ON THE PSYCHOLOGY AND NATURAL DEVELOPMENT OF GEOMETRY.

For the animal organism, the relations of the different parts of *its own body* to one another, and of physical objects to these different parts, are *primarily* of the greatest importance. Upon these relations is based its system of physiological sensations of space. More complicated conditions of life, in which the simple and direct satisfaction of needs is impossible, result in an augmentation of intelligence. The physical, and particularly the *spatial,* behavior of bodies *toward one another* may then acquire a mediate and indirect interest far transcending our interest in our momentary sensations. In this way, a spatial image of the world is created, at first instinctively, then in the practical arts, and finally scientifically, in the form of geometry. The mutual relations of bodies are geometrical in so far as they are determined by sensations of space, or find their expression in such sensations. Just as without sensations of heat there would have been no theory of heat, so also without sensations of space there would be no geometry; but both the theory of heat and the theory of geometry stand additionally in need of *experiences concerning bodies;* that is to say, both must pursue *their inquiries* beyond the

narrow boundaries of the domains of sense that constitute their peculiar foundation.

The Rôle of Bodies.

Isolated sensations have *independent* significance only in the lowest stages of animal life; as, for example, in reflex motions, in the removal of some disagreeable irritation of the skin, in the *snapping* reflex of the frog, etc. In the higher stages, attention is directed, not to space-sensation alone, but to those intricate and intimate *complexes* of other sensations with space-sensations which we call *bodies.* Bodies arouse our interest; they are the *objects* of our activities. But the *character* of our activities is coincidently determined by the *place* of the body, whether near or far, whether above or below, etc.,—in other words, by the space-sensations characterizing that body. The *mode* of reaction is thus determined by which the body can be reached, whether by extending the arms, by taking few or many steps, by hurling missiles, or what not. The *quantity* of sensitive elements which a body excites, the number of places which it covers, that is to say, the *volume* of the body, is, all other things being the same, proportional to its capacity for satisfying our needs, and possesses a consequent biological import. Although our sensations of sight and touch are primarily produced only by the *surfaces* of bodies, nevertheless powerful associations impel especially primitive man to imagine more, or, as he thinks, to *per-*

ceive more, than he actually observes. He imagines to be filled with *matter* the places enclosed by the surface which alone he perceives; and this is especially the case when he sees or seizes bodies with which he is in some measure familiar. It requires considerable power of abstraction to bring to consciousness the fact that we perceive the surfaces *only* of bodies,—a power which cannot be ascribed to primitive man.

Of importance in this regard are also the peculiar *distinctive shapes* of objects of prey and utility. Certain definite forms, that is, certain specific combinations of space-sensations, which man learns to know through intercourse with his environment, are unequivocally characterized even by purely physiological features. The straight line and the plane are distinguished from all other forms by their physiological simplicity, as are likewise the circle and the sphere. The affinity of symmetric and geometrically similar forms is revealed by purely physiological properties. The variety of shapes with which we are acquainted from our physiological experience is far from being inconsiderable. Finally, through employment with bodily objects, *physical* experience also contributes its quota of wealth to the general store.

The Notion of Constancy.

Crude physical experience impels us to attribute to bodies a certain *constancy*. Unless there are special reasons for not doing so, the same constancy is

also ascribed to the individual attributes of the complexus "body";[1] thus we also regard the color, hardness, shape, etc., of the body as constant; and particularly we look upon the body as *constant with respect to space, as indestructible.* This assumption of spatial constancy, of *spatial substantiality,* finds its direct expression in geometry. Our physiological and psychological organization is independently predisposed to emphasize constancy; for general physical constancies must necessarily have found lodgment in our organization, which is itself physical, while in the adaptation of the species very definite physical constancies were at work. Inasmuch as memory revives the images of bodies, before perceived, in their original forms and dimensions, it supplies the condition for the recognition of the same bodies, thus furnishing the first foundation for the impression of constancy. But geometry is additionally in need of certain *individual* experiences.

Let a body K move away from an observer A by being suddenly transported from the environment FGH to the environment MNO. To the optical observer A the body K decreases in size and assumes generally a different form. But to an optical observer B, who moves along with K and who always retains the same position with respect to K, K remains unaltered. An analogous sensation is experienced by the *tactual observer,* although the per-

[1] See my *Analysis of the Sensations,* introductory chapter.

spective diminution is here wanting for the reason that the sense of touch is not a telepathic sense. The experiences of *A* and *B* must now be harmonized and their contradictions eliminated,—a requirement which becomes especially imperative when *the same* observer plays alternately the parts of *A* and of *B*. And the only method by which they can be harmonized is, to attribute to *K* certain *constant* spatial properties independently of its position with respect to *other* bodies. The space-sensations determined by *K* in the observer *A* are recognized as *dependent* on other space-sensations (the position of *K* with respect to the body of the observer *A*). But these same space-sensations determined by *K* in *A* are *independent* of other space-sensations, characterizing the position of *K* with respect to *B*, or with respect to *FGH...MNO*. In *this* independence lies the *constancy* with which we are here concerned.

The fundamental assumption of geometry thus reposes on an *experience,* although on one of an idealized kind.

The Notion of Rigidity.

In order that the experience in question may assume palpable and perfectly determinate form, the body *K* must be a so-called *rigid* body. If the space-sensations associated with *three* distinct acts of sense-perception remain unaltered, then the condition is given for the invariability of the entire complexus of space-sensations determined by a rigid body. This determination of the space-sensations

produced by a body by means of *three* space-sensational *elements* accordingly characterizes the rigid body, from the point of view of the physiology of the senses. And this holds good for both the visual and the tactual sense. In so doing we are not thinking of the physical conditions of rigidity (in defining which we should be compelled to enter different sensory domains), but merely of the fact given to our spatial sense. Indeed, we are now regarding every body as rigid which possesses the property assigned, even liquids, so long as their parts are not in motion with respect to one another.

Physical Origin of Geometry.

Correct as the oft-repeated asseveration is that geometry is concerned, not with *physical,* but with *ideal* objects, it nevertheless cannot be doubted that geometry has sprung from the interest centering in the spatial relations of *physical bodies.* It bears the distinctest marks of this origin, and the course of its development is fully intelligible only on a consideration of this fact. Our knowledge of the spatial behavior of bodies is based upon a *comparison* of the space-sensations produced by them. Without the least artificial or scientific assistance we acquire abundant experience of space. We can judge approximately whether rigid bodies which we perceive alongside one another in different positions at different distances, will, when brought *successively* into the same position, produce approximately the same or dissimilar space-sensations. We know fairly well whether one body will coincide with another,—whether a pole lying flat on the ground will reach to a certain height. Our sensations of space are, however, subject to physiological circumstances, which can never be absolutely identical for the members compared. In every case, rigorously viewed, a memory-trace of a sensation is necessarily compared with a real sensation. If, therefore, it is a question of the exact spatial relationship of bodies *to one another,* we must provide characteristics that depend as little as possible on physiological conditions, which are so difficult to control.

Measurement.

This is accomplished by comparing *bodies* with *bodies.* Whether a body *A* coincides with another body *B*, whether it can be made to occupy exactly the space filled by the other—that is, whether under like circumstances both bodies produce the same space-sensations—can be estimated with great precision. We regard such bodies as spatially or geometrically equal in every respect,—*as congruent.* The *character* of the sensations is here no longer authoritative; it is now solely a question of their *equality* or *inequality.* If both bodies are rigid bodies, we can apply to the second body *B* all the experiences which we have gathered in connection with the first, more convenient, and more easily transportable, standard body *A*. We shall revert later to the circumstance that it is neither necessary nor possible to employ a special body of comparison,

or standard, for every body. The most convenient bodies of comparison, though applicable only after a crude fashion,—bodies whose invariance during transportation we always have before our eyes,—are our *hands* and *feet,* our *arms* and *legs.* The names of the oldest measures show distinctly that originally we made our measurements with hands'-breadths, forearms (*ells*), feet (*paces*), etc. Nothing but a period of *greater exactitude* in measurement began with the introduction of conventional and carefully preserved physical standards; the principle remains the same. The measure enables us to compare bodies which are difficult to move or are practically immovable.

