CHAPTER VII

THE SYSTEM OF MODAL LOGIC

§ 46. *The matrix method*

FOR a full understanding of the system of modal logic expounded in this chapter it is necessary to be acquainted with the matrix method. This method can be applied to all logical systems in which truth-functions occur, i.e. functions whose truth-values depend only on the truth-values of their arguments. The classical calculus of propositions is a two-valued system, i.e. it assumes two truth-values, 'truth' denoted here by *1*, and 'falsity' denoted by *0*. According to Philo of Megara an implication is true, unless it begins with truth and ends with falsity. That means in symbols that $C11 = C01 = C00 = 1$, and only $C10 = 0$. Obviously the negation of a true proposition is false, i.e. $N1 = 0$, and the negation of a false proposition true, i.e. $N0 = 1$. It is usual to present these symbolic equalities by means of 'truth-tables' or 'matrices', as they are called. The two-valued matrix M1 of *C* and *N* may be described as follows: the truth-values of *C* are arranged in rows and columns forming a square, and are separated by a line from the left margin and the top. The truth-values of the first argument are put on the left, those of the second on the top, and the truth-values of *C* can be found in the square, where the lines which we may imagine drawn from the truth-values on the margins of the square intersect one another. The matrix of *N* is easily comprehensible.

	C	*q*: *1*	*q*: *0*	*N*
p	*1*	*1*	*0*	*0*
p	*0*	*1*	*1*	*1*

M1

By means of this matrix any expression of the classical calculus of propositions, i.e. of the *C–N–p*-calculus, can be mechanically verified, i.e. proved when asserted and disproved when rejected. It suffices for this purpose to put the values *1* and *0* in all possible combinations for the variables, and if every combination reduced

according to equalities stated in the matrix gives *1* as final result, the expression is proved, but if not, it is disproved. For example, $CCpqCNpNq$ is disproved by M1, since when $p = 0$ and $q = 1$, we have: $CC01CN0N1 = C1C10 = C10 = 0$. By contrast, $CpCNpq$, one of our axioms of our *C–N–p*-system,[1] is proved by M1, because we have:

$$\begin{aligned} \text{For } p &= 1, q = 1: C1CN11 = C1C01 = C11 = 1,\\ \text{,, } p &= 1, q = 0: C1CN10 = C1C00 = C11 = 1,\\ \text{,, } p &= 0, q = 1: C0CN01 = C0C11 = C01 = 1,\\ \text{,, } p &= 0, q = 0: C0CN00 = C0C10 = C00 = 1. \end{aligned}$$

In the same way we can verify the other two axioms of the *C–N–p*-system, $CCpqCCqrCpr$ and $CCNppp$. As M1 is so constructed that the property of always yielding *1* is hereditary with respect to the rules of substitution and detachment for asserted expressions, all asserted formulae of the *C–N–p*-system can be proved by the matrix M1. And as similarly the property of not always yielding *1* is hereditary with respect to the rules of inference for rejected expressions, all rejected formulae of the *C–N–p*-system can be disproved by M1, if *p* is axiomatically rejected. A matrix which verifies all formulae of a system, i.e. proves the asserted and disproves the rejected ones, is called 'adequate' for the system. M1 is an adequate matrix of the classical calculus of propositions.

M1 is not the only adequate matrix of the *C–N–p*-system. We get another adequate matrix, M3, by 'multiplying' M1 by itself. The process of getting M3 can be described as follows:

First, we form ordered pairs of the values *1* and *0*, viz.: $(1, 1)$, $(1, 0)$, $(0, 1)$, $(0, 0)$; these are the elements of the new matrix. Secondly, we determine the truth-values of *C* and *N* by the equalities:

$$(y)\quad C(a, b)(c, d) = (Cac, Cbd),$$
$$(z)\quad N(a, b) = (Na, Nb).$$

Then we build up the matrix M2 according to these equalities; and finally we transform M2 into M3 by the abbreviations: $(1, 1) = 1$, $(1, 0) = 2$, $(0, 1) = 3$, and $(0, 0) = 0$.

[1] See p. 80.

C	(1, 1)	(1, 0)	(0, 1)	(0, 0)	*N*
(1, 1)	(1, 1)	(1, 0)	(0, 1)	(0, 0)	(0, 0)
(1, 0)	(1, 1)	(1, 1)	(0, 1)	(0, 1)	(0, 1)
(0, 1)	(1, 1)	(1, 0)	(1, 1)	(1, 0)	(1, 0)
(0, 0)	(1, 1)	(1, 1)	(1, 1)	(1, 1)	(1, 1)

M2

C	1	2	3	0	*N*
1	1	2	3	0	0
2	1	1	3	3	3
3	1	2	1	2	2
0	1	1	1	1	1

M3

Symbol *1* in M3 again denotes truth, and *0* falsity. The new symbols *2* and *3* may be interpreted as further signs of truth and falsity. This may be seen by identifying one of them, it does not

C	1	1	0	0	*N*
1	1	1	0	0	0
1	1	1	0	0	0
0	1	1	1	1	1
0	1	1	1	1	1

M4

C	1	0	1	0	*N*
1	1	0	1	0	0
0	1	1	1	1	1
1	1	0	1	0	1
0	1	1	1	1	1

M5

matter which, with *1*, and the other with *0*. Look at M4, where *2* = *1*, and *3* = *0*. The second row of M4 is identical with its first row, and the fourth row with its third; similarly the second column of M4 is identical with its first column, and the fourth column with its third. Cancelling the superfluous middle rows and columns we get M1. In the same way we get M1 from M5 where *2* = *0* and *3* = *1*.

M3 is a four-valued matrix. By multiplying M3 by M1 we get an eight-valued matrix, by further multiplication by M1 a sixteen-valued matrix, and, in general, a 2*n*-valued matrix. All these matrices are adequate to the *C–N–p*-system, and continue to be adequate, if we extend the system by the introduction of variable functors.

§ 47. *The C–N–δ–p-system*

We have already met two theses with a variable functor δ: the principle of extensionality *CQpqCδpδq*, and the thesis *CδpCδNpδq*. As the latter thesis is an axiom of our system of modal logic, it is necessary to explain thoroughly the *C–N–p*-system extended by δ which I call, following C. A. Meredith, the *C–N–δ–p*-system. This is the more necessary, as systems with δ are almost unknown even to logicians.

The introduction of variable functors into propositional logic is due to the Polish logician Leśniewski. By a modification of his rule of substitution for variable functors I was able to get simple and elegant proofs.[1] First, this rule must be explained.

I denote by δ a variable functor of one propositional argument, and I accept that δ*P* is a significant expression provided *P* is a significant expression. Let us see what is the meaning of the simplest significant expression with a variable functor, i.e. δ*p*.

A variable is a single letter considered with respect to a range of values that may be substituted for it. To substitute means in practice to write instead of the variable one of its values, the same value for each occurrence of the same variable. In the *C–N–p*-system the range of values of propositional variables, such as *p* or *q*, consists of all propositional expressions significant in the system; besides these two constants may be introduced, *1* and *0*, i.e. a constant true and a constant false proposition. What is the range of values of the functorial variable δ?

It is obvious that for δ we may substitute any value which gives together with *p* a significant expression of our system. Such are not only constant functors of one propositional argument, as, e.g. *N*, but also complex expressions working like functors of one argument, as *Cq* or *CCNpp*. By the substitution δ/*Cq* we get from δ*p* the expression *Cqp*, and by δ/*CCNpp* the expression *CCNppp*. It is evident, however, that this kind of substitution does not cover all possible cases. We cannot get in this way either *Cpq* or *CpCNpq* from δ*p*, because by no substitution for δ can the *p* be removed from its final position. Nevertheless there is no doubt that the two last expressions are as good substitutions of δ*p*, as *Cqp* or *CCNppp*, since δ*p*, as I understand it, represents all significant expressions which contain *p*, including *p* and δ*p* itself.

I was able to overcome this difficulty by the following device which I shall first explain by examples. In order to get *Cpq* from δ*p* by a substitution for δ I write δ/*C'q*, and I perform the substitution by dropping δ and filling up the blank marked by an apostrophe by the argument of δ, i.e. by *p*. In the same way I get from δ*p* the expression *CpCNpq* by the substitution δ/*C'CN'q*. If more than one δ occurs in an expression, as in *CδpCδNpδq*, and I want to perform on this expression the substitution δ/*C'r*, I must

[1] See Jan Łukasiewicz, 'On Variable Functors of Propositional Arguments', *Proceedings of the Royal Irish Academy*, Dublin (1951), 54 A 2.

everywhere drop the δ's and write in their stead *C'r* filling up the blanks by the respective arguments of δ. I get thus from *δp*—*Cpr*, from *δNp*—*CNpr*, from *δq*—*Cqr*, and from the whole expression—*CCprCCNprCqr*. From the same expression *CδpCδNpδq* there follows by the substitution *δ/C''* the formula *CCppCCNpNpCqq*. The substitution *δ/'* means that δ should be omitted; by this substitution we get for instance from *CδpCδNpδq* the principle of Duns Scotus *CpCNpq*. The substitution *δ/δ'* is the 'identical' substitution and does not produce any change. Speaking generally, we get from an expression containing δ's a new expression by a substitution for δ, writing for δ a significant expression with at least one blank, and filling up the blanks by the respective arguments of the δ's. This is not a new rule of substitution, but merely a description how the substitution for a variable functor should be performed.

The *C–N–δ–p*-system can be built up on the single asserted axiom known already to us:

51. *CδpCδNpδq*,

to which the axiomatically rejected expression *p* should be added to yield all rejected expressions. C. A. Meredith has shown (in an unpublished paper) that all asserted formulae of the *C–N–p*-system may be deduced from axiom 51.[1] The rules of inference are the usual rule of detachment, and the rules of substitution for propositional and functorial variables. To give an example how these rules work I shall deduce from axiom 51 the law of identity *Cpp*. Compare this deduction with the proof of *Cpp* in the *C–N–p*-system.[2]

51. *δ/'*, *q/p* × 53

53. *CpCNpp*

51. *δ/CpCNp'*, *q/Np* × C53–54

54. *CCpCNpNpCpCNpNp*

51. *δ/'*, *q/Np* × 55

[1] C. A. Meredith has proved in his paper 'On an Extended System of the Propositional Calculus', *Proceedings of the Royal Irish Academy*, Dublin (1951), 54 A 3, that the *C–O–δ–p*-calculus, i.e. the calculus with *C* and *O* as primitive terms and with functorial and propositional variables, may be completely built up from the axiom *CδδOδp*. His method of proving completeness can be applied to the *C–N–δ–p*-system with *CδpCδNpδq* as axiom. In my paper on modal logic quoted p. 133, n. 2, I deduce from axiom 51 the three asserted axioms of the *C–N–p*-system, i.e. *CCpqCCqrCpr*, *CCNppp*, *CpCNpq*, and some important theses in which δ occurs, among others the principle of extensionality.

