the total, since each term can be written once without, and once with, a negation sign. So we have, using the preceding transformation of the sum,

$$\begin{aligned}\rho = \sum_{m=1}^{n}\binom{n}{m} m \cdot 2^m = n \sum_{m-1=0}^{n-1}\binom{n-1}{m-1} \cdot 2^m = 2n \sum_{m-1=0}^{n-1}\binom{n-1}{m-1} \cdot 2^{m-1} \\ = 2n(2+1)^{n-1} = 2n \cdot 3^{n-1} \qquad (44)\end{aligned}$$

For the transition to the second line we use the binomial theorem

$$(p+q)^n = \sum_{m=0}^{n}\binom{n}{m} p^m q^{n-m} \qquad (45)$$

choosing $p = 2$, $q = 1$, and putting $n - 1$ for n and $m - 1$ for m. For $n = 1$ we have $\rho = 2$; for $n = 2$, $\rho = 12$ (see 8); for $n = 3$, $\rho = 54$.

There are many applications for the relations developed for three classes. Let B_1 be a symptom of illness; B_2, a certain disease; B_3, the case of death. The simple mutual probabilities may be known from statistics; the relation (35) shows that only five are to be ascertained, the sixth being determinable. Furthermore, one of the compound probabilities must be ascertained, for instance, $P(B_1.B_2,B_3)$. When these values are known, all statistical questions referring to the three classes are answerable except those referring to absolute probabilities or probabilities of negative reference. A psychological application obtains when B_1 means a certain stimulus; B_2, a perception; B_3, a certain reaction of a person.

§ 25. Remarks Concerning the Mathematical Formalization of the Probability Calculus

Having carried through, to a large extent, the formalization of the calculus of probability, we are now free to discuss this procedure from a logical viewpoint. The "logification" by which this construction of the calculus was introduced has, in the meantime, been transformed into a "mathematization", a notation in which the logical operations are restricted to the inner part of the P-symbols. The resulting complexes of the P-symbols, into which these symbols enter as units, have the character of mathematical equations. Thus the probability calculus acquires a form that is convenient for the purpose of carrying out calculations.

This manner of writing—the mathematical notation—has the disadvantage that it cannot express certain relations of a nonmathematical kind that hold within the probability calculus. There are three different forms of such relations:

1. The dependence of a mathematical equation on the validity of another mathematical equation, that is, the implication between equations. An exam-

ple is given by the assertion that (4, § 14) is the condition of validity for (5, § 14).

2. The dependence of a mathematical equation on a nonmathematical condition. Of this kind is the condition $(A \supset B)$ in axiom II,1 or the condition of exclusion in axiom III.

3. Logical properties of the quantities occurring, such as are expressed in the statement of univocality formulated in axiom I.

The first case is irrelevant because the existence of a logical implication between equations is easily expressible by some connecting words in the context. This is the method usually applied in mathematics. The second case is serious because here the condition on which the validity of a mathematical equation depends is not expressible in mathematical notation. The third case is irrelevant again. It concerns only the assertion of univocality; this assertion, as is usual in mathematics, may be added in words.

It will now be shown that the second difficulty can be eliminated by the use of a method that translates the logical condition into a mathematical condition. The method may be illustrated by reference to the general theorem of addition. It was seen, in § 20, that the condition of exclusion, written for this axiom in the logical notation, could be replaced by the condition that the corresponding probability becomes 0. Since this assumption, according to (9, § 13), states less than the strict condition of exclusion, a certain generalization of the special theorem of addition has thus been constructed. It will now be shown that the same procedure is feasible in some other places, so that, by its use, relations of the form 2 can be reduced to those of the form 1. We are concerned here with the following theorems written in the implicational notation:

$$(A \supset B) \supset [(A \underset{q}{\ni} C) \equiv (A . B \underset{q}{\ni} C)] \tag{1}$$

$$(A \supset B) \supset (\exists p)(A . C \underset{p}{\ni} B) . (p = 1) \tag{2}$$

$$(A \supset B) \supset [(A \underset{q}{\ni} C) \equiv (A \underset{q}{\ni} B . C)] \tag{3}$$

$$(C \supset B) \supset [(A \underset{q}{\ni} C) \equiv (A \underset{q}{\ni} B . C)] \tag{4}$$

The proof of the theorems is easily given. Let us prove immediately their generalization for the cases $P(A,B) = 1$ and $P(C,B) = 1$, respectively. From this result, of course, by the help of II,1, theorems (1)–(4) follow.