The Rôle of Volume.

As has been remarked, it is not the *spatial,* but predominantly the *material,* properties of bodies that possess the strongest interest. This fact certainly finds expression even in the beginnings of geometry. The *volume* of a body is instinctively taken into account as representing the quantity of its material properties, and so comes to form an object of *contention* long before its geometric properties receive anything approaching to profound consideration. It is here, however, that the comparison, the measurement of volumes acquires its initial import, and thus takes its place among the first and most important problems of primitive geometry.

The first measurements of volume were doubtless of liquids and fruit, and were made with hollow measures. The object was to ascertain conveniently the quantity of like matter, or the *quantity* (*number*) of homogeneous, similarly shaped (identical) *bodies.* Thus, conversely, the capacity of a store-room (granary) was in all likelihood originally estimated by the quantity or number of homogeneous bodies which it was capable of containing. The measurement of volume by a unit of volume is in all probability a much later conception, and can only have developed on a higher stage of abstraction. Estimates of areas were also doubtless made from the *number* of fruit-bearing or useful plants which a field would accommodate, or from the quantity of seed that could be sown on it; or possibly also from the *labor* which such work required.

Measurement of Surfaces.

The measurement of a surface by a surface was readily and obviously suggested in this connection when fields of the same size and shape lay near one another. There one could scarcely doubt that the field made up of n fields of the same size and form possessed also n-fold agricultural value. We shall not be inclined to underrate the significance of this intellectual step when we consider the errors in the measurement of areas which the Egyptians[1] and even the Roman *agrimensores*[2] commonly committed.

[1] Eisenlohr, *Ein mathematisches Handbuch der alten Aegypter: Papyrus Rhind,* Leipsic, 1877.

[2] M. Cantor, *Die römischen Agrimensoren,* Leipsic, 1875.

Even with a people so splendidly endowed with geometrical talent as the Greeks, and in so late a period, we meet with the sporadic expression of the idea that surfaces having equal perimeters are equal in area.[1] When the Persian "Overman," Xerxes,[2] wished to count the army which was his to destroy, and which he drove under the lash across the Hellespont against the Greeks, he adopted the following procedure. Ten thousand men were drawn up closely packed together. The area which they covered was surrounded with an enclosure, and each successive division of the army, or rather, each successive herd of slaves, that was driven into and filled the pen, counted for another ten thousand. We meet here with the converse application of the idea by which a surface is measured by the *quantity* (*number*) *of equal, identical, immediately adjacent bodies which cover it.* In abstracting, first instinctively and then consciously, from the height of these bodies, the transition is made to measuring surfaces by means of a unit of surface. The analogous step to measuring volumes by volume demands a far more practiced, geometrically disciplined intuition. It is effected later, and is even at this day less easy to the masses.

All Measurement by Bodies.

The oldest estimates of long *distances,* which were computed by days' journeys, hours of travel, etc.,

[1] Thucydides, VI., 1.

[2] Herodotus, VII., 22, 56, 103, 223.

were based doubtless upon the effort, labor, and expenditure of time necessary for covering these distances. But when lengths are measured by the repeated application of the hand, the foot, the arm, the rod, or the chain, then, accurately viewed, the measurement is made by the enumeration of like bodies, and we have again really a measurement by volume. The singularity of this conception will disappear in the course of this exposition. If, now, we abstract, first instinctively and then consciously, from the two transverse dimensions of the bodies employed in the enumeration, we reach the measuring of a line by a line.

A surface is commonly defined as the boundary of a space. Thus, the surface of a metal sphere is the boundary between the metal and the air; it is not part either of the metal or of the air; two dimensions only are ascribed to it. Analogously, the one-dimensional line is the boundary of a surface; for example, the equator is the boundary of the surface of a hemisphere. The dimensionless point is the boundary of a line; for example, of the arc of a circle. A point, by its motion, generates a one-dimensional line, a line a two-dimensional surface, and a surface a three-dimensional solid space. No difficulties are presented by this concept to minds at all skilled in abstraction. It suffers, however, from the drawback that it does not exhibit, but on the contrary artificially conceals, the natural and actual way in which the abstractions have been reached. A certain discomfort is therefore felt

when the attempt is made from this point of view to define the measure of surface or unit of area after the measurement of lengths has been discussed.[1]

A more homogeneous conception is reached if *every* measurement be regarded as a counting of space by means of immediately *adjacent*, spatially *identical*, or at least hypothetically identical, *bodies*, whether we be concerned with volumes, with surfaces, or with lines. Surfaces may be regarded as corporeal sheets, having everywhere the same constant thickness which we may make small at will, *vanishingly* small; lines, as strings or threads of constant, vanishingly small thickness. A point then becomes a small corporeal space from the extension of which we purposely abstract, whether it be part of another space, of a surface, or of a line. The bodies employed in the enumeration may be of any smallness or any form which conforms to our needs. Nothing prevents our idealizing in the usual manner these images, reached in the natural way indicated, by simply leaving out of account the thickness of the sheets and the threads.

The usual and somewhat timid mode of presenting the fundamental notions of geometry is doubtless due to the fact that the infinitesimal method which freed mathematics from the historical and accidental shackles of its early elementary form, did

[1] Hölder, *Anschauung und Denken in der Geometrie*, Leipsic, 1900, p. 18. W. Killing, *Einführung in die Grundlagen der Geometrie*, Paderborn, 1898, II., p. 22 et seq.

not begin to influence geometry until a later period of development, and that the frank and natural alliance of geometry with the *physical* sciences was not restored until still later, through Gauss. But why the elements shall not now partake of the advantages of our better insight, is not to be clearly seen. Even Leibnitz adverted to the fact that it would be more rational to begin with the *solid* in our geometrical definitions.[1]

Method of Indivisibles.

The measurement of spaces, surfaces, and lines by means of *solids* is a conception from which our refined geometrical methods have become entirely estranged. Yet this idea is not merely the forerunner of the present idealized methods, but it plays an important part in the psychology of geometry, and we find it still powerfully active at a late period of development in the workshop of the investigators and inventors in this domain.

Cavalieri's Method of Indivisibles appears best comprehensible through this idea. Taking his own illustration, let us consider the surfaces to be compared (the quadratures) as covered with equidistant parallel threads of any number we will, after the manner of the warp of woven fabrics, and the spaces to be compared (the cubatures) as filled with parallel sheets of paper. The total *length* of the

[1] Letter to Vitale Giordano, *Leibnizens mathematische Schriften*, edited by Gerhardt, Section I., Vol. I., page 199.

threads may then serve as measure of the *surfaces*, and the total *area* of the sheets as measure of the *volumes*, and the accuracy of the measurement may be carried to any point we wish. The number of *like equidistant* bodies, if close enough together and of the right form, can just as well furnish the numerical measures of surfaces and solid spaces as the number of identical bodies absolutely covering the surfaces or absolutely filling the spaces. If we cause these bodies to shrink until they become lines (straight lines) or until they become surfaces (planes), we shall obtain the division of surfaces into surface-elements and of spaces into space-elements, and coincidently the customary measurement of surfaces by surfaces and of spaces by spaces.

Cavalieri's defective exposition, which was not adapted to the state of the geometry of his time, has evoked from the historians of geometry some very harsh criticisms of his beautiful and prolific procedure.[1] The fact that a Helmholtz, his critical judgment yielding in an unguarded moment to his fancy, could, in his great youthful work,[2] regard a surface as the sum of the lines (ordinates) contained in it, is merely proof of the great depth to which this original, natural conception reaches, and of the facility with which it reasserts itself.

[1] Weissenborn, *Principien der höheren Analysis in ihrer Entwickelung*. Halle, 1856. Gerhardt, *Entdeckung der Analysis*. Halle, 1855, p. 18. Cantor, *Geschichte der Mathematik*. Leipsic, 1892, II. Bd.

[2] Helmholtz, *Erhaltung der Kraft*. Berlin, 1847, p. 14.