[2] See p. 81.

55. *CpCNpNp*

55. *p/CpCNpNp* × C55–56

56. *CNCpCNpNpNCpCNpNp*

51. *δ/C''*, *p/CpCNpNp*, *q/p* × C54–C56–57

57. *Cpp*.

I should like to emphasize that the system based on axiom 51 is much richer than the *C–N–p*-system. Among asserted consequences containing δ there are such logical laws as *CCpqCCqpCδpδq*, *CδCpqCδpδq*, *CδCpqCpδq*, all very important, but unknown to almost all logicians. The first law, for instance, is the principle of extensionality, being equivalent to *CQpqCδpδq*, the second may be taken as the sole axiom of the so-called 'implicational' system, the third as an axiom of the so-called 'positive' logic. All these laws can be verified by the matrix method according to a rule given below.

In two-valued logic there exist four and only four different functors of one argument, denoted here by *V*, *S*, *N*, and *F* (see matrix M6).

p	*V*	*S*	*N*	*F*
1	1	1	0	0
0	1	0	1	0

M6

For the verification of δ-expressions the following practical rule due in substance to Leśniewski is sufficient: Write for δ successively the functors *V*, *S*, *N*, and *F*, then drop *S*, transform *Vα* into *Cpp*, and *Fα* into *NCpp*. If you get in all cases a true *C–N*-formula, the expression should be asserted, otherwise it should be rejected. Example: *CδCpqCδpδq* must be asserted, because we have:

CSCpqCSpSq = *CCpqCpq*, *CNCpqCNpNq*,
CVCpqCVpVq = *CCppCCppCpp*, *CFCpqCFpFq* = *CNCppCNCppNCpp*.

CCpqCδpδq must be rejected, for *CCpqCNpNq* is not a true *C–N*-formula. We see thus that all expressions of the *C–N–δ–p*-system are easily proved or disproved by the matrix method.

§ 48. *δ-Definitions*

The functor δ may be successfully employed to express definitions. The authors of the *Principia Mathematica* express definitions by a special symbol consisting of the sign of equality '=' that connects the *definiens* with the *definiendum*, and of the letters 'Df' put after the definition. According to this method the definition of alternation would run thus:

$$CNpq = Hpq \qquad \text{Df},$$

where $CNpq$ ('If not p, then q') is the *definiens*, and Hpq ('either p or q') the *definiendum*.[1] The symbol '. =. Df' is associated with a special rule of inference allowing the replacement of the *definiens* by the *definiendum* and vice versa. This is the merit of this kind of definition: the result is given immediately. But it has the defect of increasing the number of primitive symbols as well as of rules of inference which should be as small as possible.

Leśniewski would write the same definition as an equivalence thereby introducing into his system no new primitive term to express definitions, because for this very purpose he chose equivalence as the primitive term of his logic of propositions enlarged by functorial variables and quantifiers, and called by him 'protothetic'. This is the merit of his standpoint. On the other hand he cannot immediately replace the *definiens* by the *definiendum* or conversely, because equivalence has its own rules which do permit such replacements.

In our C–N–δ–p-system equivalence is not a primitive term; hence it must be defined, but cannot be defined by an equivalence without a vicious circle. We shall see, however, that it is possible to express definitions by C and δ in a way which preserves the merits of both standpoints without having their defects.

The purpose of a definition is to introduce a new term which as a rule is an abbreviation of some complex expression consisting of terms already known to us. Both parts of the definition, the *definiens* as well as the *definiendum* must fulfil certain conditions in order to yield a well-formed definition. The following four conditions are necessary and sufficient for definitions of new functions introduced into our system: (a) The *definiens* as well as the *definiendum* should be propositional expressions. (b) The *definiens* should consist of primitive terms or of terms already defined by them. (c) The *definiendum* should contain the new term introduced by the definition. (d) Any free variable occurring in the *definiens*

[1] I usually denote alternation by A, but this letter has already got another meaning in my syllogistic.

should occur in the *definiendum*, and vice versa. It is easily seen that, e.g. $CNpq$ as *definiens* and Hpq as *definiendum* comply with the four above conditions.

Let us now denote by P and R two expressions that fulfil the conditions (a)–(d), so that one of them, it does not matter which, may be taken as the *definiens*, and the other as the *definiendum*. It is supposed that neither of them contains δ. I say that the asserted expression $C\delta P\delta R$ represents a definition. For instance:

58. $C\delta CNpq\delta Hpq$

represents the definition of alternation. According to 58 any expression containing $CNpq$ may be immediately transformed into another expression in which $CNpq$ is replaced by Hpq. As example we may take the principle of Duns Scotus:

59. $CpCNpq$,

from which we can get the law $CpHpq$, i.e. in words: 'If p, then either p or q', by the following deduction:

58. $\delta/Cp' \times C59$–60
60. $CpHpq$.

If we want to apply our definition to the principle of Clavius:

61. $CCNppp$,

we must first put p for q in 58 getting thus:

58. $q/p \times 62$
62. $C\delta CNpp\delta Hpp$
62. $\delta/C'p \times C61$–63
63. $CHppp$.

(Formula 63 states: 'If either p or p, then p', and is one of the 'primitive propositions' or axioms accepted by the authors of the *Principia Mathematica*. They rightly call this axiom the 'principle of tautology', as it states that to say the same (ταὐτὸ λέγειν) twice, 'p or p', is to say simply 'p'. The principle of Duns Scotus, for instance, is not a tautology in any reasonable sense.)

The converse implication of 58 $C\delta Hpq\delta CNpq$, which enables us to replace Hpq by $CNpq$ is given together with the first. We can prove, indeed, using only the rules of substitution and detachment the following general theorem:

(C) If P and R are any significant expressions not containing δ, and $C\delta P\delta R$ is asserted, then $C\delta R\delta P$ must be asserted too.

The proof:

(D) $C\delta P\delta R$

    (D) $\delta/C\delta'\delta P\times$(E)

(E) $CC\delta P\delta PC\delta R\delta P$

    (D) $\delta/CC\delta P\delta' C\delta R\delta P\times$(F)

(F) $CCC\delta P\delta PC\delta R\delta PCC\delta P\delta RC\delta R\delta P$

    (F) $\times C$(E)–C(D)–(G)

(G) $C\delta R\delta P$.

If therefore P and R do not contain δ, and one of them may be interpreted as *definiens* and the other as *definiendum*, then it is clear that any asserted expression of the form $C\delta P\delta R$ represents a definition, as P may everywhere be replaced by R, and R by P, and this is just the characteristic property of a definition.

§ 49. *The four-valued system of modal logic*

Every system of modal logic ought to include as a proper part basic modal logic, i.e. ought to have among its theses both the M-axioms $CpMp$, $*CMpp$, and $*Mp$, and the L-axioms $CLpp$, $*CpLp$, and $*NLp$. It is easily seen that both M and L are different from any of the four functors V, S, N, and F of the two-valued calculus. M cannot be V, for Mp is rejected—whereas $Vp = Cpp$ is asserted, it cannot be S, for $CMpp$ is rejected—whereas $CSpp = Cpp$ is asserted, it cannot be either N or F, for $CpMp$ is asserted—whereas $CpNp$ and $CpFp = CpNCpp$ are rejected. The same is true for L. The functors M and L have no interpretation in two-valued logic. Hence any system of modal logic must be many-valued.

There is yet another idea that leads to the same consequence. If we accept with Aristotle that some future events, e.g. a sea-fight, are contingent, then a proposition about such events enounced today can be neither true nor false, and therefore must have a third truth-value different from *1* and *0*. On the basis of this idea and by help of the matrix method with which I became acquainted through Peirce and Schröder I constructed in 1920 a three-valued system of modal logic developed later in a paper of 1930.[1] I see today that this system does not satisfy all our

[1] Jan Łukasiewicz, 'O logice trójwartościowej', *Ruch Filozoficzny*, vol. v, Lwów

intuitions concerning modalities and should be replaced by the system described below.

I am of the opinion that in any modal logic the classical calculus of propositions should be preserved. This calculus has hitherto manifested solidity and usefulness, and should not be set aside without weighty reasons. Fortunately enough the classical calculus of propositions has not only a two-valued matrix, but also many-valued adequate matrices. I tried to apply to modal logic the simplest many-valued matrix adequate to the C–N–δ–p-system, i.e. the four-valued matrix, and succeeded in obtaining the desired result.

As we have seen in § 46, the matrix M2 whose elements are pairs of values *1* and *0* follows for N from the equality:

(z) $N(a, b) = (Na, Nb)$.

The expression '(Na, Nb)' is a particular case of the general form $(\epsilon a, \zeta b)$ where ϵ and ζ have as values the functors V, S, N, and F of the two-valued calculus. As each of the four values of ϵ can be combined with each of the four values of ζ, we get 16 combinations, which define 16 functors of one argument of the four-valued calculus. I found among them two functors, either of which may represent M. Here I shall define one of them, the other I shall discuss later.

(α) $M(a, b) = (Sa, Vb) = (a, Cbb)$.

On the basis of (α) I got the matrix M7 for M which I transformed into the matrix M8 by the same abbreviations as in § 46, viz.: $(1, 1) = 1$, $(1, 0) = 2$, $(0, 1) = 3$, and $(0, 0) = 0$.

p	M
(*1*, *1*)	(*1*, *1*)
(*1*, *0*)	(*1*, *1*)
(*0*, *1*)	(*0*, *1*)
(*0*, *0*)	(*0*, *1*)

M7

p	M
1	*1*
2	*1*
3	*3*
0	*3*

M8

Having thus got the matrix of M I chose C, N, and M as

(1920). Jan Łukasiewicz, 'Philosophische Bemerkungen zu mehrwertigen Systemen des Aussagenkalküls', *Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie*, vol. xxiii, cl. 3 (1930).

primitive terms, and based my system of modal logic on the following four axioms:

51. *CδpCδNpδq* 4. *CpMp* *5. *CMpp* *7. *Mp*.

The rules of inference are the rules of substitution and detachment for asserted and rejected expressions.

Lp is introduced by a δ-definition:

64. *CδNMNpδLp*.

That means: '*NMNp*' may be everywhere replaced by '*Lp*', and conversely '*Lp*' by '*NMNp*'.

The same system of modal logic can be established using *C*, *N*, and *L* as primitive terms with the axioms:

51. *CδpCδNpδq* 3. *CLpp* *6. *CpLp* *8. *NLp*,

and the δ-definition of *M*:

65. *CδNLNpδMp*.