Instead of (1) we obtain: if $P(A,B) = 1$, then

$$P(A,C) = P(A . B,C) \tag{5}$$

This follows from the elimination theorem (2, § 19) because $P(A,\bar{B}) = 0$. Formula (5) states that, if $P(A,B) = 1$, B and any C are mutually independent with respect to A.

Instead of (2) we obtain: if $P(A,B) = 1$ and $P(A,C) > 0$, we have for any C

$$P(A.C,B) = 1 \tag{6}$$

The proof follows from the product rule (6, § 21) by the use of (5). If $P(A,C) = 0$, (6) is not derivable; in this case the value of $P(A.C,B)$ cannot be determined from $P(A,B)$, though it is possible that a determinate value $P(A.C,B)$ exists.

Instead of (3) we obtain: if $P(A,B) = 1$, then

$$P(A,C) = P(A,B.C) \tag{7}$$

The proof is given by the multiplication theorem (3, § 14) with the help of (6). This formula is also valid for the case $P(A,C) = 0$, since it then follows directly from the multiplication theorem without the use of (6).

Instead of (4) we obtain: if $P(C,B) = 1$ and $P(C,A) > 0$, then

$$P(A,C) = P(A,B.C) \tag{8}$$

The proof is given by the multiplication theorem, since, for the assumptions made, $P(A.C,B) = 1$, according to (6), if A and C are interchanged in (6).

With these proofs the mathematical formalization is carried through for the four theorems.[1] It will now be shown that in axiom II,1, too, the condition $(A \supset B)$ can be formally eliminated.

In this case we make use of the equivalence

$$([A \supset B] \equiv [B \equiv A \vee B]) \tag{9}$$

The formula is proved by solving the right side according to (7*b*, § 4), applying (4*e*, § 4) and (4*b*, § 4) and, finally, transforming the left side by (6*a*, § 4). Formula (9) is a tautology of the class calculus, that is, the expression inside the parentheses represents the universal class. The formula can be transcribed into the form

$$(A \supset B) \equiv (B \equiv A \vee B) \tag{9'}$$

This formula means that if A is a subclass of B, the joint class $A \vee B$ is identical with B.

Because of (9), axiom II,1 is equivalent to the expression

$$P(A,A \vee B) = 1 \tag{10}$$

For we have, on account of (9),

$$(A \supset B) \supset [P(A,B) = P(A,A \vee B)] \tag{11}$$

[1] The restrictions $P(A,C) > 0$ and $P(C,A) > 0$ added, respectively, to (6) and (8) are not required for the corresponding theorems (2) and (4). This is due to the fact that the logical implication represents a stronger assumption than the probability 1.

Therefore, II,1 follows from (10). That, conversely, (10) follows from II,1 can be shown by the use of (8*a*, § 4).

As a formula that cannot be formalized mathematically, there remains only axiom I, the axiom of univocality, apart from implications of the form 1. All other expressions can be formalized, and we may ask whether we can omit axiom I and construct a *mathematical axiom system* of the calculus of probability. By this term is understood a system in which the logical operations are restricted to the inner part of the P-symbols, whereas the symbols themselves enter into relations having the form of mathematical equations. Such an axiom system can be constructed; the condition of univocality is then added in words.

