lambda}$ will be the relation which never holds, and $\dot{\mathrm{V}}$ the relation which always holds. $\dot{\mathrm{V}}$ is practically never required; $\dot{\Lambda}$ will be the relation $\hat{x}\hat{y}(x \neq x . y \neq y)$. We have

$$\vdash . (x, y) . \sim(x \dot{\Lambda} y),$$

and $\quad \vdash : R = \dot{\Lambda} . \equiv . \sim\dot{\exists} ! R.$

There are no classes which contain objects of more than one type. Accordingly there is a universal class and a null-class proper to each type of object. But these symbols need not be distinguished, since it will be found that there is no possibility of confusion. Similar remarks apply to relations.

Descriptions. By a "description" we mean a phrase of the form "*the* so-and-so" or of some equivalent form. For the present, we confine our attention to *the* in the singular. We shall use this word strictly, so as to imply uniqueness; *e.g.* we should not say "A is *the* son of B" if B had other sons besides A. Thus a description of the form "the so-and-so" will only have an application in the event of there being one so-and-so and no more. Hence a description requires some propositional function $\phi\hat{x}$ which is satisfied by one value of x and by no other values; then "the x which satisfies $\phi\hat{x}$" is a description which definitely describes a certain object, though we may not know what object it describes. For example, if y is a man, "x is the father of y" must be true for one, and only one, value of x. Hence "the father of y" is a description of a certain man, though we may not know *what* man it describes. A phrase containing "the" always presupposes some initial propositional function not containing "the"; thus instead of "x is the father of y" we ought to take as our initial function "x begot y"; then "the father of y" means the one value of x which satisfies this propositional function.

If $\phi\hat{x}$ is a propositional function, the symbol "$(\iota x)(\phi x)$" is used in our symbolism in such a way that it can always be read as "the x which satisfies $\phi\hat{x}$." But we do not define "$(\iota x)(\phi x)$" as standing for "the x which satisfies $\phi\hat{x}$," thus treating this last phrase as embodying a primitive idea. Every use of "$(\iota x)(\phi x)$," where it apparently occurs as a constituent of a proposition in the place of an object, is defined in terms of the primitive ideas already on hand. An example of this definition in use is given by the proposition "$E!(\iota x)(\phi x)$" which is considered immediately. The whole subject is treated more fully in Chapter III.

The symbol should be compared and contrasted with "$\hat{x}(\phi x)$" which in use can always be read as "the x's which satisfy $\phi\hat{x}$." Both symbols are incomplete symbols defined only in use, and as such are discussed in Chapter III. The symbol "$\hat{x}(\phi x)$" always has an application, namely to the class determined by ϕx; but "$(\iota x)(\phi x)$" only has an application when $\phi\hat{x}$ is only satisfied by one value of x, neither more nor less. It should also be observed that the meaning given to the symbol by the definition, given immediately below, of $E!(\iota x)(\phi x)$ does not presuppose that we know the meaning of "one." This is also characteristic of the definition of any other use of $(\iota x)(\phi x)$.

We now proceed to define "$E!(\iota x)(\phi x)$" so that it can be read "the x satisfying ϕx exists." (It will be observed that this is a different meaning of existence from that which we express by "$\exists$.") Its definition is

$$E!(\iota x)(\phi x) . = : (\exists c) : \phi x . \equiv_x . x = c \quad \text{Df},$$

i.e. "the x satisfying $\phi\hat{x}$ exists" is to mean "there is an object c such that ϕx is true when x is c but not otherwise."

The following are equivalent forms:

$$\vdash :. E!(\iota x)(\phi x) . \equiv : (\exists c) : \phi c : \phi x . \supset_x . x = c,$$

$$\vdash :. E!(\iota x)(\phi x) . \equiv : (\exists c) . \phi c : \phi x . \phi y . \supset_{x,y} . x = y,$$

$$\vdash :. E!(\iota x)(\phi x) . \equiv : (\exists c) : \phi c : x \neq c . \supset_x . \sim\phi x.$$

The last of these states that "the x satisfying $\phi\hat{x}$ exists" is equivalent to "there is an object c satisfying $\phi\hat{x}$, and every object other than c does not satisfy $\phi\hat{x}$."