M9 represents the full adequate matrix of the system:

C	*1*	*2*	*3*	*0*	*N*	*M*	*L*
1	*1*	*2*	*3*	*0*	*0*	*1*	*2*
2	*1*	*1*	*3*	*3*	*3*	*1*	*2*
3	*1*	*2*	*1*	*2*	*2*	*3*	*0*
0	*1*	*1*	*1*	*1*	*1*	*3*	*0*

M9

I hope that after the explanations given above every reader will be able to verify by this matrix any formula belonging to the system, i.e. to prove asserted formulae, and to disprove rejected ones.

It can be proved that the system is complete in the sense that every significant expression belonging to it is decidable, being either asserted or rejected. It is also consistent, i.e. non-contradictory, in the sense that no significant expression is both asserted and rejected. The set of axioms is independent.

I should like to emphasize that the axioms of the system are perfectly evident. The axiom with δ must be acknowledged by all logicians who accept the classical calculus of propositions; the axioms with *M* must also be accepted as true; the rules of inference are evident too. All correctly derived consequences of the system must be admitted by anyone who accepts the axioms and the rules of inference. No serious objection can be maintained against this system. We shall see that this system refutes all false inferences drawn in connexion with modal logic, explains the difficulties of the Aristotelian modal syllogistic, and reveals some unexpected logical facts which are of the greatest importance for philosophy.

§ 50. *Necessity and the four-valued system of modal logic*

Two major difficulties were stated at the end of Chapter VI: the first was connected with Aristotle's acceptance of asserted apodeictic propositions, the second with his acceptance of asserted contingent propositions. Let us solve the first difficulty.

If all analytic propositions are regarded as necessarily true, then the most typical analytic proposition, the principle of identity *Jxx*, must also be regarded as necessarily true. This leads, as we have seen, to the false consequence that any two individuals are necessarily identical, if they are identical at all.

This consequence cannot be derived from our system of modal logic, because it can be proved that in this system no apodeictic proposition is true. As this proof is based on the law of extensionality *CCpqCLpLq*, we must first show that this law results from our system.

A consequence of axiom 51 runs thus:

66. *CδCpqCδpδq*.

From 66 there follows by the substitution δ/*M*' the formula:

67. *CMCpqCMpMq*,

and from 67 we get by *CCpqMCpq*, a substitution of axiom 4, and by the hypothetical syllogism the stronger *M*-law of extensionality:

19. *CCpqCMpMq*.

The stronger *L*-law of extensionality *CCpqCLpLq* is deducible from 19 by transposition. The problem left undecided in § 42, which interpretation of the Aristotelian laws of extensionality, the stronger or the weaker one, should be admitted, is thus solved in favour of the stronger interpretation. The proof that no apodeictic proposition is true will now be given with full precision.

The premisses:

*6. *CpLp*
18. *CCpqCLpLq*
33. *CCpCqrCqCpr*
68. *CCCpqrCqr.*

The deduction:

68. *r/CLpLq* × *C*18–69
69. *CqCLpLq*
33. *p/q*, *q/Lp*, *r/Lq* × *C*69–70
70. *CLpCqLq*
70. *p/α*, *q/p* × *C**71–*6
*71. *Lα*.

The Greek variable α requires an explanation. The consequent of 70, *CqLq*, which means the same as the rejected expression *CpLp*, permits according to our rules the rejection of the antecedent *Lp*, and any substitution of *Lp*. This, however, cannot be expressed by **Lp*, because from a rejected expression nothing can be got by substitution; so, for instance, *Mp* is rejected, but *MCpp*—a substitution of *Mp*—is asserted. In order to express that the antecedent of 70 is rejected for any argument of *L*, I employ Greek letters calling them 'interpretation-variables' in opposition to the 'substitution-variables' denoted by Latin letters. As the proposition α may be given any interpretation, **Lα* represents a general law and means that any expression beginning with *L*, i.e. any apodeictic proposition, should be rejected.

This result, **Lα*, is confirmed by the matrix for *L* which is constructed from the matrices for *N* and *M* according to the definition of *L*. Anyone can recognize from a glance at M9 that *L* has only *2* and *o* as its truth-values, but never *1*.

The problem of false consequences resulting from the application of modal logic to the theory of identity is now easily solved. As *LJxx* cannot be asserted, being an apodeictic proposition, it is not possible to derive by detachment from the premiss:

(*t*) *CJxyCLJxxLJxy* or *CLJxxCJxyLJxy*

the consequence: (*v*) *CJxyLJxy*. It can be matrically proved indeed that (*t*) must be asserted, giving always *1*, but (*v*) should be rejected. Since the principle of identity *Jxx* is true, i.e. *Jxx* = *1*,

we get *LJxx* = *2*, and *CJxyCLJxxLJxy* = *CJxyC2LJxy*. *Jxy* may have one of the four values, *1*, *2*, *3*, or *o*:

If *Jxy* = *1*, then *CJxyC2LJxy* = *C1C2L1* = *C1C22* = *C11* = *1*,
,, *Jxy* = *2*, ,, *CJxyC2LJxy* = *C2C2L2* = *C2C22* = *C21* = *1*,
,, *Jxy* = *3*, ,, *CJxyC2LJxy* = *C3C2L3* = *C3C2o* = *C33* = *1*,
,, *Jxy* = *o*, ,, *CJxyC2LJxy* = *CoC2Lo* = *CoC2o* = *Co3* = *1*.

Hence (*t*) is proved since the final result of its matrical reduction is always *1*. On the contrary, (*v*) is disproved, because we have for *Jxy* = *1*: *CJxyLJxy* = *C1L1* = *C12* = *2*.

A pleasing and instructive example of the above difficulty has been given by W. V. Quine who asks what is wrong with the following inference:[1]

(*a*) The Morning Star is necessarily identical with the Morning Star;
(*b*) But the Evening Star is not necessarily identical with the Morning Star (being merely identical with it in fact);
(*c*) But one and the same object cannot have contradictory properties (cannot both be *A* and not be *A*);
(*d*) Therefore the Morning Star and the Evening Star are different objects.

Given my system the solution of this difficulty is very simple. The inference is wrong, because the premisses (*a*) and (*b*) are not true and cannot be asserted, so that the conclusion (*d*) cannot be inferred from (*a*) and (*b*) in spite of the fact that the implication *C*(*a*)*C*(*b*)(*d*) is correct (the third premiss may be omitted being true). The aforesaid implication can be proved in the following way:

Let *x* denote the Morning Star, and *y* the Evening Star; then (*a*) is *LJxx*, (*b*) is *NLJyx* which is equivalent to *NLJxy*, as identity is a symmetrical relation, and (*d*) is *NJxy*. We get thus the formula *CLJxxCNLJxyNJxy* which is a correct transformation of the true thesis (*t*).

The example given by Quine can now be verified by our four-valued matrix thus: if '*x*' and '*y*' have the same meaning as before, then *Jxx* = *Jxy* = *1*; hence *LJxx* = *LJxy* = *L1* = *2*,

[1] I found this example in the mimeographed *Logic Notes*, § 160, edited by the Department of Philosophy of the Canterbury University College (Christchurch, N.Z.), and sent to me by Professor A. N. Prior.

NLJxy = *N2* = 3, and *NJxy* = *N1* = 0, so that we have according to *CLJxxCNLJxyNJxy*: *C2C30* = *C22* = 1. The implication is true, but as not both its antecedents are true, the conclusion may be false.

We shall see in the next chapter that a similar difficulty was at the bottom of a controversy between Aristotle and his friends, Theophrastus and Eudemus. The philosophical implications of the important discovery that *No apodeictic proposition is true* will be set forth in § 62.

§ 51. *Twin possibilities*

I mentioned in § 49 that there are two functors either of which may represent possibility. One of them I denoted by *M* and defined by the equality:

(α) *M*(*a*, *b*) = (*Sa*, *Vb*) = (*a*, *Cbb*),

the other I define by the equality:

(β) *W*(*a*, *b*) = (*Va*, *Sb*) = (*Caa*, *b*),

denoting it by *W* which looks like an inverted *M*. According to this definition the matrix of *W* is M10, and can be abbreviated to M11. Though *W* is different from *M* it verifies axioms of the same structure as *M*, because *CpWp* is proved by M11, like *CpMp* by M8, and **CWpp* and **Wp* are disproved by M11, as **CMpp* and **Mp* are by M8. I could have denoted the matrix of *W* by *M*.

p	*W*
(1, 1)	(1, 1)
(1, 0)	(1, 0)
(0, 1)	(1, 1)
(0, 0)	(1, 0)

M10

p	*W*
1	1
2	2
3	1
0	2

M11

It can further be shown that the difference between *M* and *W* is not a real one, but merely results from a different notation. It will be remembered that I got M3 from M2 by denoting the pair of values (1, 0) by 2, and (0, 1) by 3. As this notation was quite arbitrary, I could with equal justice denote (1, 0) by 3, and (0, 1) by 2, or choose any other figures or signs. Let us then exchange the values 2 and 3 in M9, writing everywhere 3 for 2, and 2 for 3. We get from M9 the matrix M12, and by rearrangement of the middle rows and columns of M12, the matrix M13.

C	1	2	3	0	*N*	*M*	*L*
1	1	2	3	0	0	1	2
2	1	1	3	3	3	1	2
3	1	2	1	2	2	3	0
0	1	1	1	1	1	3	0

M9

C	1	3	2	0	*N*	–	–
1	1	3	2	0	0	1	3
3	1	1	2	2	2	1	3
2	1	3	1	3	3	2	0
0	1	1	1	1	1	2	0

M12

=

C	1	2	3	0	*N*	–	–
1	1	2	3	0	0	1	3
2	1	1	3	3	3	2	0
3	1	2	1	2	2	1	3
0	1	1	1	1	1	2	0

M13

If we compare M9 with M13, we see that the matrices for *C* and *N* remain unchanged, but the matrices corresponding to *M* and *L* become different, so that I cannot denote them by *M* and *L*. The matrix in M13 corresponding to *M* in M9 is just the matrix of *W*. Nevertheless M13 is the same matrix as M9, merely written in another notation. *W* represents the same functor as *M*, and must have the same properties as *M*. If *M* denotes possibility, then *W* does so too, and there can be no difference between these two possibilities.

In spite of their identity *M* and *W* behave differently when they both occur in the same formula. They are like identical twins who cannot be distinguished when met separately, but are instantly recognized as two when seen together. To perceive this let us consider the expressions *MWp*, *WMp*, *MMp*, and *WWp*. If *M* is identical with *W*, then those four expressions should be identical with each other too. But they are not identical. It can be proved by means of our matrices that the following formulae are asserted:

72. *MWp* and 73. *WMp*,

for *Wp* has as its truth-values only 1 or 2, and *M1* as well as *M2* = 1; similarly *Mp* has as its truth-values only 1 or 3, and both *W1* = 1 and *W3* = 1. On the other hand it can be proved that the formulae:

74. *CMMpMp* and 75. *CWWpWp*

are asserted, and as both *Mp* and *Wp* are rejected, *MMp* and *WWp* must be rejected too, so that we have:

*76. *MMp* and *77. *WWp*.

