MATHEMATICAL LOGIC

BY

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a mode which consists notationally of compounding the statements by means of 'implies' *and* the two pairs of quotation marks.[1] If implication is construed as going beyond questions of truth value, this derivative mode of statement composition will not be truth-functional. Implication thus construed would then seem, after all, to interfere with a policy of admitting none but truth-functional modes of statement composition. By the same argument, indeed, a purely morphological or phonetic relation such as containing or rhyming would interfere similarly. Actually, however, derivation of modes of statement composition from relations in the suggested fashion involves abuse of quotation. The statements buried in the quotations in (9) cannot be treated in turn as constituents of (9), for a quotation figures as a single irreducible word. Similar abuse of quotation was seen in § 4 to lead from ' 'Cicero' has six letters' to ' 'Tully' has six letters'.[2]

These latter remarks serve only to show that we can construe implication as going beyond questions of truth value without *thereby* committing ourselves to any form of conditional which goes beyond questions of truth value. The need for some such strong form of conditional might still be urged on other grounds. Certainly not all uses of the subjunctive conditional submit to the easy method of paraphrase illustrated in the case of Perth and America. When this fails we may look to other devices, e.g. Carnap's method of reduction sentences ("Testability," pp. 439–453); but if any really useful cases prove to resist all such methods of analysis, then we shall perhaps have to choose pragmatically between the usefulness of those cases and the convenience and clarity of the truth functional kind of statement composition. Mathematics itself gives rise to no such recalcitrant cases; and any which seem to arise beyond the bounds of mathematics should be critically regarded.[3]

§ 6. *Quasi-Quotation*

IN DISCUSSING the modes of statement composition we are having continually to talk of expressions. Quotation suffices for the mention of any specific expression, such as 'v' or '≡' or 'Jones is away', but is not available when we want to speak generally of an unspecified expression of such and such kind. On such occasions use has been made of general locutions such as 'a conditional', 'the first component', etc.; and more difficult cases have been managed indirectly by introducing a blank '——' from time to

[1] Analogous reasoning appears in Huntington's "Note on a Recent Set", p. 11.

[2] See also Tarski, "Wahrheitsbegriff," § 1.

[3] For further discussion and references see Carnap, *Syntax*, §§ 67–71.

time. But the developments to follow call for a more elastic method of referring to unspecified expressions.

For the beginnings of such a method, the use of letters in algebra provides us with an adequate model. In algebra 'x', 'y', etc. are used as names of unspecified numbers; we may suppose them replaced by names of any specific numbers we choose. Analogously, Greek letters other than 'ϵ', 'ι', 'λ'[1] will now be used as names of unspecified *expressions;* we may suppose them replaced by names (e.g. quotations) of any specific expressions we choose.

A discussion of numbers may, for example, begin thus:

(1) Let x be a factor of y.

Throughout the discussion thus prefaced, we are to think of x and y as any specific numbers we like which satisfy the condition (1) — say the numbers 5 and 15, or 4 and 32. We are to think of the letters 'x' and 'y' as if they were names of the numbers 5 and 15, or names of the numbers 4 and 32, etc. We are to imagine the letters 'x' and 'y' replaced by the numerals (expressions) '5' and '15', or by '4' and '32', etc.

Similarly a discussion of expressions might begin thus:

(2) Let μ be part of ν.

Throughout the discussion thus prefaced, we are to think of μ and ν as any specific expressions we like which satisfy the condition (2) — say the expressions 'York' and 'New York', or '3' and '32'. We are to think of the letters 'μ' and 'ν' as if they were names of the expressions 'York' and 'New York', or names of the expressions '3' and '32', etc. We are to imagine the letters 'μ' and 'ν' replaced by the quotations ''York'' and ''New York'', or by ''3'' and ''32'', etc.

The reader is urged to compare the above short paragraph with the preceding one, word by word; also to review § 4. Roughly speaking, the letters 'x', 'y', etc. may be described as ambiguous numerals, ambiguous names of numbers, variables ambiguously designating numbers, or, in the usual technical phrase, variables taking numbers as their values. Correspondingly the letters

[1] These three letters are reserved for later purposes; cf. §§ 22, 35, 41.

'μ', 'ν', etc. may be roughly described as ambiguous quotations, ambiguous names of expressions, variables ambiguously designating expressions, or variables taking expressions as their values. This does not mean simply that 'μ' and 'ν' take the place of expressions, or are replaceable by expressions, for this is true of 'x' and 'y' as well. Rather, the letters 'μ' and 'ν' take the place of quotations or other names of expressions, just as 'x' and 'y' take the place of numerals or other names of numbers.