In order to set up this axiom system, we introduce the following changes from the axiom system written in the implicational notation. We replace II,1 by (10). Furthermore, we replace the addition theorem III by the general theorem of addition (8, § 20), so that we can free ourselves from the condition of exclusion. This requires, in the group of axioms of normalization, a further axiom, α,2, which defines the probability 0 in a way similar to that in which α,1 defines the probability 1. We thus obtain the following *mathematical axiom system of the calculus of probability:*

α) AXIOMS OF NORMALIZATION

$$1.\ P(A, A \vee B) = 1$$

$$2.\ P(A, B.\bar{B}) = 0$$

$$3.\ 0 \leqq P(A,B)$$

β) AXIOM OF ADDITION

$$P(A, B \vee C) = P(A,B) + P(A,C) - P(A,B.C)$$

γ) AXIOM OF MULTIPLICATION

$$P(A,B.C) = P(A,B) \cdot P(A.B,C)$$

Axiom α,3 needs no qualification demanding that A be nonempty, because, if A is empty, this inequality does not represent any restriction on the numerical values of probabilities. According to the convention concerning the use of the P-symbol for empty reference classes (see p. 59), the inequality α,3 expresses, in this case, merely a trivial existential statement. Thus the axiom does not depend on a special condition to be added in words. The only condition of this kind is the axiom of univocality. It may be convenient to formulate this axiom, together with the rule of existence and the convention about the use of P-symbol, as a rule given in the metalanguage.

It will be shown briefly how the rule of the complement (7, § 13) can be derived from these axioms. We substitute first, in α,1, $B \vee \bar{B}$ for B; because, according to (8c, § 4), the relation $(A \supset B \vee \bar{B})$ is always valid, we obtain, by the use of (11),

$$P(A,B \vee \bar{B}) = 1$$

Dissolving the term on the left side according to axiom β, we obtain, by the use of α,2,

$$P(A,B \vee \bar{B}) = P(A,B) + P(A,\bar{B}) = 1 \tag{12}$$

From this result we derive the special theorem of addition for the mutually exclusive events B and C, that is, for $(B \supset \bar{C})$. Since $(B \supset \bar{C}) \equiv (\overline{B.C})$, the relation $(A \supset \overline{B.C})$ follows from $(B \supset \bar{C})$ with the help of (8c, § 4); substituting in (11) $\overline{B.C}$ for B and using axiom α,1 we thus derive

$$P(A,\overline{B.C}) = P(A,A \vee \overline{B.C}) = 1 \tag{13}$$

With the help of (12) we now derive $P(A,B.C) = 0$ and thus obtain from axiom β the special theorem of addition.

Regarding the theorem of multiplication, the previous remarks hold good, according to which this theorem can be replaced by the weaker assumption of § 15. It is possible, furthermore, to replace the axioms β and γ by a compound axiom, as was shown by William Gustin. According to Gustin, the following mathematical axiom system is sufficient:

a) NORMALIZATION

$$1.\ P(A.B,B) = 1$$

$$2.\ 0 \leqq P(A,B)$$

b) AXIOM OF THE COMPLEMENT OF THE PRODUCT

$$P(A,\overline{B.C}) = 1 - P(A,B) \cdot P(A.B,C)$$

The postulate of univocality must be added in words, as in the preceding system. The Gustin system shows that the axiom of addition can be replaced by the rule of the complement and that the latter can be combined with the axiom of multiplication in one axiom. In this system the rule of the complement is derivable as follows:

$$P(A,\bar{B}) = P(A,\overline{B.B}) = 1 - P(A,B) \cdot P(A.B,B) = 1 - P(A,B) \tag{14}$$

Using this result, we immediately derive from axiom b the general theorem of multiplication. The general theorem of addition is proved as follows:

$$\begin{aligned} P(A,B \vee C) &= P(A,\overline{\bar{B}.\bar{C}}) = 1 - P(A,\bar{B}) \cdot P(A.\bar{B},\bar{C}) \\ &= 1 - P(A,\bar{B}) \cdot [1 - P(A.\bar{B},C)] \\ &= P(A,B) + P(A,\bar{B}.C) \\ &= P(A,B) + P(A,C) \cdot P(A.C,\bar{B}) \\ &= P(A,B) + P(A,C) \cdot [1 - P(A.C,B)] \\ &= P(A,B) + P(A,C) - P(A,B.C) \end{aligned} \tag{15}$$

Axiom α,2 follows when $\bar{B}$ is substituted for C in the theorem of addition, and the rule of the complement is used. Axiom α,1 follows by substituting A for C in the theorem of addition. The Gustin system is thus proved to be equivalent to the system of axioms α–γ.