We cannot therefore, in 72 or 73, replace *M* by *W* or *W* by *M*, because we should get a rejected formula from an asserted one.

The curious logical fact of twin possibilities (and of twin necessities connected with them), which hitherto has not been observed by anybody, is another important discovery I owe to my four-valued modal system. It is too subtle and requires too great a development of formal logic to have been known to ancient logicians. The existence of these twins will both account for Aristotle's mistakes and difficulties in the theory of problematic syllogisms, and justify his intuitive notions about contingency.

§ 52. *Contingency and the four-valued system of modal logic*

We know already that the second major difficulty of Aristotle's modal logic is connected with his supposing that some contingent propositions were true. On the ground of the thesis:

52. *CKδpδNpδq*,

which is a transformation of our axiom 51, we get the following consequences:

52. $\delta/M,\ p/\alpha,\ q/p \times 78$

78. *CKMαMNαMp*

78. $C{*}79{-}{*}7$

*79. *KMαMNα*.

This means that 79 is rejected for any proposition α, as α is here an interpretation-variable. Consequently there exists no α that would verify both of the propositions: 'It is possible that α' and 'It is possible that not α', i.e. there exists no true contingent proposition *Tα*, if *Tp* is defined, with Aristotle, by the conjunction of *Mp* and *MNp*, i.e. by:

80. *CδKMpMNpδTp*.

This result is confirmed by the matrix method. Accepting the usual definition of *Kpq*:

81. *CδNCpNqδKpq*

we get for *K* the matrix M14, and we have:

K	*1*	*2*	*3*	*0*
1	*1*	*2*	*3*	*0*
2	*2*	*2*	*0*	*0*
3	*3*	*0*	*3*	*0*
0	*0*	*0*	*0*	*0*

M14

For $p = 1$: $KMpMNp = KM1MN1 = K1M0 = K13 = 3$

„ $p = 2$: „ $= KM2MN2 = K1M3 = K13 = 3$

„ $p = 3$: „ $= KM3MN3 = K3M2 = K31 = 3$

„ $p = 0$: „ $= KM0MN0 = K3M1 = K31 = 3$.

We see that the conjunction *KMpMNp* has the constant value 3, and is therefore never true. Hence $Tp = 3$, i.e. there exists no true contingent proposition in the sense given by definition 80.

Aristotle, however, thinks that the propositions 'It is possible that there will be a sea-fight tomorrow' and 'It is possible that there will not be a sea-fight tomorrow' may both be true today. Thus, according to his idea of contingency, there may be true contingent propositions.

There are two ways of avoiding this contradiction between Aristotle's view and our system of modal logic: we must either deny that any propositions are both contingent and true, or modify the Aristotelian definition of contingency. I choose the second way, making use of the twin types of possibility discovered above.

Tossing a coin we may throw either a head or a tail; in other words, it is possible to throw a head, and it is possible not to throw a head. We are inclined to regard both propositions as true. But they cannot be both true, if the first 'possible' is denoted by the same functor as the second. The first possibility is just the same as the second, but it does not follow that it should be denoted in the same way. The possibility of throwing a head is different from the possibility of not throwing a head. We may denote the one by *M*, and the other by *W*. The proposition with the affirmative argument 'It is possible that *p*' may be translated by *Mp*, the proposition with the negative argument 'It is possible that not *p*' by *WNp*; or the first by *Wp*, and the second by *MNp*. We get thus two functors of contingency, say *X* and *Y*, defined as follows:

82. *CδKMpWNpδXp* and 83. *CδKWpMNpδYp*.

It is impossible to translate these definitions into words, as we have no names for the two kinds of possibility and contingency. Let us call them '*M*-possible' and '*W*-possible', '*X*-contingent' and '*Y*-contingent'. We may then roughly say that '*p* is *X*-contingent' means 'p is M-possible and Np is W-possible', and 'p is Υ-contingent' means 'p is W-possible and Np is M-possible'.

From definitions 82 and 83 We can derive the matrices of X and Υ. We get:

For $p = 1$:
$X1 = KM1WN1 = K1W0 = K12 = 2$; $\Upsilon 1 = KW1MN1 = K1M0 = K13 = 3$.
For $p = 2$:
$X2 = KM2WN2 = K1W3 = K11 = 1$; $\Upsilon 2 = KW2MN2 = K2M3 = K23 = 0$.
For $p = 3$:
$X3 = KM3WN3 = K3W2 = K32 = 0$; $\Upsilon 3 = KW3MN3 = K1M2 = K11 = 1$.
For $p = 0$:
$X0 = KM0WN0 = K3W1 = K31 = 3$; $\Upsilon 0 = KW0MN0 = K2M1 = K21 = 2$.

p	X	Υ
1	2	3
2	1	0
3	0	1
0	3	2

M15

Matrix M15 shows that Xp as well as Υp turns out to be true for some value of p: Xp for $p = 2$, Υp for $p = 3$. Now it has been proved that $KMpMNp$ has the constant value 3; similarly it can be shown that $KWpWNp$ has the constant value 2. We get thus two asserted formulae:

84. $XKWpWNp$ and 85. $\Upsilon KMpMNp$.

This means that there exists in our system a true X-contingent and a true Υ-contingent proposition. We can accommodate contingency in Aristotle's sense within our four-valued modal logic.

It also follows from M15 that the X-contingency and the Υ-contingency are twins. If we replace in M15 2 by 3, and 3 by 2, X becomes Υ, and Υ becomes X. Nevertheless X is different from Υ, and more different than M is from W, because the propositions Xp and Υp are contradictory. It can be easily seen by M15 that the following equalities hold:

(γ) $Xp = \Upsilon Np = N\Upsilon p$ and (δ) $\Upsilon p = XNp = NXp$.

The laws of contradiction and of the excluded middle are true for Xp and Υp, i.e. we have:

86. $NKXp\Upsilon p$ and 87. $HXp\Upsilon p$.

This means: no proposition can be both X-contingent and Υ-contingent, and any proposition is either X-contingent or Υ-contingent. The negation of an X-contingent proposition is a Υ-contingent proposition, and conversely the negation of a Υ-contingent proposition is an X-contingent proposition. This sounds like a paradox, because we are accustomed to think that, what is not contingent is either impossible or necessary, relating the impossible and the necessary to the same kind of possibility. But it is not true to say that, what is not X-contingent is either M-impossible or M-necessary; it should rather be said that, what is not X-contingent is either M-impossible or W-necessary, and that being either M-impossible or W-necessary is equivalent to being Υ-contingent.

The same misunderstanding lies at the bottom of the controversy about the thesis:

88. $CKMpMqMKpq$

which is asserted in our system. C. I. Lewis in some of his modal systems accepts the formula:

89. $CMKpqKMpMq$,

but rejects its converse, i.e. 88, by the following argument:[1] 'If it is possible that p and q are both true, then p is possible and q is possible. This implication is not reversible. For example: it is possible that the reader will see this at once. It is also possible that he will not see it at once. But it is not possible that he will both see it at once and not see it at once.' The persuasiveness of this argument is illusory. What is meant by 'the reader'? If an individual reader, say R, is meant, then R either will see this at once, or R will not see this at once. In the first case the first premiss 'It is possible that R will see this at once' is true; but the second premiss is false, and how can a false proposition be possibly true? In the second case the second premiss is true, but the first is false, and a false proposition cannot be possibly true. The two premisses of the formula 88 are not both provable, and the formula cannot be refuted in this way.

If again by 'the reader' some reader is meant, then the premisses 'It is possible that some reader will see this at once' and 'It is possible that some reader will not see this at once' may be both true, but in this case the conclusion 'It is possible that some

[1] C. I. Lewis and C. H. Langford, *Symbolic Logic*, New York and London (1932), p. 167.

reader will see this at once and some reader will not see this at once' is obviously also true. It is, of course, not the same reader who will see this and not see this at once. The example given by Lewis does not refute formula 88; on the contrary it supports its correctness.

It seems, however, that this example has not been properly chosen. By the addition of the words 'at once' the premisses have lost the character of contingency. Saying that the reader will see this, or not, 'at once', we refer to something which is decided at the moment of seeing. The true contingent refers to undecided events. Let us take the example with the coin which is of the same sort as Aristotle's example with the sea-fight. Both examples concern events that are undecided at present, but will be decided in the future. Hence the premisses 'It is possible to throw a head' and 'It is possible not to throw a head' may at present be both true, whereas the conclusion 'It is possible to throw a head and not to throw a head' is never true. We know, however, that contingency cannot be defined by the conjunction of *Mp* and *MNp*, but either by *Mp* and *WNp* or by *Wp* and *MNp*, so that the example quoted above does not fall under the thesis 88. It cannot therefore disprove it. This was not known to Lewis and the other logicians, and on the basis of a wrong conception of contingency they have rejected the discussed thesis.

§ 53. *Some further problems*

Although the axioms and the rules of inference of our four-valued system of modal logic are perfectly evident, some consequences of the system may look paradoxical. We have already met the paradoxical thesis that the negation of a contingent proposition is also contingent; as another thesis of this kind I may quote the law of 'double contingency' according to which the following formulae are true:

90. *QpXXp* and 91. *QpYYp*.

The problem is to find some interpretation of these formulae which will be intuitively satisfactory and will explain away their apparent oddness. When the classical calculus of propositions was only recently known there was heated opposition to some of its principles too, chiefly to *CpCqp* and *CpCNpq*, which embody two logical laws known to medieval logicians and formulated by

them in the words: *Verum sequitur ad quodlibet* and *Ad falsum sequitur quodlibet*. So far as I see, these principles are now universally acknowledged.

At any rate our modal system is not in a worse position in this respect than other systems of modal logic. Some of them contain such non-intuitive formulae, as:

*92. *QMNMpNMp*

where a problematic proposition 'It is possible that *p* is impossible' is equivalent to an apodeictic proposition 'It is impossible that *p*'. Instead of this odd formula which has to be rejected we have in our system the thesis:

93. *QMNMpMNp* which together with
94. *QMMpMp*

enables us to reduce all combinations of modal functors consisting of *M* and *N* to four irreducible combinations known to Aristotle, viz. *M* = possible, *NM* = impossible, *MN* = non-necessary, and *NMN* = necessary.