The following simple illustration of Cavalieri's method may be helpful to readers not thoroughly conversant with geometry. Imagine a right circular cylinder of horizontal base cut out of a stack of paper sheets resting on a table and conceive inscribed in the cylinder a cone of the same base and altitude. While the sheets cut out by the cylinder are all equal, those forming the cone increase in size as the squares of their distances from the vertex. Now from elementary geometry we know that the

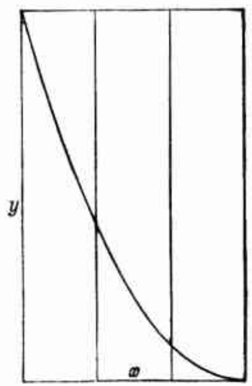

Fig. 3.

volume of such a cone is one-third that of the cylinder. This result may be applied at once to the quadrature of the parabola (Fig. 3). Let a rectangle be described about a portion of a parabola, its sides coinciding with the axis and the tangent to the curve at the origin. Conceiving the rectangle to be covered with a system of threads running parallel to y, every

thread of the rectangle will be divided into two parts, of which that lying outside the parabola is proportional to x^2. Therefore, the area outside the parabola is to the total area of the rectangle precisely as is the *volume* of the cone to that of the cylinder, viz., as 1 is to 3.

It is significant of the naturalness of Cavalieri's view that the writer of these lines, hearing of the higher geometry when a student at the Gymnasium, but without any training in it, lighted on very similar conceptions,—a performance not attended with any difficulty in the nineteenth century. By the aid of these he made a number of little discoveries, which were of course already long known, found Guldin's theorem, calculated some of Kepler's solids of rotation, etc.

Practical Origin of Geometry.

We have then, first, the general experience that *movable bodies* exist, to which, in spite of their mobility, a certain *spatial constancy* in the sense above described, a permanently *identical property,* must be attributed,—a property which constitutes the foundation of all notions of measurement. But in addition to this there has been gathered instinctively, in the pursuit of the trades and the arts, a considerable variety of *special* experiences, which have contributed their share to the development of geometry. Appearing in part in unexpected form, in part harmonizing with one another, and sometimes, when incautiously applied, even becoming involved in what appears to be paradoxical contradictions, these experiences disturb the course of thought and incite it to the pursuit of the orderly logical connection of these experiences. We shall now devote our attention to some of these processes.

Even though the well known statement of Herodotus[1] were wanting, in which he ascribes the origin of geometry to land-surveying among the Egyptians; and even though the account were totally lost[2] which Eudemus has left regarding the early history of geometry, and which is known to us from an extract in Proclus, it would be impossible for us to doubt that a pre-scientific period of geometry existed. The first geometrical knowledge was acquired accidentally and without design by way of practical experience, and in connection with the most varied employments. It was gained at a time when the scientific spirit, or interest in the interconnection of the experiences in question, was but little developed. This is plain even in our meager history of the beginnings of geometry, but still more so in the history of primitive civilization at large, where technical geometrical appliances are known to have existed at so early and barbaric a day as to exclude absolutely the assumption of scientific effort.

All savage tribes practice the art of weaving, and here, as in their drawing, painting, and wood-cut-

[1] *Herodotus*, II., 109.

[2] James Gow, *A Short History of Greek Mathematics*, Cambridge, 1884, p. 134.

ting, the ornamental themes employed consist of the simplest geometrical forms. For such forms, like the drawings of our children, correspond best to their simplified, typical, schematic conception of the objects which they are desirous of representing and

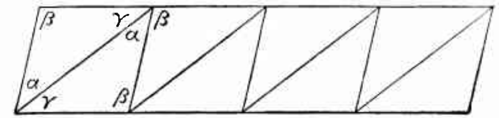

Fig. 4.

it is also these forms that are most easily produced with their primitive implements and lack of manual dexterity. Such an ornament consisting of a series of similarly shaped triangles alternately inverted, or of a series of parallelograms (Fig. 4), clearly suggests the idea, that the sum of the three angles of a triangle, when their vertices are placed together, makes up two right angles. Also this fact could not possibly have escaped the clay and stone workers of Assyria, Egypt, Greece, etc., in constructing their mosaics and pavements from differently colored stones of the same shape. The theorem of the Pythagoreans that the plane space about a point can be completely filled by only three regular polygons, viz., by six equilateral triangles, by four squares, and by three regular hexagons, points to the same source.[1] A like origin of this truth is revealed also in the early Greek method of demonstrat-

[1] This theorem is attributed to the Pythagoreans by Proclus. Cf. Gow, *A Short History of Greek Mathematics*, p. 143, footnote.

ing the theorem regarding the angle-sum of any triangle by dividing it (by drawing the altitude) into two right-angled triangles and completing the rectangles corresponding to the parts so obtained.[1]

The same experiences arise on many other occasions. If a surveyor walk round a polygonal piece of land, he will observe, on arriving at the starting

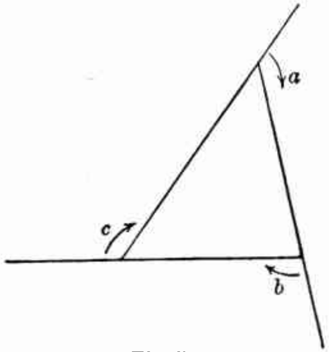

Fig. 5.

point, that he has performed a complete revolution, consisting of four right angles. In the case of a triangle, accordingly, of the six right angles constituting the interior and exterior angles (Fig. 5) there will remain, after subtracting the three exterior angles of revolution, *a, b, c,* two right angles as the sum of the interior angles. This deduction of the theorem was employed by Thibaut,[2] a con-

[1] Hankel, *Geschichte der Mathematik,* Leipsic, 1874, p. 96.

[2] Thibaut, *Grundriss der reinen Mathematik,* Göttingen, 1809, p. 177. The objections which may be raised to this and the following deductions will be considered later. [The same proof is also given by Playfair (1813). See Halsted's translation of Bolyai's *Science Absolute of Space,* p. 67.—*Tr.*]

temporary of Gauss. If a draughtsman draw a triangle by successively turning his ruler round the interior angles, always in the same direction (Fig. 6), he will find on reaching the first side again that if the edge of his ruler lay toward the outside of the triangle on starting, it will now lie toward the inside. In this procedure the ruler has swept out the in-

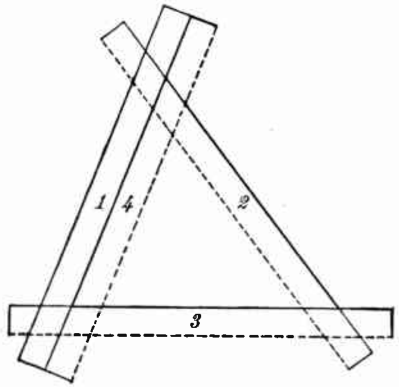

Fig. 6.

terior angles of the triangle in the same direction, and in doing so has performed half a revolution.[1] Tylor[2] remarks that cloth or paper-folding may have led to the same results. If we fold a triangular piece of paper in the manner shown in Fig. 7, we shall obtain a double rectangle, equal in area to one-half the triangle, where it will be seen that the sum of the angles of the triangle coinciding at *a* is two

[1] Noticed by the writer of this article while drawing.

[2] Tylor, *Anthropology, An Introduction to the Study of Man,* etc., German trans., Brunswick, 1883, p. 383.

right angles. Although some very astonishing results may be obtained by paper-folding,[1] it can scarcely be assumed that these processes were *historically* very productive for geometry. The mate-

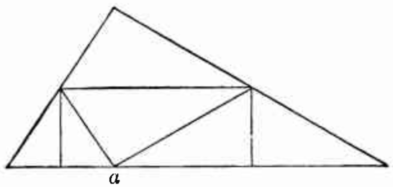

Fig. 7.

rial is of too limited application, and artisans employed with it have too little incentive to exact observation.

Experimental Knowledge of Geometry.