Occasionally Greek letters will be used with accents or subscripts attached: 'μ'', 'μ''', 'μ_1', 'μ_2', 'μ_n', etc. Such variants may be regarded simply as so many further Greek letters. Three Greek letters, 'ϕ', 'ψ', and 'χ', together with their accented and subscripted variants, will be limited in their use to those cases where the expression designated is intended to be a statement. They serve as names of unspecified statements, and are replaceable by statement quotations or other names of specific statements.[1]

There is need also of a convenient way of speaking of specific contexts of unspecified expressions: speaking, e.g., of the result of enclosing the unspecified expression μ in parentheses, or the result of joining the unspecified statements ϕ and ψ in that order by the sign '$\equiv$'. Note that quotation is not available here. The quotations:

$$\text{`}(\mu)\text{'}, \qquad \text{`}\phi \equiv \psi\text{'}$$

designate only the specific expressions therein depicted, containing the specific Greek letters 'μ', 'ϕ', and 'ψ'. Reference to the intended contexts of the unspecified expressions μ, ϕ, and ψ will be accomplished by a new notation of *corners*, thus:

$$(3) \qquad \ulcorner(\mu)\urcorner, \qquad \ulcorner\phi \equiv \psi\urcorner.$$

Because of the close relationship which it bears to quotation, this device may be called *quasi-quotation*. It amounts to quoting the constant contextual backgrounds, '( )' and ' $\equiv$ ', and imagining the unspecified expressions μ, ϕ, and ψ written in the blanks. If in particular we take the expression 'Jones' as μ,

[1] The three letters have indeed already appeared in this use in § 5, where the sense intended was apparent.

'Jones is away' as ϕ, and 'Smith is ill' as ψ, then $\ulcorner(\mu)\urcorner$ is '(Jones)' and $\ulcorner\phi \equiv \psi\urcorner$ is 'Jones is away ≡ Smith is ill'.

The quasi-quotations (3) are synonymous with the following verbal descriptions:

> The result of writing '(' and then μ and then ')',
> The result of writing ϕ and then '≡' and then ψ;

or, equivalently:

> The result of putting μ in the blank of '(   )',
> The result of putting ϕ and ψ in the respective blanks of ' ≡ ';

or, equivalently:

> The result of putting μ for 'μ' in '(μ)',
> The result of putting ϕ for 'ϕ' and ψ for 'ψ' in '$\phi \equiv \psi$'.

We may translate any quasi-quotation:

$$\ulcorner\rule{2em}{0.4pt}\urcorner$$

into words in corresponding fashion:

> The result of putting μ for 'μ', ν for 'ν', . . . , ϕ for 'ϕ', ψ for 'ψ', . . . in '——'.

Described in another way: a quasi-quotation designates that (unspecified) expression which is obtained from the contents of the corners by replacing the Greek letters (other than 'ϵ', 'ι', 'λ') by the (unspecified) expressions which they designate.

When a Greek letter stands alone in corners, quasi-quotation is vacuous: $\ulcorner\mu\urcorner$ is μ. For, by the foregoing general description, $\ulcorner\mu\urcorner$ is the result of putting μ for 'μ' in 'μ'; $\ulcorner\mu\urcorner$ is what the letter 'μ' becomes when that letter itself is replaced by the (unspecified) expression μ; in other words, $\ulcorner\mu\urcorner$ is simply that expression μ.

Quasi-quotation would have been convenient at earlier points, but was withheld for fear of obscuring fundamentals with excess machinery. Now, however, it may be a useful exercise to recapitulate some sample points from §§ 1–5 in terms of this device. A conjunction $\ulcorner\phi \,.\, \psi\urcorner$ is true just in case ϕ and ψ are both true, and an alternation $\ulcorner\phi \vee \psi\urcorner$ is false just in case ϕ and ψ are both false. A conditional $\ulcorner\phi \supset \psi\urcorner$ is true if ϕ is false or ψ true, and false if ϕ is dots, if any, and otherwise at the limit of the whole symbolic context. E.g., since '$\equiv$' in

$$\ulcorner \phi \vee : \psi \supset \chi \,.\!\equiv \phi \,.:\supset . \,\psi \supset \phi \urcorner$$

has just one dot prefixed, the first component of the biconditional begins at the last previous group of two or more dots — hence, as it happens, at the group of two. Again, since '$\equiv$' has no dots suffixed, the second component of the biconditional stops as soon as it meets any dots at all — hence at the group of three. The components of the biconditional are thus $\ulcorner \psi \supset \chi \urcorner$ and ϕ. The alternation has $\ulcorner \psi \supset \chi \,.\!\equiv \phi \urcorner$ as second component, for this runs from the pair of dots to the first larger group. The conditional corresponding to the second occurrence of '$\supset$' has $\ulcorner \phi \vee : \psi \supset \chi \,.\!\equiv \phi \urcorner$ in its entirety as antecedent, since the group of dots prefixed to the second occurrence of '$\supset$' exceeds any previous group. Similarly that conditional has $\ulcorner \psi \supset \phi \urcorner$ as consequent. The whole would appear in terms of parentheses as

$$\ulcorner((\phi \vee ((\psi \supset \chi) \equiv \phi)) \supset (\psi \supset \phi))\urcorner.$$