The second problem concerns the extension of the four-valued modal logic into higher systems. The eight-valued system may serve as an example. We get the matrix M16 of this system by multiplying the matrix M9 by the matrix M1. As elements of the new matrix we form the pairs of values: $(1, 1) = 1$, $(1, 0) = 2$, $(2, 1) = 3$, $(2, 0) = 4$, $(3, 1) = 5$, $(3, 0) = 6$, $(0, 1) = 7$, $(0, 0) = 0$, and then we determine the truth-values of *C*, *N*, and *M* according to the equalities (*y*), (*z*), and (α).

C	1	2	3	4	5	6	7	0	*N*	*M*
1	1	2	3	4	5	6	7	0	0	1
2	1	1	3	3	5	5	7	7	7	1
3	1	2	1	2	5	6	5	6	6	3
4	1	1	1	1	5	5	5	5	5	3
5	1	2	3	4	1	2	3	4	4	5
6	1	1	3	3	1	1	3	3	3	5
7	1	2	1	2	1	2	1	2	2	7
0	1	1	1	1	1	1	1	1	1	7

M16

Figure *1* denotes, as usually, truth; *0* falsity; and the other figures are intermediate values between truth and falsity. If we attentively consider the matrix M16 we shall find that the second row of C is identical with the column of M. This row consequently represents the matrix of possibility. In the same way all the other rows of C, except the first and the last, represent some kinds of possibility. If we denote them by M_2 to M_7, we can state that M_i for $2 \leqslant i \leqslant 7$ satisfies all the axioms of possibility, viz.

95. CpM_ip, *96. CM_ipp, *97. M_ip.

Among these different kinds of possibility there are some 'stronger' and 'weaker', because we have, for instance, CM_2pM_4p or CM_3pM_6p, but not conversely. We may say therefore that in eight-valued modal logic there exist possibilities of different degrees. I have always thought that only two modal systems are of possible philosophic and scientific importance: the simplest modal system, in which possibility is regarded as having no degrees at all, that is our four-valued modal system, and the $\aleph_0$-valued system in which there exist infinitely many degrees of possibility. It would be interesting to investigate this problem further, as we may find here a link between modal logic and the theory of probability.

CHAPTER VIII
ARISTOTLE'S MODAL SYLLOGISTIC

Aristotle's modal syllogistic has, in my opinion, less importance in comparison with his assertoric syllogistic or his contributions to propositional modal logic. This system looks like a logical exercise which in spite of its seeming subtlety is full of careless mistakes and does not have any useful application to scientific problems. Nevertheless two controversial questions of this syllogistic are worth studying, chiefly for historical reasons: the question of syllogisms with one assertoric and one apodeictic premiss, and the question of syllogisms with contingent premisses.

§ 54. *Moods with two apodeictic premisses*

Aristotle deals with modal syllogisms after the pattern of his assertoric syllogistic. The syllogisms are divided into figures and moods, some moods are accepted as perfect and these need no proof as being self-evident, the imperfect moods are proved by conversion, *reductio ad absurdum*, or by 'ecthesis', as it is called. The invalid moods are rejected by interpretation through concrete terms. It is strange that with one exception Aristotle makes no use of his theorems of propositional modal logic. We shall see that this would yield in several cases better and simpler proofs than those given by him.

The laws of conversion for apodeictic propositions are analogous to those for assertoric ones. The following theses are accordingly true: 'If it is necessary that no b should be an a, it is necessary that no a should be a b', in symbols:

98. *CLEbaLEab*,

and 'If it is necessary that every b or some b should be an a, it is necessary that some a should be a b', in symbols:

99. *CLAbaLIab* and 100. *CLIbaLIab*.[1]

The proofs given by Aristotle are not satisfactory.[2] He did not see

[1] *An. pr.* i. 3, 25a29 εἰ μὲν γὰρ ἀνάγκη τὸ *A* τῷ *B* μηδενὶ ὑπάρχειν, ἀνάγκη καὶ τὸ *B* τῷ *A* μηδενὶ ὑπάρχειν. —32 εἰ δὲ ἐξ ἀνάγκης τὸ *A* παντὶ ἢ τινὶ τῷ *B* ὑπάρχει, καὶ τὸ *B* τινὶ τῷ *A* ἀνάγκη ὑπάρχειν.

[2] Cf. A. Becker, loc. cit., p. 90.

that the laws 98–100 may be immediately deduced from the analogous laws of the assertoric syllogistic by means of the theorem:

18. *CCpqCLpLq*.

For instance, from 18, by putting *Eba* for *p* and *Eab* for *q*, we get the assertoric law of conversion in the antecedent, hence we can detach the consequent, i.e. law 98.

Syllogisms with two apodeictic premisses are, according to Aristotle, identical with assertoric syllogisms, except that the sign of necessity must be added to the premisses as well as to the conclusion.[1] The formula for the mood Barbara will accordingly run:

101. *CKLAbaLAcbLAca*.

Aristotle tacitly accepts that the moods of the first figure are perfect and need not be proved. The moods of the other figures, which are imperfect, should be proved according to the proofs of assertoric syllogisms except Baroco and Bocardo, which are proved in the assertoric syllogistic by *reductio ad absurdum*, and should here be proved by ecthesis.[2] Once again, for all these proofs it would be easier to use theorem 18, as will appear from the following example.

By means of the laws of exportation and importation, *CCKpqr-CpCqr* and *CCpCqrCKpqr*, it can be shown that 15, the assertoric mood Barbara, is equivalent to the formula:

102. *CAbaCAcbAca*.

This purely implicational form is more convenient for deriving consequences than the conjunctional form. According to the thesis 3 *CLpp* we have:

103. *CLAbaAba*,

and from 103 and 102 we get by the hypothetical syllogism:

104. *CLAbaCAcbAca*.

On the other hand we have as substitution of 18:

[1] *An. pr.* i. 8, 29^b35 ἐπὶ μὲν οὖν τῶν ἀναγκαίων σχεδὸν ὁμοίως ἔχει καὶ ἐπὶ τῶν ὑπαρχόντων· ὡσαύτως γὰρ τιθεμένων τῶν ὅρων ἔν τε τῷ ὑπάρχειν καὶ τῷ ἐξ ἀνάγκης ὑπάρχειν ἢ μὴ ὑπάρχειν ἔσται τε καὶ οὐκ ἔσται συλλογισμός, πλὴν διοίσει τῷ προσκεῖσθαι τοῖς ὅροις τὸ ἐξ ἀνάγκης ὑπάρχειν ἢ μὴ ὑπάρχειν.

[2] Ibid. 30^a3–14.

105. *CCAcbAcaCLAcbLAca*,

and from 104 and 105 there follows the consequence:

106. *CLAbaCLAcbLAca*,

which is equivalent to 101. All the other syllogistic moods with two apodeictic premisses can be proved in the same way without new axioms, laws of conversion, *reductio ad absurdum*, or arguments by ecthesis.

§ 55. *Moods with one apodeictic and one assertoric premiss*[1]

Syllogistic moods of the first figure with one apodeictic and one assertoric premiss are treated by Aristotle differently according to which premiss, the major or the minor, is apodeictic. He says that when the major is apodeictic and the minor assertoric we get an apodeictic conclusion, but when the minor is apodeictic and the major assertoric we can have only an assertoric conclusion.[2] This difference will be made clear by the following examples of the mood Barbara. Aristotle asserts the syllogism: 'If it is necessary that every *b* should be an *a*, then if every *c* is a *b*, it is necessary that every *c* should be an *a*.' He rejects, however, the syllogism: 'If every *b* is an *a*, then if it is necessary that every *c* should be a *b*, it is necessary that every *c* should be an *a*.' In symbols:

(ε) *CLAbaCAcbLAca* is asserted,
(ζ) *CAbaCLAcbLAca* is rejected.

Aristotle considers the syllogism (ε) as self-evident. He says: 'Since every *b* is necessarily an *a* or not an *a*, and *c* is one of the *b*'s, it is evident (φανερόν) that *c* too will be necessarily an *a* or not an *a*.'[3] For reasons that will be explained later it is difficult to show this by examples. But the following picture will perhaps make the syllogism (ε) more acceptable to intuition. Let us

[1] Cf. J. Łukasiewicz, 'On a Controversial Problem of Aristotle's Modal Syllogistic', *Dominican Studies*, vol. vii (1954), pp. 114–28.

[2] *An. pr.* i. 9, 30^a15–25 συμβαίνει δέ ποτε καὶ τῆς ἑτέρας προτάσεως ἀναγκαίας οὔσης ἀναγκαῖον γίνεσθαι τὸν συλλογισμόν, πλὴν οὐχ ὁποτέρας ἔτυχεν, ἀλλὰ τῆς πρὸς τὸ μεῖζον ἄκρον, οἷον εἰ τὸ μὲν Α τῷ Β ἐξ ἀνάγκης εἴληπται ὑπάρχον ἢ μὴ ὑπάρχον, τὸ δὲ Β τῷ Γ ὑπάρχον μόνον· οὕτως γὰρ εἰλημμένων τῶν προτάσεων ἐξ ἀνάγκης τὸ Α τῷ Γ ὑπάρξει ἢ οὐχ ὑπάρξει. (Here follows the sentence quoted in the next note.) εἰ δὲ τὸ μὲν ΑΒ μὴ ἔστιν ἀναγκαῖον, τὸ δὲ ΒΓ ἀναγκαῖον, οὐκ ἔσται τὸ συμπέρασμα ἀναγκαῖον.

[3] Ibid. 30^a21 ἐπεὶ γὰρ παντὶ τῷ Β ἐξ ἀνάγκης ὑπάρχει ἢ οὐχ ὑπάρχει τὸ Α, τὸ δὲ Γ τι τῶν Β ἐστί, φανερὸν ὅτι καὶ τῷ Γ ἐξ ἀνάγκης ἔσται θάτερον τούτων.

imagine that the expression *LAba* means: 'Every *b* is connected by a wire with an *a*.' Hence it is evident that also every *c* (since every *c* is a *b*) is connected by a wire with an *a*, i.e. *LAca*. For whatever is true in some way of every *b*, is also true in the same way of every *c*, if every *c* is a *b*. The evidence of the last proposition is beyond any doubt.