The knowledge that the angle-sum of the plane triangle is equal to a *determinate quantity*, namely, to two right angles, has thus been reached by experience, not otherwise than the law of the lever or Boyle and Mariotte's law of gases. It is true that neither the unaided eye nor measurements with the most delicate instruments can demonstrate *absolutely* that the sum of the angles of a plane triangle is *exactly* equal to two right angles. But the case is precisely the same with the law of the lever and with Boyle's law. All these theorems are therefore idealized and schematized experiences; for real

[1] See, for example, Sundara Row's *Geometric Exercises in Paper-Folding*. Chicago: The Open Court Publishing Co., 1901. —*Tr.*

measurements will always show slight deviations from them. But whereas the law of gases has been proved by further experimentation to be approximate only and to stand in need of modification when the facts are to be represented with great exactness, the law of the lever and the theorem regarding the angle-sum of a triangle have remained in as exact accord with the facts as the inevitable errors of experimenting would lead us to expect; and the same statement may be made of all the consequences that have been based on these two laws as preliminary assumptions.

Equal and similar triangles placed in paving alongside one another with their bases *in one and*

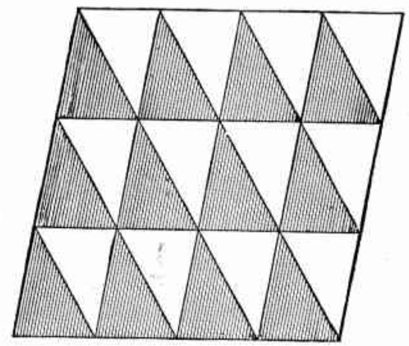

Fig. 8.

the same straight line must also have led to a very important piece of geometrical knowledge. (Fig. 8.) If a triangle be displaced in a plane along a straight line (without rotation), all its points, including those of its bounding lines, will describe equal paths. The same bounding line will furnish, therefore, in any two different positions, a system of two straight lines *equally distant* from one another at all points, and the operation coincidently vouches for the equality of the angles made by the line of displacement on corresponding sides of the two straight lines. The sum of the interior angles on the same side of the line of displacement was consequently determined to be two right angles, and thus Euclid's theorem of parallels was reached. We may add that the possibility of extending a pavement of this kind indefinitely, necessarily lent increased obviousness to this discovery. The sliding of a triangle along a ruler has remained to this day the simplest and most natural method of drawing parallel lines. It is scarcely necessary to remark that the theorem of parallels and the theorem of the angle-sum of a triangle are inseparably connected and represent merely different aspects of the same experience.

The stone masons above referred to must have readily made the discovery that a regular hexagon can be composed of equilateral triangles. Thus resulted immediately the simplest instances of the division of a circle into parts,—namely its division into six parts by the radius, its division into three parts, etc. Every carpenter knows instinctively and almost without reflection that a beam of rectangular symmetric cross-section may, owing to the perfect symmetry of the circle, be cut out from a cylindrical

tree-trunk in an infinite number of different ways. The edges of the beam will all lie in the cylindrical surface, and the diagonals of a section will pass through the center. It was in this manner, according to Hankel[1] and Tylor,[2] that the discovery was probably made that all angles inscribed in a semicircle are right angles.

Rôle of Physical Experiences.

A stretched thread furnishes the distinguishing *visualization*[3] of the *straight line*. The straight line is characterized by its physiological simplicity. All its parts induce the *same* sensation of direction; every point evokes the mean of the space-sensations of the neighboring points; every part, however small, is similar to every other part, however great. But, though it has influenced the definitions of many writers,[4] the geometer can accomplish little with this physiological characterization. The visual image must be enriched by physical experience concerning corporeal objects, to be geometrically available. Let a string be fastened by one extremity at *A*, and let its other extremity be passed through a ring fastened at *B*. If we pull on the extremity at *B*, we shall *see* parts of the string which before lay between *A* and *B* pass out at *B*, while at the same

[1] *Loc. cit.*, pp. 206-207.

[2] *Lot cit.*

[3] *Anschauung.*

[4] Euclid, *Elements*, I., Definition 3.

time the string will approach the form of a straight line. A smaller number of like parts of the string, *identical bodies,* suffices to compose the straight line joining *A* and *B* than to compose a curved line.

It is erroneous to assert that the straight line is recognized as the shortest line *by mere visualization.* It is quite true we can, so far as quality is concerned, reproduce in *imagination* with perfect accuracy and reliability, the simultaneous change of form and length which the string undergoes. But this is nothing more than a reviviscence of a *prior experience with bodies,—an experiment in thought.* The mere *passive contemplation of space* would never lead to such a result. Measurement is experience involving a physical reaction, an experiment of superposition. Visualized or imagined lines having different directions and lengths cannot be applied to one another forthwith. The possibility of such a procedure must be actually experienced with material objects accounted as unalterable. It is erroneous to attribute to animals an instinctive knowledge of the straight line as the shortest distance between two points. If a stimulus excites an animal's attention, and if the animal has so turned that its plane of symmetry passes through the stimulating object, then the straight line is the path of motion *uniquely* determined by the stimulus. This is distinctly shown in Loeb's investigations on the tropisms of animals.

Further, visualization alone does not prove that any two sides of a triangle are together greater

than the third side. It is true that if the two sides be laid upon the base by rotation round the vertices of the basal angles, it will be seen by an act of *imagination* alone that the two sides with their free ends moving in arcs of circles will ultimately overlap, thus more than filling up the base. But we should not have attained to this representation had not the procedure been actually witnessed in connection with corporeal objects. Euclid[1] deduces this truth circuitously and artificially from the fact that the greater side of every triangle is opposite to the greater angle. But the source of our knowledge here also is experience,—experience of the motion of the side of a physical triangle; this source has, however, been laboriously concealed by the form of the deduction,—and this not to the enhancement of perspicuity or brevity.

But the properties of the straight line are not exhausted with the preceding empirical truths. If a wire of any arbitrary shape be laid on a board in contact with two upright nails, and slid along so as to be always in contact with the nails, the form and position of the parts of the wire between the nails will be constantly changing. The straighter the wire is, the slighter the alteration will be. A straight wire submitted to the same operation slides *in itself.* Rotated round two of its own fixed points, a crooked wire will keep constantly changing its position, but a straight wire will maintain its position, it will ro-

[1] Euclid, *Elements,* Book I., Prop. 20.

tate within itself.[1] When we define, now, a straight line as the line which is completely determined by two of its points, there is nothing in this *concept* except the *idealization* of the empirical notion derived from the physical experience mentioned,—a notion by no means directly furnished by the physiological act of visualization.

The plane, like the straight line, is physiologically characterized by its simplicity. It appears the same at all parts.[2] Every point evokes the mean of the space-sensations of the neighboring points. Every part, however small, is like every other part, however great. But experiences gained in connection with physical objects are also required, if these properties are to be put to geometrical account. The plane, like the straight line, is physiologically symmetrical with respect to itself, if it coincides with the median plane of the body or stands at right angles to the same. But to discover that symmetry is a *permanent* geometrical property of the plane and the straight line, both concepts must be given as movable, unalterable physical objects. The connection of physiological symmetry with metrical

[1] In a letter to Vitale Giordano (*Leibnizens mathematische Schriften, herausgegeben v. Gerhardt, erste Abtheilung, Bd. I., S.* 195-196), Leibnitz makes use of the above-mentioned property of a straight line for its definition. The straight line shares the property of displaceability in itself with the circle and the circular cylindrical spiral. But the property of rotatability within itself and that of being determined by *two* points, are exclusively its own.

[2] Compare Euclid, *Elements* I., Definition 7.

properties also is in need of special metrical demonstration.