The kind of existence just defined covers a great many cases. Thus for example "the most perfect Being exists" will mean:

$$(\exists c) : x \text{ is most perfect} . \equiv_x . x = c,$$

which, taking the last of the above equivalences, is equivalent to

$$(\exists c) : c \text{ is most perfect} : x \neq c . \supset_x . x \text{ is not most perfect}.$$

A proposition such as "Apollo exists" is really of the same logical form, although it does not explicitly contain the word *the*. For "Apollo" means really "the object having such-and-such properties," say "the object having the properties enumerated in the Classical Dictionary*." If these properties make up the propositional function ϕx, then "Apollo" means "$(\iota x)(\phi x)$," and "Apollo exists" means "$E!(\iota x)(\phi x)$." To take another illustration, "the author of Waverley" means "the man who (or rather, the object which) wrote Waverley." Thus "Scott is the author of Waverley" is

$$\text{Scott} = (\iota x)(x \text{ wrote Waverley}).$$

Here (as we observed before) the importance of *identity* in connection with descriptions plainly appears.

The notation "$(\iota x)(\phi x)$," which is long and inconvenient, is seldom used, being chiefly required to lead up to another notation, namely "$R`y$," meaning "the object having the relation R to y." That is, we put

$$R`y = (\iota x)(xRy) \quad \text{Df.}$$

The inverted comma may be read "of." Thus "$R`y$" is read "the $R$ of $y$." Thus if $R$ is the relation of father to son, "$R`y$" means "the father of y"; if R is the relation of son to father, "$R`y$" means "the son of $y$," which will only "exist" if $y$ has one son and no more. $R`y$ is a function of y, but not a propositional function; we shall call it a *descriptive* function. All the ordinary functions of mathematics are of this kind, as will appear more fully in the sequel. Thus in our notation, "sin y" would be written "sin $`y$," and "sin" would stand for the relation which sin $`y$ has to y. Instead of a variable descriptive function fy, we put $R`y$, where the variable relation R takes the

* The same principle applies to many uses of the proper names of existent objects, *e.g.* to all uses of proper names for objects known to the speaker only by report, and not by personal acquaintance.

place of the variable function f. A descriptive function will in general exist while y belongs to a certain domain, but not outside that domain; thus if we are dealing with positive rationals, $\sqrt{y}$ will be significant if y is a perfect square, but not otherwise; if we are dealing with real numbers, and agree that "$\sqrt{y}$" is to mean the *positive* square root (or, is to mean the negative square root), $\sqrt{y}$ will be significant provided y is positive, but not otherwise; and so on. Thus every descriptive function has what we may call a "domain of definition" or a "domain of existence," which may be thus defined: If the function in question is $R\text{‘}y$, its domain of definition or of existence will be the class of those arguments y for which we have $E!\,R\text{‘}y$, *i.e.* for which $E!\,(\imath x)(xRy)$, *i.e.* for which there is one x, and no more, having the relation R to y.

If R is any relation, we will speak of $R\text{‘}y$ as the "associated descriptive function." A great many of the constant relations which we shall have occasion to introduce are only or chiefly important on account of their associated descriptive functions. In such cases, it is easier (though less correct) to begin by assigning the meaning of the descriptive function, and to deduce the meaning of the relation from that of the descriptive function. This will be done in the following explanations of notation.

Various descriptive functions of relations. If R is any relation, the *converse* of R is the relation which holds between y and x whenever R holds between x and y. Thus *greater* is the converse of *less*, *before* of *after*, *cause* of *effect* *husband* of *wife*, etc. The converse of R is written* $\text{Cnv}\text{‘}R$ or $\breve{R}$. The definition is

$$\breve{R} = \hat{x}\hat{y}\,(yRx) \quad \text{Df,}$$

$$\text{Cnv}\text{‘}R = \breve{R} \qquad \text{Df.}$$

The second of these is not a formally correct definition, since we ought to define "Cnv" and deduce the meaning of $\text{Cnv}\text{‘}R$. But it is not worth while to adopt this plan in our present introductory account, which aims at simplicity rather than formal correctness.