We know, however, from Alexander that the evidence of the syllogism (ϵ) which Aristotle asserted, was not convincing enough for his friends who were pupils of Theophrastus and Eudemus.[1] As opposed to Aristotle, they held the doctrine that if either premiss is assertoric the conclusion must be so, just as if either premiss is negative the conclusion must be so and if either premiss is particular the conclusion must be so, according to a general rule formulated later by the scholastics: *Peiorem sequitur semper conclusio partem.*

This argument can be easily refuted. The syllogism (ϵ) is deductively equivalent to the problematic mood Bocardo of the third figure: 'If it is possible that some *c* should not be an *a*, then if every *c* is a *b*, it is possible that some *b* should not be an *a*.' In symbols:

(η) *CMOcaCAcbMOba.*

Syllogism (η) is as evident as (ϵ). Its evidence can be illustrated by examples. Let us suppose that a box contains ballots numbered from 1 to 90, and let *c* mean 'number drawn from the box', *b* 'even number drawn from the box', and *a* 'number divisible by 3'. We assume that in a certain case five even numbers have been drawn from the box, so that the premiss: 'Every number drawn from the box is an even number drawn from the box', i.e. *Acb*, is factually true. From this we can safely infer that, if it is possible in our case that some number drawn from the box should not be divisible by 3, i.e. *MOca*, it is also possible in our case that some *even* number drawn from the box should not be divisible by 3, i.e. *MOba*.

Aristotle accepts the syllogism (η) and proves it by a *reductio*

[1] Commenting on the passage quoted in n. 2, p. 183, Alexander says 124. 8 *οὗτος μὲν οὕτως λέγει. οἱ δέ γε ἑταῖροι αὐτοῦ οἱ περὶ Εὔδημόν τε καὶ Θεόφραστον οὐχ οὕτως λέγουσι, ἀλλά φασιν ἐν πάσαις ταῖς ἐξ ἀναγκαίας τε καὶ ὑπαρχούσης συζυγίαις, ἐὰν ὦσι συγκείμεναι συλλογιστικῶς, ὑπάρχον γίνεσθαι τὸ συμπέρασμα . . .* 17 *τῷ ἔλαττον εἶναι τὸ ὑπάρχον τοῦ ἀναγκαίου.*

ad absurdum from the syllogism (ϵ).[1] He does not, however, deduce (ϵ) from (η), though he certainly knew that this could be done. Alexander saw this point and explicitly proves (ϵ) from (η) by a *reductio ad absurdum* saying that this argument should be held as the soundest proof in favour of Aristotle's doctrine.[2] As according to him Aristotle's friends accept the syllogism (η) which fulfils peiorem rule, and (ϵ) is deducible from (η), they cannot reject (ϵ) on the ground of this rule, which becomes false when applied to modalities.

We shall see in the next Section that there was yet another argument raised by Theophrastus and Eudemus against syllogism (ϵ) which could not be refuted by Alexander, as it stands or falls with an Aristotelian argument. In spite of Alexander's talk about the 'soundest proof' one feels that some doubt is left in his mind, for he finally remarks after having presented several arguments in support of Aristotle's opinion, of which the argument quoted above is the last, that he has shown with greater rigour in other works which of those arguments are sound and which are not.[3] Alexander is referring here to his work 'On the Disagreement concerning Mixed Moods between Aristotle and his Friends', and to his 'Logical Scholia'.[4] Unfortunately both works are lost.

Our times have seen a revival of this controversy. Sir David Ross, commenting on syllogism (ϵ) and its proof from syllogism (η), states decidedly:[5] 'Yet Aristotle's doctrine is plainly wrong. For what he is seeking to show is that the premisses prove not only that all C is A, but also that it is necessarily A, just as all B is

[1] *An. pr.* i. 21, 39^b33–39 *ὑπαρχέτω γὰρ τὸ μὲν Β παντὶ τῷ Γ, τὸ δὲ Α ἐνδεχέσθω τινὶ τῷ Γ μὴ ὑπάρχειν· ἀνάγκη δὴ τὸ Α ἐνδέχεσθαι τινὶ τῷ Β μὴ ὑπάρχειν. εἰ γὰρ παντὶ τῷ Β τὸ Α ὑπάρχει ἐξ ἀνάγκης, τὸ δὲ Β παντὶ τῷ Γ κεῖται ὑπάρχειν, τὸ Α παντὶ τῷ Γ ἐξ ἀνάγκης ὑπάρξει· τοῦτο γὰρ δέδεικται πρότερον. ἀλλ' ὑπέκειτο τινὶ ἐνδέχεσθαι μὴ ὑπάρχειν.*

[2] Alexander says, commenting on syllogism (ϵ), 127. 3 *ἔστι δὲ πιστώσασθαι, ὅτι τὸ λεγόμενον ὑπὸ Ἀριστοτέλους ὑγιές ἐστι, μάλιστα διὰ τῆς εἰς ἀδύνατον ἀπαγωγῆς τῆς γινομένης ἐν τρίτῳ σχήματι . . .* 12 *ἐν γὰρ τῇ τοιαύτῃ συζυγίᾳ τῇ ἐν τρίτῳ σχήματι καὶ Ἀριστοτέλει δοκεῖ καὶ τοῖς ἑταίροις αὐτοῦ ἐπὶ μέρος ἐνδεχόμενον ἀποφατικὸν γίνεσθαι τὸ συμπέρασμα.*

[3] Alexander 127. 14 *τοσούτοις καὶ τοιούτοις ἄν τις χρήσαιτο παριστάμενος τῇ περὶ τούτων Ἀριστοτέλους δόξῃ. τί δὲ τούτων ὑγιῶς ἢ μὴ ὑγιῶς λέγεσθαι δοκεῖ, ἐν ἄλλοις ἡμῖν, ὡς ἔφην, μετὰ ἀκριβείας εἴρηται.*

[4] The title of the first work reads (Alexander 125. 30): *Περὶ τῆς κατὰ τὰς μίξεις διαφορᾶς Ἀριστοτέλους τε καὶ τῶν ἑταίρων αὐτοῦ*. Cf. Alexander 249. 38–250. 2, where *διαφωνίας* is used instead of *διαφορᾶς*, and the other work is cited as *Σχόλια λογικά*.

[5] W. D. Ross, loc. cit., p. 43.

necessarily A, i.e. by a permanent necessity of its own nature; while what they do show is only that so long as all C is B, it is A, not by a permanent necessity of its own nature, but by a temporary necessity arising from its temporary sharing in the nature of B.'

This argument is a metaphysical one, as the terms 'nature of a thing' and 'permanent necessity of its nature' belong to metaphysics. But behind this metaphysical terminology a logical problem is hidden which can be solved by our four-valued modal logic. Let us now turn to the syllogism rejected by Aristotle.

§ 56. *Rejected moods with one apodeictic and one assertoric premiss*

Syllogism (ζ) is as evident as syllogism (ϵ). It is strange that Aristotle rejects the syllogism

(ζ) *CAbaCLAcbLAca*,

though it is clear that this syllogism is on the same footing as the asserted syllogism (ϵ). In order to show its evidence let us employ the same picture as before. If *LAcb* means that every *c* is connected by a wire with a *b*, and every *b* is an *a*, i.e. *Aba*, it is evident that every *c* is connected by a wire with an *a*, i.e. *LAca*. Speaking generally, if every *b* is an *a*, then if every *c* is connected with a *b* in any way whatever, it must be connected with an *a* in just the same way. This seems to be obvious.

The most convincing argument that syllogism (ζ) is sound results from its deductive equivalence with the problematic mood Baroco of the second figure:

(θ) *CAbaCMOcaMOcb*, in words:

'If every *b* is an *a*, then if it is possible that some *c* should not be an *a*, it is possible that some *c* should not be a *b*.' This can be illustrated by an example. Let us turn to our box from which five numbers have been drawn, and let us suppose that every even number drawn from the box (*b*) is divisible by 3 (*a*), i.e. *Aba*. From this factual truth we can safely infer that, if it is possible that some number drawn from the box (*c*) should not be divisible by 3, i.e. *MOca*, it is also possible that some number drawn from the box should not be an *even* number, i.e. *MOcb*. This syllogism seems to be perfectly evident. In spite of its seeming so Aristotle

disproves syllogism (ζ), first by a purely logical argument which will be considered later, and then by the following example: Let *c* mean 'man', *b* 'animal', and *a* 'being in movement'. He accepts that the proposition 'Every man is an animal' is necessarily true, i.e. *LAcb*; but it is not necessary that every animal should be in movement, this may be only accepted as a factual truth, i.e. *Aba*, and so it is not necessary that every man should be in movement, i.e. *LAca* is not true.[1]

Aristotle's example is not convincing enough, as we cannot admit as a factual truth that every animal is in movement. A better example is provided by our box. Let *c* mean 'number drawn from the box and divisible by 4', *b* 'even number drawn from the box', and *a* 'divisible by 3'. Aristotle would agree that the proposition 'Every number drawn from the box and divisible by 4 is an even number drawn from the box' is a necessary truth, i.e. *LAcb*, while the premiss 'Every even number drawn from the box is divisible by 3' can be only accepted as a factual truth, i.e. *Aba*, and the conclusion 'Every number drawn from the box and divisible by 4 is divisible by 3' is also only a factual truth, i.e. *Aca*, and not *LAca*. The 'nature' of a number drawn from the box and divisible by 4 does not involve any 'permanent necessity' for it to be divisible by 3.

It would seem, therefore, that Aristotle is right in rejecting syllogism (ζ). The matter, however, becomes complicated, for it can be shown that just the same argument can be raised against syllogism

(ϵ) *CLAbaCAcbLAca*.

This was seen by Theophrastus and Eudemus who refute (ϵ) using in another order the same terms which were applied by Aristotle for disproving (ζ). Let *b* mean 'man', *a*—'animal', and *c*—'being in movement'. They agree with Aristotle that the proposition 'Every man is an animal' is necessarily true, i.e. *LAba*, and they accept as factually true that 'Everything in movement is a man', i.e. *Acb*. The premisses of (ϵ) are thus verified, but it is obvious that the conclusion 'Everything in movement is an animal', i.e. *Aca*, is not necessarily true.[2] This example is as

[1] *An. pr.* i. 9, 30ᵃ28 ἔτι καὶ ἐκ τῶν ὅρων φανερὸν ὅτι οὐκ ἔσται τὸ συμπέρασμα ἀναγκαῖον, οἷον εἰ τὸ μὲν A εἴη κίνησις, τὸ δὲ B ζῷον, ἐφ' ᾧ δὲ τὸ Γ ἄνθρωπος· ζῷον μὲν γὰρ ὁ ἄνθρωπος ἐξ ἀνάγκης ἐστί, κινεῖται δὲ τὸ ζῷον οὐκ ἐξ ἀνάγκης, οὐδ' ὁ ἄνθρωπος.

[2] Alexander 124.21 ἀλλὰ καὶ ἐπὶ τῆς ὕλης δεικνύουσι τοῦτο ἔχον οὕτως . . . 24 τὸ γὰρ

unconvincing as the corresponding one in Aristotle, for we cannot admit that the premiss *Acb* is factually true.