Physically a plane is constructed by rubbing three bodies together until three surfaces, *A, B, C,* are obtained, each of which exactly fits the others,—a result which can be accomplished, as Fig. 9 shows, with neither convex nor concave surfaces, but with plane surfaces only. The convexities and concavities are, in fact, removed by the rubbing. Similarly, a truer straight line can be obtained with the aid of an imperfect ruler, by first placing it with its ends against the points *A, B,* then turning it through an angle of 180° out of its plane and again placing it against *A, B,* afterwards taking the *mean* between the two lines so obtained as a more perfect straight line, and repeating the operation with the line last

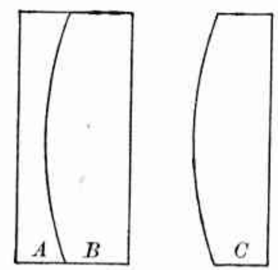

Fig. 9.

obtained. Having by rubbing, produced a plane, that is to say, a surface having the same form at *all points* and on *both* sides, experience furnishes additional results. Placing two such planes one on the other, it will be learned that the plane is *displaceable* into itself, and *rotatable* within itself, just as a straight line is. A thread stretched between any two points in the plane falls entirely within the plane. A piece of cloth drawn tight across any bounded portion of a plane coincides with it. Hence the plane represents the minimum of surface within its boundaries. If the plane be laid on two sharp points, it can still be rotated around the straight line joining the points, but any third point outside of this straight line fixes the plane, that is, determines it completely.

In the letter to Vitale Giordano, above referred to, Leibnitz makes the frankest use of this experience with corporeal objects, when he defines a plane as a surface which divides an unbounded solid into two congruent parts, and a straight line as a line which divides an unbounded plane into two congruent parts.[1]

If attention be directed to the symmetry of the plane with respect to itself, and two points be assumed, one on each side of it, each symmetrical to the other, it will be found that every point in the plane is equidistant from these two points, and Leibnitz's definition of the plane is reached.[1] The uniformity and symmetry of the straight line and the plane are consequences of their being *absolute* minima of length and area respectively. For the boundaries given the minimum must exist, no other collateral condition being involved. The minimum is unique, *single in its kind;* hence the *symmetry* with respect to the bounding points. Owing to the *absoluteness* of the minimum, every portion, however small, again exhibits the same minimal property; hence the uniformity.

[1] The passage reads literally: "Et difficulter absolvi poterit demonstratio, nisi quis assumat notionem rectæ, qualis est qua ego uti soleo, quod corpore aliquo duobus punctis immotis revoluto locus omnium punctorum quiescentium sit recta, vel saltem quod recta sit linea secans planum interminatum in duas partes congruas; et planum sit superficies secans solidum interminatum in duas partes congruas." For similar definitions, see, for example, Halsted's *Elements of Geometry*, 6th edition. New York, 1895, p. 9.—*T. J. McC.*

Empirical Origin of Geometry.

Empirical truths organically connected may make their appearance independently of one another, and doubtless were so discovered long before the fact of their connection was known. But this does not preclude their being afterwards recognized as involved in, and *determined* by, one another, as being *deducible* from one another. For example, supposing we are acquainted with the symmetry and uniformity of the straight line and the plane, we easily deduce that the intersection of two planes is a *straight line,* that any two points of the plane can be joined by a straight line lying wholly within the plane, etc. The fact that only a *minimum* of inconspicuous and unobtrusive experiences is requisite for such deductions should not lure us into the error of regarding this minimum as wholly super-

[1] Leibnitz, *in re* "geometrical characteristic," letter to Huygens, Sept. 8, 1679 (Gerhardt, *loc. cit., erste Abth., Bd. II., S.* 23).

fluous, and of believing that visualization and reasoning are alone sufficient for the construction of geometry.

Like the concrete visual images of the straight line and the plane, so also our visualizations of the circle, the sphere, the cylinder, etc., are enriched by metrical experiences, and in this manner first rendered amenable to fruitful geometrical treatment. The same economic impulse that prompts our children to retain only the *typical* features in their concepts and drawings, leads us also to the *schematization* and conceptual *idealization* of the images derived from our experience. Although we never come across in nature a perfect straight line or an exact circle, in our thinking we nevertheless designedly abstract from the deviations which thus occur. Geometry, therefore, is concerned with *ideal objects* produced by the schematization of *experiential objects.* I have remarked elsewhere that it is wrong in elementary geometrical instruction to cultivate predominantly the logical side of the subject, and to neglect to throw open to young students the wells of knowledge contained in experience. It is gratifying to note that the Americans who are less dominated than we by tradition, have recently broken with this system and are introducing a sort of experimental geometry as introductory to systematic geometric instruction.*

*See the essays and books of Hanus, Campbell, Speer, Myers, Hall and many others noticed in the reviews of *School Science and Mathematics* (Chicago) during the last few years.—*T. J. McC.*

Technical and Scientific Development of Geometry.

No sharp line can be drawn between the instinctive, the technical, and the scientific acquisition of geometric notions. Generally speaking, we may say, perhaps, that with division of labor in the industrial and economic fields, with increasing employment with very definite objects, the instinctive acquisition of knowledge falls into the background, and the technical begins. Finally, *when measurement becomes an aim and business in itself,* the *connection* obtaining between the various operations of measuring acquires a powerful *economic* interest, and we reach the period of the scientific development of geometry, to which we now proceed.

The insight that the measures of geometry depend on one another, was reached in divers ways. After surfaces came to be measured by surfaces, further progress was almost inevitable. In a parallelogrammatic field permitting a division into equal partial parallelogrammatic fields so that n rows of partial fields each containing m fields lay alongside one another, the counting of these fields was unnecessary. By multiplying together the numbers measuring the sides, the area of the field was found to be equal to mn such fields, and the area of each of the two triangles formed by drawing the diagonal was readily discovered to be equal to $\frac{mn}{2}$ such fields. This was the first and simplest application of arithmetic to geometry. Coincidently, the dependence of measures of area on other measures, linear and angular, was discovered. The area of a rectangle was found to be larger than that of an oblique parallelogram having sides of the same length; the area, consequently, depended not only on the length of the sides, but also on the angles. On the other hand, a rectangle constructed of strips of wood running parallel to the base, can, as is easily seen, be converted by displacement into any parallelogram of the same height and base without altering its area. Quadrilaterals having their sides given are still undetermined in their angles, as every carpenter knows. He adds diagonals, and converts his quadrilateral into triangles, which, the sides being given, are rigid, that is to say, are unalterable as to their angles also.

With the perception that measures were dependent on one another, the real problem of geometry was introduced. Steiner has aptly and justly entitled his principal work "Systematic Development of the Dependence of Geometrical Figures on One Another."[1] In Snell's original but unappreciated treatise on Elementary Geometry, the problem in question is made obvious even to the beginner.[2]

A plane physical triangle is constructed of wires. If one of the sides be rotated around a vertex, so as to increase the interior angle at that point, the side

[1] J. Steiner, ***Systematische Entwicklung der Abhängigkeit der geometrischen Gestalten von einander.***

[2] **Snell, *Lehrbuch der Geometrie*, Leipsic, 1869.**

moved will be seen to change its position and the side opposite to grow *larger* with the angle. *New* pieces of wire besides those before present will be required to complete the last-mentioned side. This and other similar experiments can be repeated in thought, but the mental experiment is never anything more than a copy of the physical experiment. The mental experiment would be impossible if physical experience had not antecedently led us to a knowledge of *spatially unalterable physical bodies,*[1] —to the concept of measure.

The Geometry of the Triangle.

By experiences of this character, we are conducted to the truth that of the six metrical magnitudes discoverable in a triangle (three sides and three angles) three, including at least *one* side, suffice to determine the triangle. If *one* angle only be given among the parts determining the triangle, the angle in question must be either the angle included by the given sides, or that which is opposite to the greater side,—at least if the determination is to be *unique*. Having reached the perception that a triangle is determined by three sides and that its form is independent of its position, it follows that in an equilateral triangle all three angles and in an isosceles triangle the two angles opposite the equal sides, must be equal, in

[1] **The whole construction of the Euclidean geometry shows traces of this foundation. It is still more conspicuous in the "geometric characteristic" of Leibnitz already mentioned. We shall revert to this topic later.**

whatever manner the angles and sides may depend on one another. This is logically certain. But the empirical foundation on which it rests is for that reason not a whit more superfluous than it is in the analogous cases of physics.