A relation is called *symmetrical* if $R = \breve{R}$, *i.e.* if it holds between y and x whenever it holds between x and y (and therefore vice versa). Identity, diversity, agreement or disagreement in any respect, are symmetrical relations. A relation is called *asymmetrical* when it is incompatible with its converse, *i.e.* when $R \dot{\frown} \breve{R} = \dot{\Lambda}$, or, what is equivalent,

$$xRy \,.\supset_{x,y}.\, \sim (yRx).$$

Before and after, greater and less, ancestor and descendant, are asymmetrical, as are all other relations of the sort that lead to *series*. But there are many asymmetrical relations which do not lead to series, for instance, that of

* The second of these notations is taken from Schröder's *Algebra und Logik der Relative*.

wife's brother*. A relation may be neither symmetrical nor asymmetrical; for example, this holds of the relation of inclusion between classes: $\alpha \subset \beta$ and $\beta \subset \alpha$ will both be true if $\alpha = \beta$, but otherwise only one of them, at most, will be true. The relation *brother* is neither symmetrical nor asymmetrical, for if x is the brother of y, y may be either the brother or the sister of x.

In the propositional function xRy, we call x the *referent* and y the *relatum*. The class $\hat{x}(xRy)$, consisting of all the x's which have the relation R to y, is called the class of referents of y with respect to R; the class $\hat{y}(xRy)$, consisting of all the y's to which x has the relation R, is called the class of relata of x with respect to R. These two classes are denoted respectively by $\overrightarrow{R}\text{‘}y$ and $\overleftarrow{R}\text{‘}x$. Thus

$$\overrightarrow{R}\text{‘}y = \hat{x}\,(xRy) \quad \text{Df,}$$

$$\overleftarrow{R}\text{‘}x = \hat{y}\,(xRy) \quad \text{Df.}$$

The arrow runs towards y in the first case, to show that we are concerned with things having the relation R *to* y; it runs away from x in the second case, to show that the relation R goes *from* x to the members of $\overleftarrow{R}\text{‘}x$. It runs in fact *from* a referent and *towards* a relatum.

The notations $\overrightarrow{R}\text{‘}y$, $\overleftarrow{R}\text{‘}x$ are very important, and are used constantly. If R is the relation of parent to child, $\overrightarrow{R}\text{‘}y$ = the parents of y, $\overleftarrow{R}\text{‘}x$ = the children of x. We have

$$\vdash :\, x \in \overrightarrow{R}\text{‘}y \,.\equiv .\, xRy$$

and

$$\vdash :\, y \in \overleftarrow{R}\text{‘}x \,.\equiv .\, xRy.$$

These equivalences are often embodied in common language. For example, we say indiscriminately "x is an inhabitant of London" or "x inhabits London." If we put "R" for "inhabits," "x inhabits London" is "xR London," while "x is an inhabitant of London" is "$x \in \overrightarrow{R}\text{‘}$ London."

Instead of $\overrightarrow{R}$ and $\overleftarrow{R}$ we sometimes use $\text{sg}\text{‘}R$, $\text{gs}\text{‘}R$, where "sg" stands for "sagitta," and "gs" is "sg" backwards. Thus we put

$$\text{sg}\text{‘}R = \overrightarrow{R} \quad \text{Df,}$$

$$\text{gs}\text{‘}R = \overleftarrow{R} \quad \text{Df.}$$

These notations are sometimes more convenient than an arrow when the relation concerned is represented by a combination of letters, instead of a single letter such as R. Thus *e.g.* we should write $\text{sg}\text{‘}(R \dot{\frown} S)$, rather than put an arrow over the whole length of $(R \dot{\frown} S)$.

The class of all terms that have the relation R to something or other is called the *domain* of R. Thus if R is the relation of parent and child, the

* This relation is not strictly asymmetrical, but is so except when the wife's brother is also the sister's husband. In the Greek Church the relation is strictly asymmetrical.