We can give a better example from our box. Let *b* mean 'number divisible by 6', *a*—'number divisible by 3', and *c*—'even number drawn from the box'. Aristotle would accept that the proposition 'Every number divisible by 6 is divisible by 3' is necessarily true, i.e. *LAba*, but it can be only factually true that 'Every even number drawn from the box is divisible by 6', i.e. *Acb*, and so it is only factually true that 'Every even number drawn from the box is divisible by 3', i.e. *Aca*. The propositions *Acb* and *Aca* are clearly equivalent to each other, and if one of them is only factually true, then the other cannot be necessarily true.

The controversy between Aristotle and Theophrastus about moods with one apodeictic and one assertoric premiss has led us to a paradoxical situation: there are apparently equally strong arguments for and against the syllogisms (ε) and (ζ). The controversy shown by the example of the mood Barbara can be extended to all other moods of this kind. This points to an error that lurks in the very foundations of modal logic, and has its source in a false conception of necessity.

§ 57. *Solution of the controversy*

The paradoxical situation expounded above is quite analogous to the difficulties we have met in the application of modal logic to the theory of identity. On the one hand, the syllogisms in question are not only self-evident, but can be demonstrated in our system of modal logic. I give here a full proof of the syllogisms (ε) and (ζ) based among others on the stronger *L*-law of extensionality known to Aristotle.

The premisses:

3. *CLpp*
18. *CCpqCLpLq*
24. *CCpqCCqrCpr*
33. *CCpCqrCqCpr*
102. *CAbaCAcbAca*.

ζῷον παντὶ ἀνθρώπῳ ἐξ ἀνάγκης, ὁ ἄνθρωπος παντὶ κινουμένῳ ὑπαρχέτω· οὐκέτι τὸ ζῷον παντὶ κινουμένῳ ἐξ ἀνάγκης.

The deduction:

18. *p/Aba*, *q/Aca* × 107

107. *CCAbaAcaCLAbaLAca*

33. *p/Aba*, *q/Acb*, *r/Aca* × C102–108

108. *CAcbCAbaAca*

24. *p/Acb*, *q/CAbaAca*, *r/CLAbaLAca* × C108–C107–109

109. *CAcbCLAbaLAca*

33. *p/Acb*, *q/LAba*, *r/LAca* × C109–110

110. *CLAbaCAcbLAca* (ε)

18. *p/Acb*, *q/Aca* × 111

111. *CCAcbAcaCLAcbLAca*

24. *p/Aba*, *q/CAcbAca*, *r/CLAcbLAca* × C102–C111–112

112. *CAbaCLAcbLAca* (ζ).

We see that the syllogisms (ε) and (ζ) denoted here by 110 and 112, are asserted expressions of our modal logic.

On the other hand, we get the thesis 113 from 110 by the substitution *b/a*, and the thesis 114 from 112 by the substitution *b/c* and commutation of the antecedents:

113. *CLAaaCAcaLAca* 114. *CLAccCAcaLAca*.

Both theses have in the consequent the expression *CAcaLAca*, i.e. the proposition 'If every *c* is an *a*, then it is necessary that every *c* should be an *a*'. If this proposition were asserted, all true universally-affirmative propositions would be necessarily true which is contrary to intuition. Moreover, as *CAcaLAca* is equivalent to *CNLAcaNAca*, and *Aca* means the same as *NOca*, we should have *CNLNOcaNNOca* or *CMOcaOca*. This last proposition which means 'If it is possible that some *c* should not be an *a*, then some *c* is not an *a*' is not true, for it is certainly possible that a number drawn from the box should not be even; so that, if the proposition is true, every set of drawings would contain an odd number—a result plainly contrary to the facts.

The expression *CAcaLAca* must be therefore rejected, and we get:

*115. *CAcaLAca*,

from which there follows according to our rules for rejected expressions the consequence:

113. $\times C$*116–*115
*116. *LAaa.*

The apodeictic Aristotelian law of identity must be rejected like the apodeictic principle of identity *LJxx*. This is conformable to our general view according to which no apodeictic proposition is true. The consequent of 113, i.e. *CAcaLAca*, cannot be detached, and the incompatibility between the acceptance of true apodeictic propositions and the assertion of the stronger *L*-law of extensionality is solved in favour of the law of extensionality. I do not believe that any other system of modal logic could satisfactorily solve this ancient controversy.

I mentioned earlier that Aristotle tries to refute the syllogism (ζ) not only by examples, but also by a purely logical argument. Asserting that the premisses *Aba* and *LAcb* do not give an apodeictic conclusion he says: 'If the conclusion were necessary, there would follow from it by a syllogism of the first or the third figure that some *b* is necessarily an *a*; but this is false, because *b* may be such that possibly no *b* is an *a*.'[1] Aristotle refers here to the apodeictic moods Darii and Darapti, since from (ζ) combined with either of these moods we can derive the consequence *CAbaCLAcbLIba*. The proof from Darapti runs:

117. *CCpCqrCCrCqsCpCqs*
112. *CAbaCLAcbLAca* (ζ)
118. *CLAcaCLAcbLIba* (Darapti)
117. p/Aba, $q/LAcb$, $r/LAca$, $s/LIba \times C112–C118–119$
119. *CAbaCLAcbLIba.*

The proof from Darii gives the same consequence, but is more complicated. Aristotle seems to disregard the premiss *LAcb*, and interprets this consequence as a simple implication:

*120. *CAbaLIba*,

which is obviously false and must be rejected. Or perhaps he thought that *LAcb* could be made true by a suitable substitution for *c* and dropped. If so he was wrong and his proof is a failure. We see besides by this example how difficult it is to confirm the validity of such theses, as 119, 112, or 110, through terms yielding

[1] *An. pr.* i. 9, 30^a25 (continuation of n. 2, p. 183) *εἰ γὰρ ἔστι, συμβήσεται τὸ Α τινὶ τῷ Β ὑπάρχειν ἐξ ἀνάγκης διά τε τοῦ πρώτου καὶ διὰ τοῦ τρίτου σχήματος. τοῦτο δὲ ψεῦδος· ἐνδέχεται γὰρ τοιοῦτον εἶναι τὸ Β ᾧ ἐγχωρεῖ τὸ Α μηδενὶ ὑπάρχειν.*

some would-be true apodeictic premisses. As many logicians believe that such propositions are really true, it is impossible to convince them of the validity of those syllogisms by examples.

Concluding this discussion we may say that Aristotle is right in asserting (ε), but wrong in rejecting (ζ). Theophrastus and Eudemus are wrong in both ways.

§ 58. *Moods with possible premisses*

The Aristotelian theory of problematic syllogisms displays a very strange gap: moods with possible premisses are entirely neglected in favour of moods with contingent premisses. According to Sir David Ross, 'Aristotle always takes ἐνδέχεται in a *premiss* as meaning "is neither impossible nor necessary"; where the only valid *conclusion* is one in which ἐνδέχεται means "is not impossible", he is as a rule careful to point this out'.[1] Aristotle, indeed, seems to be careful to distinguish the two meanings of ἐνδέχεσθαι when he says, expounding for instance the moods with two problematic premisses of the first figure, that ἐνδέχεσθαι in these moods should be understood according to the definition he has given, i.e. as 'contingent', and not in the sense of 'possible'. He adds, however, that this is sometimes overlooked.[2] Who may have overlooked this? Aristotle himself, of course, or some of his pupils just because of the ambiguity of the term ἐνδέχεσθαι. In the *De Interpretatione* ἐνδεχόμενον means the same as δυνατόν,[3] while in the *Prior Analytics* it has two meanings. It is always dangerous to use the same word in two meanings which may be unconsciously confused; as also to use two different words with the same meaning. Aristotle sometimes says ἐγχωρεῖ instead of ἐνδέχεται, and also uses the latter in two meanings.[4] We cannot be always sure what he means by ἐνδέχεται. The ambiguity of this term probably contributed to the controversies between himself and his friends Theophrastus and Eudemus. It is therefore a pity that he did not treat moods with possible premisses separately before introducing contingency. We shall supply this deficiency which has hitherto escaped the notice of scholars.

[1] W. D. Ross, loc. cit., p. 44; see also the table of the valid moods, facing p. 286.
[2] *An. pr.* i. 14, 33^b21 *δεῖ δὲ τὸ ἐνδέχεσθαι λαμβάνειν μὴ ἐν τοῖς ἀναγκαίοις, ἀλλὰ κατὰ τὸν εἰρημένον διορισμόν. ἐνίοτε δὲ λανθάνει τὸ τοιοῦτον.* [3] See n. 1, p. 134.
[4] Cf. for instance *An. pr.* i. 3, 25^b10 (n. 1, p. 192) and i. 9, 30^a27 (n. 1, p. 190) with i. 13, 32^b30 (n. 1, p. 193).

Let us first consider the laws of conversion. Aristotle begins the exposition of these laws in Book I, chapter 3 of the *Prior Analytics* with the statement that the term ἐνδέχεσθαι has several meanings. He then says, without explaining the various meanings of this term, that the laws of conversion of affirmative propositions are the same for all kinds of ἐνδέχεσθαι, but those of negative propositions differ. He states explicitly that the problematic propositions 'Every *b* may be an *a*' and 'Some *b* may be an *a*' (I use the word 'may' to cover both kinds of the problematic proposition) are convertible into the proposition 'Some *a* may be a *b*' which gives for possibility the formulae:

121. *CMAbaMIab* and 122. *CMIbaMIab*.

The law of conversion for universally-negative propositions is explained only by examples from which we may infer the formula:

123. *CMEbaMEab*.

It is tacitly assumed that particularly-negative possible propositions are not convertible.[1] We see from this that the laws of conversion of possible propositions are somewhat negligently treated by Aristotle. He apparently does not attach any great importance to the concept of possibility.

Formulae 121–3 are correct and are easily deducible from the analogous laws of conversion for assertoric propositions by means of the theorem:

19. *CCpqCMpMq*.

The same theorem, i.e. the stronger *M*-law of extensionality, enables us to establish the whole theory of syllogisms with possible premisses. By means of the classical calculus of propositions we get from 19 the formulae:

124. *CCpCqrCMpCMqMr* and 125. *CCpCqrCpCMqMr*.