The mode in which the sides and angles depend on one another is, naturally, first recognized in special instances. In computing the areas of rectangles and of the triangles formed by their diagonals, the fact must have been noticed that a rectangle having sides 3 and 4 units in length gives a right-angled triangle having sides, 3, 4, and 5 units in length. Rectangularity was thus shown to be connected with a definite, rational ratio between the sides. The knowledge of this truth was employed to stake off right angles, by means of three connected ropes respectively 3, 4, and 5 units in length.[1] The equation $3^2 + 4^2 = 5^2$, the analogue of which was proved to be valid for all right-angled triangles having sides of lengths a, b, c (the general formula being $a^2 + b^2 = c^2$), now riveted the attention. It is well known how profoundly this relation enters into metrical geometry, and how all indirect measurements of distance may be traced back to it. We shall endeavor to disclose the foundation of this relation.

It is to be remarked first that neither the Greek geometrical nor the Hindu arithmetical deductions of the so-called *Pythagorean Theorem* could avoid the consideration of areas. One essential point on

[1] Cantor, *Geschichte der Mathematik*, Leipsic, 1880. I., pp. 53, 56.

which all the deductions rest and which appears more or less distinctly in different forms in all of them, is the following. If a triangle, a, b, c (Fig. 10) be slid along a short distance in its own plane, it is as-

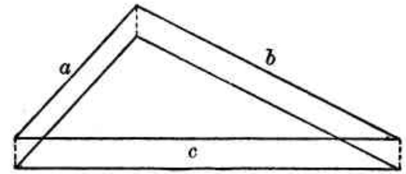

Fig. 10.

sumed that the space which it leaves behind is compensated for by the new space on which it enters. That is to say, the area swept out by *two* of the sides during the displacement is equal to the area swept out by the *third* side. The basis of this conception is the assumption of the *conservation of the area* of the triangle. If we consider a surface as a body of very minute but unvarying thickness of third dimension (which for that reason is uninfluential in the present connection), we shall again have the *conservation of the volume of bodies* as our fundamental assumption. The same conception may be applied to the translation of a tetrahedron, but it does not lead in this instance to new points of view. Conservation of volume is a property which rigid and liquid bodies possess in common, and was idealized by the old physics as *impenetrability*. In the case of rigid bodies, we have the additional attribute that the distances between all the parts are preserved, while in the case of liquids, the properties of rigid bodies exist only for the smallest time and space elements.

If an oblique-angled triangle having the sides a, b, and c be displaced in the direction of the side b, only a and c will, by the principle above stated, describe equivalent parallelograms, which are alike in an equal pair of parallel sides on the same parallels. If a make with b a right angle, and the triangle be displaced at right angles to c, the distance c, the side c will describe the square c^2, while the two other sides will describe parallelograms the combined areas of which are equal to the area of the

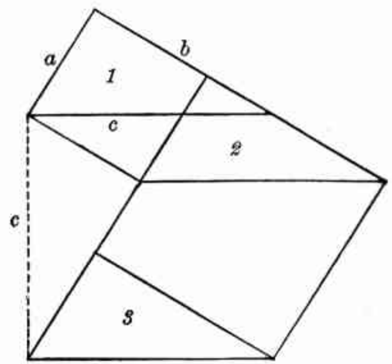

Fig. 11.

square. But the two parallelograms are, by the observation which just precedes, equivalent respectively to a^2 and b^2,—and with this the Pythagorean theorem is reached.

The same result may also be attained (Fig. 11) by first sliding the triangle a distance a at right angles to a, and then a distance b at right angles to

b, where $a^2 + b^2$ will be equal to the sum of the surfaces swept out by c, which is obviously c^2. Taking an oblique-angled triangle, the same procedure just as easily and obviously gives the more general proposition, $c^2 = a^2 + b^2 - 2ab\cos\gamma$.

The dependence of the third side of the triangle on the two other sides is accordingly determined by the area of the enclosed triangle; or, in our conception, by a condition involving *volume*. It will also be directly seen that the equations in question express relations of area. It is true that the angle included between two of the sides may also be regarded as determinative of the third side, in which case the equations will aparently assume an entirely different form.

Let us look a little more closely at these different measures. If the extremities of two straight lines of lengths a and b meet in a point, the length of the line c joining their free extremities will be included between definite limits. We shall have $c \leqq a + b$, and $c \geqq a - b$. Visualization alone cannot inform us of this fact; we can learn it only from *experimenting in thought*,—a procedure which both reposes on physical experience and reproduces it. This will be seen by holding a fast, for example, and turning b, first, until it forms the prolongation of a, and, secondly, until it coincides with a. A straight line is primarily a unique concrete image characterized by physiological properties,—an image which we have obtained from a *physical* body of a definite specific character, which in the form of a string or wire of indefinitely small but constant thickness interposes a *minimum of volume* between the positions of its extremities,—which can be accomplished only in *one uniquely-determined* manner. If several straight lines pass through a point, we distinguish between them *physiologically* by their directions. But in *abstract space* obtained by metrical experiences with physical objects, differences of direction do not exist. A straight line passing through a point can be completely determined in abstract space only by assigning a second *physical* point on it. To define a straight line as a line which is constant in direction, or an angle as a *difference between directions,* or parallel straight lines as straight lines having the *same* direction, is to define these concepts *physiologically.*

The Measurement of the Angle.

Different methods are at our disposal when we come to characterize or determine *geometrically* angles which are *visually* given. An angle is determined when the distance is assigned between any two fixed points lying each on a separate side of the angle outside the point of intersection. To render the definition uniform, points situated at the same fixed and invariable distance from the vertex might be chosen. The inconvenience that then equimultiples of a given angle placed alongside one another in the same plane with their vertices coinci-

dent, would not be measured by the same equimultiples of the distance between those points, is the reason that this method of determining angles was not introduced into elementary geometry.[1] A simpler measure, a simpler characterization of an angle, is obtained by taking the aliquot part of the *circumference* or the *area* of a circle which the angle intercepts when laid in the plane of the circle with its vertex at the center. The convention here involved is more convenient.[2]

In employing an arc of a circle to determine an angle, we are again merely measuring a volume,—viz., the volume occupied by a body of simple definite form introduced between two points on the arms of the angle equidistant from the vertex. But a circle can be characterized by simple rectilinear distances. It is a matter of perspicuity, of immediacy, and of the facility and convenience resulting therefrom, that two measures, viz., the rectilinear measure of length and the angular measure, are principally employed as fundamental measures, and that the others are derived from them. It is in no sense necessary. For example (Fig. 12), it is possible without a special angular measure to determine the straight line that cuts another straight line at right angles by making all its points equidistant from two points in the first straight line lying at equal distances from the point of intersection. The

[1] A closely allied principle of measurement is, however, applied in trigonometry.

[2] So also the superficial portion of a sphere intercepted by the including planes is used as the measure of a solid angle.

bisector of an angle can be determined in a quite similar manner, and by continued bisection an angular unit can be derived of any smallness we wish. A straight line *parallel* to another straight

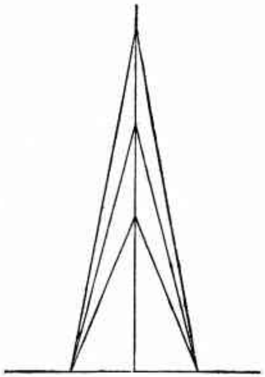

Fig. 12.

line can be defined as one, all of whose points can be translated by congruent curved or *straight* paths into points of the first straight line.[1]

LENGTH AS THE FUNDAMENTAL MEASURE.

It is quite possible to start with the straight length *alone* as our fundamental measure. Let a fixed physical point a be given. Another point, m, has the distance r_a from the first point. Then this last point can still lie in any part of the spherical surface described about a with radius r_a. If we know still a second fixed point b, from which m is removed by the distance r_b, the triangle abm will

[1] If this form had been adopted, the doubts as to the Euclidean theorem of parallels would probably have risen much later.

be rigid, determined; but m can still revolve round in the circle described by the rotation of the triangle around the axis ab. If now the point m be held fast in any position, then also the whole rigid body to which the three points in question, a, b, m, belong will be fixed.