The mathematical axiom systems presented here are sufficient to prove all the theorems of the calculus of probability. They do not carry through the formalization completely; the condition of univocality and the implication between equations must be added verbally in the context. But the nonformalized residue is relatively small. It is true that my mathematical axiom systems require the use of symbolic logic for the inner part of the probability symbols, but I hope that the presentation shows that this feature only facilitates operations within the calculus. With the help of symbolic logic a probability calculus has been constructed that exhibits not only the mathematical but also the logical structure of its subject matter. I should be happy if the unification of mathematics and symbolic logic thus achieved would stimulate other authors to attempt similar constructions in other fields of research.

Historical remark concerning the axiomatic construction of the calculus of probability.—Axiomatic foundations of the calculus of probabilities have been given repeatedly within the last few decades. Corresponding to my division into a formal and an interpreted theory of probability, two groups may be distinguished. The interpreted form of axiomatic construction regards probability, from the beginning, as a frequency, and derives from this interpretation, by the possible inclusion of additional postulates, the rules of the theory. This group began with Richard von Mises' analyses[1] (1919) and was continued by the inquiries of Karl Dörge[2] (1930), Erhard Tornier[3] (1930), and Erich Kamke[4] (1932); it includes, also, the investigations by Arthur H. Copeland[5] (1928).

The formal conception introduces the concept of probability by the method of implicit definitions, and uses no properties of the concept other than those expressed in a set of formal relations placed as axioms at the beginning of the theory, leaving open various possibilities for its interpretation. The group includes the axiom system given in 1901 by Georg Bohlmann[6] and the analyses published by S. Bernstein[7] (1917) and Emile Borel[8] (1925). To it belongs also my own axiomatic presentation, which was first published in 1932.[9] It was followed by an axiomatic construction by A. N. Kolmogoroff[10] in 1933.

[1] "Grundlagen der Wahrscheinlichkeitsrechnung," in *Math. Zs.*, Vol. V (1919), pp. 52–99; *Vorlesungen aus dem Gebiete der angewandten Mathematik*, Vol. I: *Wahrscheinlichkeitsrechnung* . . . (Leipzig, 1931).

[2] "Zu der von R. v. Mises gegebenen Begründung der Wahrscheinlichkeitsrechnung," in *Math. Zs.*, Vol. 32 (1930), pp. 232–258.

[3] "Eine neue Grundlegung der Wahrscheinlichkeitsrechnung," in *Zs. f. Physik*, Vol. 63 (1930), p. 697; "Grundlagen der Wahrscheinlichkeitsrechnung," in *Acta Math.*, Vol. 60 (1933), pp. 239–380.

[4] *Einführung in die Wahrscheinlichkeitstheorie* (Leipzig, 1932).

[5] "Admissible Numbers in the Theory of Probability," in *Amer. Jour. Math.*, Vol. 50, No. 4 (1928), p. 535; and later papers.

[6] *Encykl. d. math. Wiss.*, Vol. I, Part 2 D 4*b* (1901), pp. 852–917.

[7] "Versuch einer axiomatischen Begründung der Wahrscheinlichkeitsrechnung," in *Mitt. d. math. Ges. Charkow* (1917), pp. 209–274.

[8] *Traité du calcul des probabilités* (Paris, 1924–), Vol. I, Part 1; *Principes et formules classiques du calcul des probabilités* (Paris, 1925).

[9] "Axiomatik der Wahrscheinlichkeitsrechnung," in *Math. Zs.*, Vol. 34 (1932), pp. 568–619.

[10] *Grundbegriffe der Wahrscheinlichkeitsrechnung* (Berlin, 1933).

Most inquiries of the formal group omit the development of a theory of the order of probability sequences. The problem was first attacked by von Mises, Dörge, and Copeland in articles applying the interpreted conception, whereas my presentation has shown that the problem can be dealt with even within the formal conception. The next chapter deals with the differences between my presentation and those of von Mises, Dörge, Tornier, and other authors. These differences result from the fact that my theory develops a system comprising all types of order, whereas the other systems are restricted to special types.