domain of R will be the class of parents. We represent the domain of R by "$\mathrm{D}‘R$." Thus we put

$$\mathrm{D}‘R = \hat{x}\{(\exists y) . xRy\} \quad \mathrm{Df.}$$

Similarly the class of all terms to which something or other has the relation R is called the *converse domain* of R; it is the same as the domain of the converse of R. The converse domain of R is represented by "$\text{Ɑ}‘R$"; thus

$$\text{Ɑ}‘R = \hat{y}\{(\exists x) . xRy\} \quad \mathrm{Df.}$$

The sum of the domain and the converse domain is called the *field*, and is represented by $C‘R$: thus

$$C‘R = \mathrm{D}‘R \cup \text{Ɑ}‘R \quad \mathrm{Df.}$$

The *field* is chiefly important in connection with series. If R is the ordering relation of a series, $C‘R$ will be the class of terms of the series, $\mathrm{D}‘R$ will be all the terms except the last (if any), and $\text{Ɑ}‘R$ will be all the terms except the first (if any). The first term, if it exists, is the only member of $\mathrm{D}‘R \cap -\text{Ɑ}‘R$, since it is the only term which is a predecessor but not a follower. Similarly the last term (if any) is the only member of $\text{Ɑ}‘R \cap -\mathrm{D}‘R$. The condition that a series should have no end is $\text{Ɑ}‘R \subset \mathrm{D}‘R$, *i.e.* "every follower is a predecessor"; the condition for no beginning is $\mathrm{D}‘R \subset \text{Ɑ}‘R$. These conditions are equivalent respectively to $\mathrm{D}‘R = C‘R$ and $\text{Ɑ}‘R = C‘R$.

The *relative product* of two relations R and S is the relation which holds between x and z when there is an intermediate term y such that x has the relation R to y and y has the relation S to z. The relative product of R and S is represented by $R \mid S$; thus we put

$$R \mid S = \hat{x}\hat{z}\{(\exists y) . xRy . ySz\} \quad \mathrm{Df,}$$

whence $\vdash : x(R \mid S)z . \equiv . (\exists y) . xRy . ySz.$

Thus "paternal aunt" is the relative product of *sister* and *father*; "paternal grandmother" is the relative product of *mother* and *father*; "maternal grandfather" is the relative product of *father* and *mother*. The relative product is not commutative, but it obeys the associative law, *i.e.*

$$\vdash . (P \mid Q) \mid R = P \mid (Q \mid R).$$

It also obeys the distributive law with regard to the logical addition of relations, *i.e.* we have

$$\vdash . P \mid (Q \dot{\cup} R) = (P \mid Q) \dot{\cup} (P \mid R),$$
$$\vdash . (Q \dot{\cup} R) \mid P = (Q \mid P) \dot{\cup} (R \mid P).$$

But with regard to the logical product, we have only

$$\vdash . P \mid (Q \dot{\cap} R) \mathrel{\dot{\subset}} (P \mid Q) \dot{\cap} (P \mid R),$$
$$\vdash . (Q \dot{\cap} R) \mid P \mathrel{\dot{\subset}} (Q \mid P) \dot{\cap} (Q \mid R).$$

The relative product does not obey the law of tautology, *i.e.* we do not have in general $R \mid R = R$. We put

$$R^2 = R \mid R \quad \mathrm{Df.}$$

Thus paternal grandfather = (father)^2,
maternal grandmother = (mother)^2.

A relation is called *transitive* when $R^2 \mathrel{\dot{\subset}} R$, *i.e.* when, if xRy and yRz, we always have xRz, *i.e.* when

$$xRy . yRz . \supset_{x,y,z} . xRz.$$

Relations which generate series are always transitive; thus *e.g.*

$$x > y . y > z . \supset_{x,y,z} . x > z.$$

If P is a relation which generates a series, P may conveniently be read "precedes"; thus "$xPy . yPz . \supset_{x,y,z} . xPz$" becomes "if x precedes y and y precedes z, then x always precedes z." The class of relations which generate series are partially characterized by the fact that they are transitive and asymmetrical, and never relate a term to itself.

If P is a relation which generates a series, and if we have not merely $P^2 \mathrel{\dot{\subset}} P$, but $P^2 = P$, then P generates a series which is *compact* (*überall dicht*), *i.e.* such that there are terms between any two. For in this case we have

$$xPz . \supset . (\exists y) . xPy . yPz,$$

i.e. if x precedes z, there is a term y such that x precedes y and y precedes z, *i.e.* there is a term between x and z. Thus among relations which generate series, those which generate compact series are those for which $P^2 = P$.