A point m is spatially determined, accordingly, by the distances r_a, r_b, r_c from at least three fixed points in space, a, b, c. But this determination is still not unique, for the pyramid with the edges r_a, r_b, r_c, in the vertex of which m lies, can be constructed as well on the one as on the other side of the plane a, b, c. If we were to fix the side, say by a special sign, we should be resorting to a *physiological* determination, for *geometrically* the two sides of the plane are not different. If the point m is to be uniquely determined, its distance, r_d, from a fourth point, d, lying *outside* the plane abc, must in addition be given. Another point, m', is determined with like completeness by four distances, r'_a, r'_b, r'_c, r'_d. Hence, the distance of m from m' is also given by this determination. And the same holds true of any number of other points as severally determined by four distances. Between four points $\frac{4(4-1)}{1 \cdot 2} = 6$ distances are conceivable, and precisely this number must be given to determine the form of the point complex. For $4 + z = n$ points, $6 + 4z$ or $4n - 10$ distances are needed for the determination, while a still larger number, viz.,

$\frac{n(n-1)}{1 \cdot 2}$ distances exist, so that the excess of the distances is also coincidently determined.*

If we start from *three* points and prescribe that the distances of all points to be further determined shall hold for one side only of the plane determined by the three points, then $3n - 6$ distances will suffice to determine the form, magnitude, and position of a system of n points with respect to the three initial points. But if there be no condition as to the side of the plane to be taken,—a condition which involves sensuous and physiological, but not abstract metrical characteristics,—the system of points, instead of the intended form and position, may assume that symmetrical to the first, or be combined of the points of both. *Symmetric* geometrical figures are, owing to our symmetric *physiological organization*, very easily taken to be identical, whereas *metrically* and *physically* they are entirely different. A screw with its spiral winding to the right and one with its spiral winding to the left, two bodies rotating in contrary directions, etc., appear very much alike to the eye. But we are for this reason not permitted to regard them as geometrically or physically equivalent. Attention to this fact would avert many paradoxical questions. Think only of the trouble that such problems gave Kant!

*For an interesting attempt to found both the Euclidean and non-Euclidean geometrics on the pure notion of distance, see De Tilly, "Essai sur les principes fondamentaux de la géométrie et de la mécanique" (*Mémoires de la Société de Bordeaux*, 1880).

Sensuous physiological attributes are determined by relationship to *our body*, to a corporeal system of *specific* constitution; while metrical attributes are determined by relations to the world of physical bodies *at large*. The latter can be ascertained only by experiments of coincidence,—by measurements.

Volume the Basis of Measurement.

As we see, every geometrical measurement is at bottom reducible to measurements of *volumes*, to the *enumeration of bodies*. Measurements of lengths, like measurements of areas, repose on the comparison of the volumes of very thin strings, sticks, and leaves of constant thickness. This is not at variance with the fact that measures of area may be *arithmetically* derived from measures of length, or solid measures from measures of length alone, or from these in combination with measures of area. This is merely proof that *different* measures of volume are dependent on one another. To ascertain the forms of this interdependence is the *fundamental object of geometry*, as it is the province of arithmetic to ascertain the manner in which the various numerical operations, or ordinative activities of the mind, are connected together.

The Visual Sense in Geometry.

It is extremely probable that the experiences of the visual sense were the cause of the rapidity with which geometry developed. But our great familiarity with the properties of rays of light gained from the present advanced state of optical technique, should not mislead us into regarding our *experimental knowledge of rays of light* as the principal foundation of geometry. Rays of light in dust or smoke-laden air furnish admirable *visualizations* of straight lines. But we can derive the *metrical properties* of straight lines from rays of light just as little as we can derive them from *imaged* straight lines. For this purpose experiences with *physical* objects are absolutely necessary. The *rope-stretching* of the practical geometers is certainly older than the use of the theodolite. But once knowing the physical straight line, the ray of light furnishes a very distinct and handy means of reaching new points of view. A blind man could scarcely have invented modern synthetic geometry. But the oldest and the most powerful of the experiences lying at the basis of geometry are just as accessible to the blind man, through his sense of touch, as they are to the person who can see. Both are acquainted with the spatial *permanency of bodies* despite their *mobility;* both acquire a conception of *volume* by *taking hold* of objects. The creator of primitive geometry disregards, first instinctively and then intentionally and consciously, those physical properties that are unessential to his operations and that for the moment do not concern him. In this manner, and by gradual growth, the idealized concepts of geometry arise on the basis of experience.

Various Sources of Our Geometric Knowledge.

Our geometrical knowledge is thus derived from various sources. We are *physiologically* acquainted, from direct visual and tactual contact, with many and various spatial forms. With these are associated physical (*metrical*) experiences (involving comparison of the space-sensations evoked by different bodies under the same circumstances), which experiences are in their turn also but the expressions of other relations obtaining between sensations. These diverse orders of experience are so intimately interwoven with one another that they can be separated only by the most thoroughgoing scrutiny and analysis. Hence originate the widely divergent views concerning geometry. Here it is based on pure visualization (*Anschauung*), there on physical experience, according as the one or the other factor is overrated or disregarded. But both factors entered into the development of geometry and are still active in it to-day; for, as we have seen, geometry by no means exclusively employs purely metrical concepts.

If we were to ask an unbiased, candid person under what form he pictured space, referred, for example, to the Cartesian system of co-ordinates, he would doubtless say: I have the image of a system of rigid (form-fixed), transparent, penetrable, contiguous cubes, having their bounding surfaces marked only by nebulous visual and tactual percepts,—a species of phantom cubes. Over and through these phantom constructions the real bodies or their phantom counterparts move, conserving their spatial permanency (as above defined), whether we are concerned with practical or theoretical geometry, or phoronomy. Gauss's famous investigation of curved surfaces, for instance, is really concerned with the application of infinitely thin laminate and hence flexible bodies to one another. That diverse orders of experience have co-operated in the formation of the fundamental conceptions under consideration, cannot be gainsaid.

The Fundamental Facts and Concepts.

Yet, varied as the special experiences are from which geometry has sprung, they may be reduced to a minimum of facts: Movable bodies exist having definite spatial permanency,—viz., rigid bodies exist. But the movability is characterized as follows: we draw from a point three lines not all in the same plane but otherwise undetermined. By three movements along these straight lines any point can be reached from any other. Hence, three measurements or dimensions, physiologically and metrically characterized as the simplest, are sufficient for all spatial determinations. These are the fundamental facts.[1]

The physical metrical experiences, like all experiences forming the basis of experimental sciences, are conceptualized,—idealized. The *need* of representing the facts by simple perspicuous concepts under easy logical control, is the reason for this. Absolutely rigid, spatially invariable bodies, perfect straight lines and planes, no more exist than a perfect gas or a perfect liquid. Nevertheless, deferring the consideration of the deviations, we prefer to work, and we also work more readily, with these concepts than with others that conform more closely to the actual properties of the objects. *Theoretical* geometry does not even need to consider these deviations, inasmuch as it assumes objects that fulfil the requirements of the theory absolutely, just as theoretical physics does. But in *practical* geometry, where we are concerned with actual objects, we are obliged, as in practical physics, to consider the deviations from the theoretical assumptions. But geometry has still the advantage that every deviation of its objects from the assumptions of the theory *which may be detected* can be *removed;* whereas physics for obvious reasons cannot construct more perfect gases than actually exist in nature. For, in the latter case, we are concerned not with a *single* arbitrarily constructible spatial property alone, but with a relation (occurring in nature and independent of our will) between pressure, volume, and temperature.

The choice of the concepts is suggested by the facts; yet, seeing that this choice is the outcome of our *voluntary* reproduction of the facts in thought,

[1] The historical development of this conception will be considered in another place.

some free scope is left in the matter. The importance of the concepts is estimated by their range of application. This is why the concepts of the straight line and the plane are placed in the foreground, for every geometrical object can be split up with sufficient approximateness into elements bounded by planes and straight lines. The particular properties of the straight line, plane, etc., which we decide to emphasize, are matters of our own free choice, and this truth has found expression in the various definitions that have been given of the same concept.[1]

Experimenting in Thought.