A third line of development, going much further back historically than the axiomatic inquiries, connects the treatment of probability with the methods of symbolic logic. This line can be traced to Leibniz,[11] whose program of a mathematical logic included that of a logic of probability. The idea of construing probability as a relation between statements, which includes logical implication as a special case, was proposed in 1837 by Bernard Bolzano.[12] British and American logicians have followed a similar course. In his fundamental work introducing the period of modern logic, George Boole[13] (1854) included a logic of probability; he was followed by John Venn[14] (1866), Charles S. Peirce[15] (1878), and John M. Keynes[16] (1921). The latter work, besides combining symbolic logic with the calculus of probability, contains a report on earlier attempts at constructing such a calculus. In this group belong also the publications of Harold Jeffreys.[17]

My own presentation undertakes to unite the axiomatic method with the construction of a logico-mathematical calculus, which I developed without a knowledge of the calculi published much earlier by the authors cited. My theory of probability implication originated within the context of inquiries into the nature of causality.[18] The table of rules of probability implication given there is to be replaced by my present formulation. A summary of my theory of probability was published in French.[19]

---

[11] See the presentation by Louis Couturat, *La Logique de Leibniz . . .* (Paris, 1901), pp. 239–250.

[12] *Wissenschaftslehre* (1837; Leipzig, 1929–), § 161; see also the remark by Walter Dubislav, in *Erkenntnis*, Vol. I (1930), p. 264.

[13] *The Laws of Thought* (London, 1854).

[14] *The Logic of Chance* (London and Cambridge, 1866).

[15] *The Doctrine of Chances* (1878), printed in *Collected Papers* (Cambridge, Mass., 1932), Vol. II, p. 395.

[16] *A Treatise on Probability* (London, 1921).

[17] *Theory of Probability* (Oxford, 1939).

[18] "Die Kausalstruktur der Welt und der Unterschied von Vergangenheit und Zukunft," in *Ber. d. bayer. Akad., Math.-phys. Kl.* (1925), p. 144.

[19] "Les Fondements logiques du calcul des probabilités," in *Ann. de l'Inst. Henri Poincaré*, Vol. VII, Part 5 (1937), pp. 267–348.

APPENDIX TO CHAPTER 3

Exercises

Problem 1

According to official statistics from 1937, published by the National Socialist government, Germany had 66,031,580 inhabitants (A); among them were 502,799 Jews (J) and 325,541 sentenced criminals (C). Among the latter category were 1,794 Jews. What is the probability

a) that a German Jew is a criminal?
b) that a non-Jewish German is a criminal?
c) that a German criminal is a Jew?
d) that a non-criminal German is a Jew?

For the solution use the frequency interpretation directly.

Problem 2

Out of 1,000 unmarried men who are 20 years old (A), 28.3 die in that year (D). Among these, 6.1 die from tuberculosis (T), and 6.6 die from accidents (C) (German statistics of 1937). What is the probability

a) that a man 20 years old dies from tuberculosis *or* accident?
b) that a reported case of death of a man 20 years old is due to tuberculosis?
c) that a reported case of death of a man 20 years old is due to tuberculosis *or* accident?

For the solution use the frequency interpretation directly.

Problem 3

Throwing (A) with two dice distinguished as B and C, what is the probability of getting a number smaller than 5 on die B *or* a number greater than 4 on die C?

Problem 4

Urn A contains 10 slips showing the number 1, 20 slips showing the number 2, 30 slips showing the number 3. Urn B_1 contains 30 black and 50 white balls; urn B_2 contains 50 black and 50 white balls; urn B_3 contains 60 black and 20 white balls. The drawing is made as follows. A slip is drawn from urn A. The number obtained determines with which urn B_i to continue, and a ball is then drawn from that urn.

a) Determine the probability of getting a white ball (C).
b) If a white ball has been drawn, it being unknown from which of the urns B_i it was obtained, what is the probability that it was drawn, respectively, from urns B_1, B_2, B_3?

Problem 5

Mr. Smith's gardener is not dependable; the chances are 2 to 1 that he will forget to water the rosebush during Smith's absence. The rosebush is in a questionable condition; it has even chances of recovery if it is watered, but only 25% chances of recovery if it is not watered. Upon returning, Smith finds that the rosebush has withered. What is the probability that the gardener did not water the rosebush?