Dots serve in this fashion to reinforce the connectives '$\vee$', '$\supset$', and '$\equiv$'. In the case of conjunction, already represented by a dot, such reinforcement will be accomplished simply by using a group of dots instead of the one; thus the conjunction of ϕ with $\ulcorner \psi \vee \chi \,.\!\supset \phi \urcorner$ appears as $\ulcorner \phi : \psi \vee \chi \,.\!\supset \phi \urcorner$. Use of dots for conjunction is distinguishable from the other uses by the absence of any adjacent connective '$\vee$', '$\supset$', or '$\equiv$'. A group of n dots standing as a conjunction sign will be regarded as indicating a smaller break than that indicated by a group of n dots alongside '$\vee$', '$\supset$', or '$\equiv$', though of course a greater break than that indicated by n–1 dots. Thus, in

$$\ulcorner \phi \,.\, \psi \,.\!\supset \chi : \psi \vee \phi :\equiv \chi \urcorner$$

the conjunction indicated by the single dot has ϕ and ψ as components; the conjunction indicated by the pair of dots has $\ulcorner \phi \,.\, \psi \,.\!\supset \chi \urcorner$ and $\ulcorner \psi \vee \phi \urcorner$ as components; the conditional has $\ulcorner \phi \,.\, \psi \urcorner$ and χ as components; and the biconditional has $\ulcorner \phi \,.\, \psi \,.\!\supset \chi : \psi \vee \phi \urcorner$ and χ. The whole would appear in terms of parentheses as:

true and ψ false. A biconditional $\ulcorner \phi \equiv \psi \urcorner$ is true just in case ϕ and ψ are alike in truth value. A denial $\ulcorner \sim \phi \urcorner$ is true just in case ϕ is false. ϕ logically implies ψ or is logically equivalent to ψ according as $\ulcorner \phi \supset \psi \urcorner$ or $\ulcorner \phi \equiv \psi \urcorner$ is logically true, and ϕ materially implies ψ or is materially equivalent to ψ according as $\ulcorner \phi \supset \psi \urcorner$ or $\ulcorner \phi \equiv \psi \urcorner$ is true.

§ 7. *Parentheses and Dots*

PARENTHESES, taken for granted thus far as an auxiliary notation, are most simply construed as forming integral parts of the binary connectives. The notations of conjunction, alternation, the conditional, and the biconditional will no longer be regarded as $\ulcorner \phi \,.\, \psi \urcorner$, $\ulcorner \phi \vee \psi \urcorner$, $\ulcorner \phi \supset \psi \urcorner$, and $\ulcorner \phi \equiv \psi \urcorner$, but as $\ulcorner (\phi \,.\, \psi) \urcorner$, $\ulcorner (\phi \vee \psi) \urcorner$, $\ulcorner (\phi \supset \psi) \urcorner$, and $\ulcorner (\phi \equiv \psi) \urcorner$. But the notation of denial remains simply $\ulcorner \sim \phi \urcorner$. This formulation yields just the usage of parentheses which we have hitherto followed, except in one respect: a conjunction, alternation, conditional, or biconditional comes now to bear an outside pair of parentheses even when it stands apart from any further symbolic context. Such outside parentheses have hitherto been omitted.

When the binary modes of composition are thus construed, no auxiliary technique of grouping is needed. The syntactical simplicity thus gained proves useful in certain abstract studies (e.g. § 10; also later chapters). In applications, however, such simplicity is less important than facility of reading; and excess of parentheses is a hindrance, for we have to count them off in pairs to know which ones are mates. It is hence convenient in practice to omit the outside parentheses as hitherto, and furthermore to suppress most of the remaining parentheses in favor of a more graphic notation of *dots*.

Parentheses mark the outer limits of a binary compound; dots determine those same limits less directly. Roughly speaking, a group of dots placed beside '$\vee$', '$\supset$', or '$\equiv$' indicates that the component on that side has its other end at the nearest larger group of