Many relations which do not generate series are transitive, for example, identity, or the relation of inclusion between classes. Such cases arise when the relations are not asymmetrical. Relations which are transitive and symmetrical are an important class: they may be regarded as consisting in the possession of some common property.

Plural descriptive functions. The class of terms x which have the relation R to some member of a class α is denoted by $R‘‘\alpha$ or $R_\epsilon‘\alpha$. The definition is

$$R‘‘\alpha = \hat{x}\{(\exists y) . y \epsilon \alpha . xRy\} \quad \mathrm{Df.}$$

Thus for example let R be the relation of *inhabiting*, and α the class of towns; then $R‘‘\alpha$ = inhabitants of towns. Let R be the relation "less than" among rationals, and α the class of those rationals which are of the form $1 - 2^{-n}$, for integral values of n; then $R‘‘\alpha$ will be all rationals less than some member of α, *i.e.* all rationals less than 1. If P is the generating relation of a series, and α is any class of members of the series, $P‘‘\alpha$ will be predecessors of α's, *i.e.* the segment defined by α. If P is a relation such that $P‘y$ always exists when $y \epsilon \alpha$, $P‘‘\alpha$ will be the class of all terms of the form $P‘y$ for values of y which are members of α; *i.e.*

$$P‘‘\alpha = \hat{x}\{(\exists y) . y \epsilon \alpha . x = P‘y\}.$$

Thus a member of the class "fathers of great men" will be the father of y, where y is some great man. In other cases, this will not hold; for instance, let P be the relation of a number to any number of which it is a factor; then

P" (even numbers) = factors of even numbers, but this class is not composed of terms of the form "*the* factor of x," where x is an even number, because numbers do not have only one factor apiece.

Unit classes. The class whose only member is x might be thought to be identical with x, but Peano and Frege have shown that this is not the case. (The reasons why this is not the case will be explained in a preliminary way in Chapter II of the Introduction.) We denote by "ι'x" the class whose only member is x: thus

$$\iota\text{'}x = \hat{y}\,(y = x) \quad \text{Df},$$

i.e. "ι'x" means "the class of objects which are identical with x."

The class consisting of x and y will be $\iota\text{'}x \cup \iota\text{'}y$; the class got by adding x to a class α will be $\alpha \cup \iota\text{'}x$; the class got by taking away x from a class α will be $\alpha - \iota\text{'}x$. (We write $\alpha - \beta$ as an abbreviation for $\alpha \cap -\beta$.)

It will be observed that unit classes have been defined without reference to the number 1; in fact, we use unit classes to define the number 1. This number is defined as the class of unit classes, *i.e.*

$$1 = \hat{\alpha}\,\{(\exists x) \,.\, \alpha = \iota\text{'}x\} \quad \text{Df}.$$

This leads to

$$\vdash :.\, \alpha \in 1 \,.\equiv :\, (\exists x) : y \in \alpha \,.\equiv_y .\, y = x.$$

From this it appears further that

$$\vdash : \alpha \in 1 \,.\equiv .\, \text{E}\,!\,(\imath x)\,(x \in \alpha),$$

whence $$\vdash : \hat{z}\,(\phi z) \in 1 \,.\equiv .\, \text{E}\,!\,(\imath x)\,(\phi x),$$

i.e. "$\hat{z}\,(\phi z)$ is a unit class" is equivalent to "the x satisfying $\phi\hat{x}$ exists."

If $\alpha \in 1$, $\breve{\iota}\text{'}\alpha$ is the only member of α, for the only member of α is the only term to which α has the relation ι. Thus "$\breve{\iota}\text{'}\alpha$" takes the place of "$(\imath x)\,(\phi x)$," if α stands for $\hat{z}\,(\phi z)$. In practice, "$\breve{\iota}\text{'}\alpha$" is a more convenient notation than "$(\imath x)\,(\phi x)$," and is generally used instead of "$(\imath x)\,(\phi x)$."

The above account has explained most of the logical notation employed in the present work. In the applications to various parts of mathematics, other definitions are introduced; but the objects defined by these later definitions belong, for the most part, rather to mathematics than to logic. The reader who has mastered the symbols explained above will find that any later formulae can be deciphered by the help of comparatively few additional definitions.