Problem 6

Lady Catherine's poodle has been missing for five days. There are only three explanations. Either the poodle went to the town, in which case there is a 3% chance that he was run over by a car, and a 50% chance that he was taken to the dog pound; or he went astray in the woods and was accidentally hit by a hunter, the chance of such an accident being 1%; or he went to the village and was stolen by gypsies, who in the last year had stolen five out of ten stray dogs. On a dozen previous occasions the dog had been found four times in the town, twice in the village, and six times in the woods. An inquiry at the dog pound showed that he was not there. He had never been absent more than three days at a time.

a) What is the probability that the dog was stolen by gypsies?

b) If no inquiry at the dog pound had been made, what would be the probability that the dog was stolen by gypsies?

Problem 7

Under the present conditions (A) the chances for election as governor are $\frac{1}{4}$ for Brown (B), $\frac{5}{8}$ for Jones (J), $\frac{1}{8}$ for Robinson (R). Should Brown be elected, the chances for the construction of a certain highway (C) are 60%; the chances are 80% if Jones is elected; and 20% if Robinson is elected.

a) On the basis of the present conditions, what is the chance that the highway will be constructed?

b) On the evening of the election, before the counting of the votes, Jones dies of apoplexy because of excitement. A simple majority will decide the election. What is the chance now that the highway will be constructed?

Solutions

Problem 1

a) $P(A.J,C) = \dfrac{N^n(A.J.C)}{N^n(A.J)} = \dfrac{1{,}794}{502{,}799} = 3.57$ per thousand

b) $P(A.\bar{J},C) = \dfrac{N^n(A.\bar{J}.C)}{N^n(A.\bar{J})} = \dfrac{(325{,}541 - 1{,}794)}{66{,}031{,}580 - 502{,}799}$

$= 4.94$ per thousand

c) $P(A.C,J) = \dfrac{N^n(A.C.J)}{N^n(A.C)} = \dfrac{1{,}794}{325{,}541} = 5.52$ per thousand

d) $P(A.\bar{C},J) = \dfrac{N^n(A.\bar{C}.J)}{N^n(A.\bar{C})} = \dfrac{502{,}799 - 1{,}794}{66{,}031{,}580 - 325{,}541}$

$= 7.64$ per thousand

The figures show that criminality among Jews is smaller than among non-Jews.

Problem 2

a) $P(A,D.T \vee D.C) = \dfrac{N^n(A.[D.T \vee D.C])}{N^n(A)} = \dfrac{6.1 + 6.6}{10{,}000} = 0.00127$

b) $P(A.D,T) = \dfrac{N^n(A.D.T)}{N^n(A.D)} = \dfrac{6.1}{28.3} = 0.216$

c) $P(A.D,T \vee C) = \dfrac{N^n(A.D.[T \vee C])}{N^n(A.D)} = \dfrac{6.1 + 6.6}{28.3} = 0.45$

Problem 3

Notation: B_5 = number smaller than 5 on die B; C_4 = number greater than 4 on die C.

$$P(A,B_5 \vee C_4) = P(A,B_5) + P(A,C_4) - P(A,B_5.C_4)$$
$$= \tfrac{2}{3} + \tfrac{1}{3} - \tfrac{2}{3} \cdot \tfrac{1}{3} = \tfrac{7}{9}$$

Problem 4

$$P(A,C) = \sum_{i=1}^{3} P(A,B_i) \cdot P(A.B_i,C)$$
$$= \tfrac{1}{6} \cdot \tfrac{5}{8} + \tfrac{1}{3} \cdot \tfrac{1}{2} + \tfrac{1}{2} \cdot \tfrac{1}{4} = \tfrac{19}{48}$$

$$P(A.C,B_k) = \frac{P(A,B_k) \cdot P(A.B_k,C)}{\sum_{i=1}^{3} P(A,B_i) \cdot P(A.B_i,C)}$$

$$P(A.C,B_1) = \tfrac{5}{19} \qquad P(A.C,B_2) = \tfrac{8}{19} \qquad P(A.C,B_3) = \tfrac{6}{19}$$

Problem 5

This and problem 6 are examples of a formalization of an informal use of probability rules—in particular, of the rule of Bayes. The numerical values used should be considered as rough estimates of the probabilities concerned. The inference then leads to values that have some significance, at least qualitatively, and correspond to instinctive appraisals of probabilities made, for example, by detectives or other experts in indirect evidence, in situations of the kind described.