The fundamental truths of geometry have thus, unquestionably, been derived from physical experience, if only for the reason that our visualizations and sensations of space are absolutely inaccessible to measurement and cannot possibly be made the subject of metrical experience. But it is no less indubitable that when the relations connecting our visualizations of space with the simplest metrical experiences have been made familiar, then geometrical facts can be reproduced with great facility and certainty in the imagination alone,—that is by purely *mental experiment*. The very fact that a continuous change in our space-sensation corresponds to a continuous metrical change in physical bodies, enables

[1] Compare, for example, the definitions of the straight line given by Euclid and by Archimedes.

us to ascertain by imagination alone the particular metrical elements that depend on one another. Now, if such metrical elements are observed to enter different constructions having different positions in precisely the same manner, then the metrical results will be regarded as *equal*. The case of the isosceles and equilateral triangles, above mentioned, may serve as an example. The *geometric* mental experiment has advantage over the physical, only in the respect that it can be performed with far simpler experiences and with such as have been more easily and almost unconsciously acquired.

Our sensuous imagings and visualizations of space are *qualitative*, not quantitative nor metrical. We derive from them coincidences and differences of extension, but never real magnitudes. Conceive, for example, Fig. 13, a coin rolling clockwise down and around the rim of another fixed coin of the same size, without sliding. Be our imagination as vivid as it will, it is impossible by a pure feat of reproductive imagery alone, to determine here the angle described in a full revolution. But if it be considered that at the beginning of the motion the radii *a*, *a'* lie in one straight line, but that after a *quarter* revolution the radii *b*, *b'* lie in a straight line, it will be seen at once that the radius *a'* now points vertically upwards and has consequently performed *half* a revolution. The *measure* of the revolution is obtained from metrical concepts, which fix idealized experiences on definite physical objects, but the *direction* of the revolution is retained in the *sensuous* imagination. The metrical concepts simply determine that in equal circles equal angles are subtended by equal arcs, that the radii to the point of contact lie in a straight line, etc.

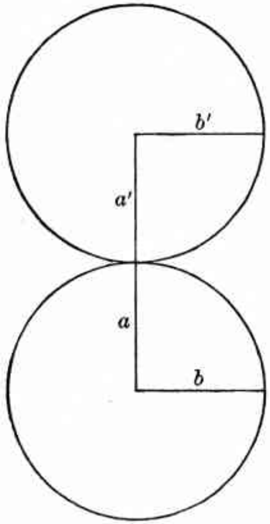

Fig. 13.

If I picture to myself a triangle with one of its angles increasing, I shall also see the side opposite the angle increasing. The impression thus arises that the interdependence in question follows *a priori* from a feat of imagination alone. But the imagination has here merely reproduced a fact of experience. Measure of angle and measure of side are *two physical* concepts applicable to *the same* fact,—concepts that have grown so familiar to us that they have come to be regarded as merely *two* different attributes of the *same* imaged group of facts, and hence appear as linked together of sheer necessity. Yet we should never have acquired these concepts without physical experience.

The combined action of the sensuous imagination with idealized concepts derived from experience is apparent in every geometrical deduction. Let us consider, for example, the simple theorem that the perpendicular bisectors of the sides of a triangle ABC meet in a common point. Experiment and imagination both doubtless led to the theorem. But the more carefully the construction is executed, the more one becomes convinced that the third perpendicular does not pass exactly through the point of intersection of the first two, and that in any actual construction, therefore, three points of intersection will be found closely adjacent to one another. For in reality neither perfect straight lines nor perfect perpendiculars can be drawn; nor can the latter be erected exactly at the midpoints; and so on. *Only* on the assumption of these *ideal* conditions does the perpendicular bisector of AB contain all points equally distant from A and B, and the perpendicular bisector of BC all points equidistant from B and C. From which it follows that the point of intersection of the two is equidistant from A, B, and C, and by reason of its equidistance from A and C is also a point of the *third* perpendicular bisector, of AC. The theorem asserts therefore that the more accurately the assumptions are fulfilled the more nearly will the three points of intersection coincide.

Kant's Theory.

The importance of the combined action of the sensuous imagination [viz., of the *Anschauung* or intuition so called] and of concepts, will doubtless have been rendered clear by these examples. Kant says: "Thoughts without contents are empty, intuitions without concepts are blind."[1] Possibly we might more appropriately say: "Concepts without intuitions are blind, intuitions without concepts are lame." For it would appear to be not so absolutely correct to call intuitions [viz., sensuous images] blind and concepts empty. When Kant further says that "there is in every branch of natural knowledge only so much science as there is mathematics contained in it,"[2] one might possibly also assert of all sciences, *including* mathematics, "that they are only in so far sciences as they operate with concepts." For our logical mastery extends only to those concepts of which we have ourselves determined the contents.

The Present Form of Geometry.

The two facts that bodies are rigid and movable would be sufficient for an understanding of any geometrical fact, no matter how complicated,—sufficient, that is to say, to derive it from the two facts

[1] *Kritik der reinen Vernunft*, 1787, p. 75. Max Müller's translation, 2nd ed., 1896, p. 41.

[2] *Metaphysische Anfangsgründe der Naturwissenschaft. Vorwort.*

mentioned. But geometry is obliged, both in its own interests and in its rôle as an auxiliary science, as well as in the pursuit of practical ends, to answer questions that *recur repeatedly in the same form.* Now it would be uneconomical, in such a contingency, to begin each time with the most elementary facts and to go to the bottom of each new case that presented itself. It is preferable, rather, to select a few simple, familiar, and indubitable theorems, in our choice of which caprice is by no means excluded,[1] and to formulate from these, *once for all*, for application to practical ends, general propositions answering the questions that most frequently recur. From this point of view we understand at once the *form* geometry has assumed,—the emphasis, for example, that it lays upon its propositions concerning triangles. For the purpose designated it is desirable to collect the most general possible propositions having the widest range of application. From history we know that propositions of this character have been obtained by embracing various special cases of knowledge under single general cases. We are forced even today to resort to this procedure when we treat the relationship of two geometrical figures, or when the different special cases of form and position compel us to modify our modes of deduction. We may cite as the most familiar instance of this in elementary geometry, the

[1] Zindler. *Zur Theorie der mathematischen Erkenntniss. Sitzungsberichte der Wiener Akademie. Philos-histor. Abth.* Bd. 118. 1889.

mode of deducing the relation obtaining between angles at the centre and angles at the circumference.

UNIVERSAL VALIDITY.

Kroman[1] has put the question, Why do we regard a demonstration made with a special figure (a special triangle) as universally valid for all figures? and finds his answer in the supposition that we are able by rapid variations to impart all possible forms to the figure in thought and so convince ourselves of the admissibility of the same mode of inference in all special cases. History and introspection declare this idea to be in all essentials correct. But we may not assume, as Kroman does, that in each special case every individual student of geometry acquires this perfect comprehension "with the rapidity of lightning," and reaches immediately the lucidity and intensity of geometric conviction in question. Frequently the required operation is absolutely impracticable, and errors prove that in other cases it was actually not performed but that the inquirer rested content with a conjecture based on analogy.[2]

But that which the individual does not or cannot achieve in a jiffy, he may achieve in the course of his life. Whole generations labor on the verification of geometry. And the conviction of its certitude is unquestionably strengthened by their collective exertions. I once knew an otherwise excellent teacher who compelled his students to perform all their demonstrations with *incorrect* figures, on the theory that it was the *logical* connection of the concepts, not the figure, that was essential. But the experiences imbedded in the concepts cleave to our sensuous images. Only the actually visualized or imaged figure can tell us what particular concepts are to be employed in a given case. The method of this teacher is admirably adapted for rendering palpable the degree to which logical operations participate in reaching a given perception. But to employ it habitually is to miss utterly the truth that abstract concepts draw their ulitmate power from sensuous sources.

[1] *Unsere Naturerkenntniss.* Copenhagen, 1883, pp. 74 et seq.

[2] Hoelder, *Anschauung und Denken in der Geometrie,* p. 12.