Notation: A = the situation before Smith's voyage: W = the watering of the rosebush; D = the withering of the rosebush.

$$P(A,W) = \tfrac{1}{3} \quad P(A,\bar{W}) = \tfrac{2}{3} \quad P(A\,.\,W,D) = \tfrac{1}{2} \quad P(A\,.\,\bar{W},D) = \tfrac{3}{4}$$

$$P(A\,.\,D,\bar{W}) = \frac{P(A,\bar{W}) \cdot P(A\,.\,\bar{W},D)}{P(A,W) \cdot P(A\,.\,W,D) + P(A,\bar{W}) \cdot P(A\,.\,\bar{W},D)} = \tfrac{3}{4}$$

Problem 6

Notation: A = general situation after the poodle's disappearance, but not yet including a statement that an accident has occurred; T = the poodle's going to the town; V = the poodle's going to the village; W = the poodle's going to the woods; D = the poodle's being in the dog pound; C = the poodle's having an accident of any kind, including the case of his being stolen by gypsies. The following values are given:

$$P(A,T) = \tfrac{4}{12} \qquad P(A,V) = \tfrac{2}{12} \qquad P(A,W) = \tfrac{6}{12}$$

$$P(A\,.\,T,C) = \tfrac{3}{100} \qquad P(A\,.\,T,D) = \tfrac{50}{100} \qquad P(A\,.\,V,C) = \tfrac{5}{10}$$

$$P(A\,.\,W,C) = \tfrac{1}{100}$$

Question a: The poodle is not in the dog pound. Because he has never been absent for more than three days but has now been missing for five days, we consider the assumption of an accident as true. The assumption that he was stolen by gypsies is equivalent to his having gone to the village *and* having an accident, that is to $V\,.\,C$. Therefore the probability sought for is given by

$$P(A\,.\,C,V) = \frac{P(A,V) \cdot P(A\,.\,V,C)}{P(A,C)} = 85\%$$

where $P(A,C) = P(A,T) \cdot P(A\,.\,T,C) + P(A,V) \cdot P(A\,.\,V,C) + P(A,W)$

$$\cdot\, P(A\,.\,W,C) = \tfrac{59}{600}$$

Question b: Here the situation is characterized by $C \vee D$, and the rule of reduction must be applied:

$$P(A.[C \vee D],V) = \frac{P(A,C) \cdot P(A.C,V) + P(A,D) \cdot P(A.D,V)}{P(A,C) + P(A,D)}$$

Now $P(A.D,V) = 0$ because the dog pound is not in the village but in the town. Furthermore, we have

$$P(A,D) = P(A,T) \cdot P(A.T,D) + P(A,\bar{T}) \cdot P(A.\bar{T},D)$$

Since the dog pound is in the town, $P(A.\bar{T},D) = 0$. Therefore

$$P(A,D) = P(A,T) \cdot P(A.T,D) = \tfrac{4}{12} \cdot \tfrac{50}{100} = \tfrac{1}{6}$$

and the probability asked for is given by

$$P(A.[C \vee D],V) = 31.5\%$$

This result shows that the probability of an accident is considerably smaller so long as there is a chance that the poodle is in the dog pound.

Problem 7

Question a

$$\begin{aligned} &P(A,C) \\ &= P(A,B) \cdot P(A.B,C) + P(A,J) \cdot P(A.J,C) + P(A,R) \cdot P(A.R,C) \\ &= \tfrac{27}{40} = 67.5\% \end{aligned}$$

Question b

$$\begin{aligned} P(A.[B \vee R],C) &= \frac{P(A,B) \cdot P(A.B,C) + P(A,R) \cdot P(A.R,C)}{P(A,B) + P(A,R)} \\ &= \tfrac{7}{15} = 46.6\% \end{aligned}$$