Formula 124 yields moods with two possible premisses and a possible conclusion: we merely have to add the mark of possibility to the premisses and to the conclusion of valid assertoric

[1] *An. pr.* i. 3, $25^{a}37$–$^{b}14$ ἐπειδὴ πολλαχῶς λέγεται τὸ ἐνδέχεσθαι, . . . ἐν μὲν τοῖς καταφατικοῖς ὁμοίως ἕξει κατὰ τὴν ἀντιστροφὴν ἐν ἅπασιν. εἰ γὰρ τὸ *A* παντὶ ἢ τινὶ τῷ *B* ἐνδέχεται, καὶ τὸ *B* τινὶ τῷ *A* ἐνδέχοιτο ἄν. . . . ($^{b}3$) ἐν δὲ τοῖς ἀποφατικοῖς οὐχ ὡσαύτως, ἀλλ' ὅσα μὲν ἐνδέχεσθαι λέγεται ἢ τῷ ἐξ ἀνάγκης ὑπάρχειν ἢ τῷ μὴ ἐξ ἀνάγκης μὴ ὑπάρχειν, ὁμοίως, οἷον . . . ($^{b}9$)εἰ . . . ἐνδέχεται μηδενὶ ἀνθρώπῳ ἵππον, καὶ ἄνθρωπον ἐγχωρεῖ μηδενὶ ἵππῳ, . . . ($^{b}13$) ὁμοίως δὲ καὶ ἐπὶ τῆς ἐν μέρει ἀποφατικῆς.

moods. So, for instance, we get according to 124 from the assertoric mood Barbara by the substitution *p/Aba*, *q/Acb*, *r/Aca* the syllogism:

126. *CMAbaCMAcbMAca*.

Formula 125 yields moods with one assertoric and one possible premiss, it does not matter which, e.g.

127. *CAbaCMAcbMAca* 128. *CMAbaCAcbMAca*.

The system is extremely rich. Any premiss may be strengthened by replacing the assertoric or problematic proposition by the corresponding apodeictic proposition. Besides, there are moods with one problematic and one apodeictic premiss which yield apodeictic conclusions according to the formula:

129. *CCpCqrCMpCLqLr*.

Thus we have, for instance, the mood:

130. *CMAbaCLAcbLAca*

which is contrary to the *peiorem* rule accepted by Theophrastus and Eudemus.

I think that Aristotle would have accepted—not, of course, the last syllogistic mood—but the moods with possible premisses, in particular 126 and 128. There is, indeed, in the *Prior Analytics* an interesting introductory remark to the theory of problematic syllogisms which, in my opinion, may be applied to possibility as well as to contingency. Aristotle says that the expression 'Of anything, of which *b* is predicated, *a* may be predicated' has two meanings the best translation of which seems to be this: 'For all *c*, if every *c* is a *b*, then every *c* may be an *a*', and 'For all *c*, if every *c* may be a *b*, then every *c* may be an *a*'. Then he adds that the expression 'Of anything, of which *b* is predicated, *a* may be predicated' means the same as 'Every *b* may be an *a*'.[1] We have thus two equivalences: 'Every *b* may be an *a*' means either 'For all *c*, if every *c* is a *b*, then every *c* may be an *a*', or 'For all *c*, if every *c* may be a *b*, then every *c* may be an *a*'. If we interpret 'may' in the sense of possibility, we get the formulae:

[1] *An. pr.* i. 13, $32^{b}27$ τὸ γάρ, 'καθ' οὗ τὸ *B*, τὸ *A* ἐνδέχεσθαι' τούτων σημαίνει θάτερον, ἢ 'καθ' οὗ λέγεται τὸ *B*' ἢ 'καθ' οὗ ἐνδέχεται λέγεσθαι'. τὸ δέ, 'καθ' οὗ τὸ *B*, τὸ *A* ἐνδέχεσθαι' ἢ 'παντὶ τῷ *B* τὸ *A* ἐγχωρεῖν' οὐδὲν διαφέρει.

131. *QMAbaΠcCAcbMAca* and 132. *QMAbaΠcCMAcbMAca*

which are true in our system of modal logic, and from which the moods 128 and 126 are easily deducible. If, however, 'may' is interpreted in the sense of contingency which seems to be the intention of Aristotle, then the formulae given above become false.

§ 59. *Laws of conversion of contingent propositions*

Continuing his exposition of the laws of conversion of modal propositions Aristotle says at the beginning of the *Prior Analytics* that universally-negative contingent propositions are not convertible, whereas particularly-negative ones are.[1]

This curious statement demands careful examination. I shall first discuss it critically not from the point of view of my modal system, but from that of the basic modal logic accepted by Aristotle and all logicians.

According to Aristotle, contingency is that which is neither necessary nor impossible. This meaning of the contingent is clearly implicit in the somewhat clumsy definition of Aristotle, and is expressly corroborated by Alexander.[2] Let us repeat in order to ensure complete clearness: '*p* is contingent—means the same as—*p* is not necessary and *p* is not impossible', or in symbols:

48. *QTpKNLpNLNp.*

This formula is obviously equivalent to the expression:

50. *QTpKMpMNp,*

i.e. the contingent is both capable of being and capable of not being.

Formulae 48 and 50 are quite general and applicable to any proposition *p*. Let us apply them to the universally-negative proposition *Eba*. We get from 50:

133. *QTEbaKMEbaMNEba.*

As *NEba* is equivalent to *Iba*, we also have:

[1] *An. pr.* i. 3, 25^{b}14 (continuation of the text quoted in n. 1, p. 192) ὅσα δὲ τῷ ὡς ἐπὶ τὸ πολὺ καὶ τῷ πεφυκέναι λέγεται ἐνδέχεσθαι, . . . οὐχ ὁμοίως ἕξει ἐν ταῖς στερητικαῖς ἀντιστροφαῖς, ἀλλ' ἡ μὲν καθόλου στερητικὴ πρότασις οὐκ ἀντιστρέφει, ἡ δὲ ἐν μέρει ἀντιστρέφει.

[2] See above, § 45, in particular nn. 3, p. 154 and 1, p. 155.

134. *QTEbaKMEbaMIba.*

Now we can derive from the laws of conversion:

123. *CMEbaMEab* and 122. *CMIbaMIab*

that *MEba* is equivalent to *MEab*, and *MIba* to *MIab*; hence we have:

135. *QKMEbaMIbaKMEabMIab.*

The first part of this formula *KMEbaMIba* is equivalent to *TEba*, the second *KMEabMIab* to *TEab*; so we get the result:

136. *QTEbaTEab.*

This means that contingent universally-negative propositions are convertible.

How was it possible for Aristotle not to see this simple proof, when he had all its premisses at his disposal? Here we touch on another infected portion of his modal logic, even more difficult to cure than the wound which his ideas about necessity inflicted on it. Let us see how he tries to disprove formula 136.

Aristotle states quite generally that contingent propositions with opposite arguments are convertible with one another in respect of their arguments. The following examples will explain this not very clear formulation. 'It is contingent that *b* should be an *a*' is convertible with 'It is contingent that *b* should not be an *a*'; 'It is contingent that every *b* should be an *a*' is convertible with 'It is contingent that not every *b* should be an *a*'; and 'It is contingent that some *b* should be an *a*' is convertible with 'It is contingent that some *b* should not be an *a*'.[1] This kind of conversion I shall call, following Sir David Ross, 'complementary conversion'.[2]

Aristotle would assert accordingly that the proposition 'It is contingent that every *b* should be an *a*' is convertible with the proposition 'It is contingent that no *b* should be an *a*', in symbols:

(ι) *QTAbaTEba* (asserted by Aristotle).

This is the starting-point of his proof, which is performed by

[1] *An. pr.* i. 13, 32^{a}29 συμβαίνει δὲ πάσας τὰς κατὰ τὸ ἐνδέχεσθαι προτάσεις ἀντιστρέφειν ἀλλήλαις. λέγω δὲ οὐ τὰς καταφατικὰς ταῖς ἀποφατικαῖς, ἀλλ' ὅσαι καταφατικὸν ἔχουσι τὸ σχῆμα κατὰ τὴν ἀντίθεσιν, οἷον τὸ ἐνδέχεσθαι ὑπάρχειν τῷ ἐνδέχεσθαι μὴ ὑπάρχειν, καὶ τὸ παντὶ ἐνδέχεσθαι τῷ ἐνδέχεσθαι μηδενὶ καὶ μὴ παντί, καὶ τὸ τινὶ τῷ μὴ τινί.

[2] W. D. Ross, loc. cit., p. 44.

reductio ad absurdum. He argues in substance thus: If *TEba* were convertible with *TEab*, then *TAba* would be convertible with *TEab*, and as *TEab* is convertible with *TAab*, we should get the false consequence:

(κ) *QTAbaTAab* (rejected by Aristotle).[1]

What should we say to this argument? It is quite obvious that the definition of contingency adopted by Aristotle entails the convertibility of contingent universally-negative propositions. Consequently the disproof of this convertibility must be wrong. Since it is formally correct, the error must lie in the premisses, and as there are two premisses on which the disproof is based, the asserted formula (ι), and the rejected (κ), then either it is wrong to assert (ι) or it is wrong to reject (κ). This, however, cannot be decided within basic modal logic.

Within those limits we can merely say that the truth of the asserted formula (ι) is not justified by the accepted definition of contingency. From the definition:

50. *QTpKMpMNp*

we get by the substitution *p/Np* the formula *QTNpKMNpMNNp*, and as *MNNp* is equivalent to *Mp* according to thesis 9 of basic modal logic, we have:

137. *QTNpKMpMNp.*

From 50 and 137 there results the consequence:

138. *QTpTNp,*

and applying this consequence to the premiss *Eba* we get:

139. *QTEbaTNEba* or 140. *QTEbaTIba,*

as *NEba* means the same as *Iba*. We see that *QTEbaTIba* is justified by the definition of contingency, but that *QTEbaTAba* is not. This last formula has been accepted by Aristotle by a mistake.

We shall understand this error better if we examine Aristotle's

[1] *An. pr.* i. 17, $36^{b}35$ *πρῶτον οὖν δεικτέον ὅτι οὐκ ἀντιστρέφει τὸ ἐν τῷ ἐνδέχεσθαι στερητικόν, οἷον εἰ τὸ A ἐνδέχεται μηδενὶ τῷ B, οὐκ ἀνάγκη καὶ τὸ B ἐνδέχεσθαι μηδενὶ τῷ A. κείσθω γὰρ τοῦτο, καὶ ἐνδεχέσθω τὸ B μηδενὶ τῷ A ὑπάρχειν. οὐκοῦν ἐπεὶ ἀντιστρέφουσιν αἱ ἐν τῷ ἐνδέχεσθαι καταφάσεις ταῖς ἀποφάσεσι, καὶ αἱ ἐναντίαι καὶ αἱ ἀντικείμεναι, τὸ δὲ B τῷ A ἐνδέχεται μηδενὶ ὑπάρχειν, φανερὸν ὅτι καὶ παντὶ ἂν ἐνδέχοιτο τῷ A ὑπάρχειν. τοῦτο δὲ ψεῦδος· οὐ γὰρ εἰ τόδε τῷδε παντὶ ἐνδέχεται, καὶ τόδε τῷδε ἀναγκαῖον· ὥστ' οὐκ ἀντιστρέφει τὸ στερητικόν.*

refutation of an attempt to prove the law of conversion for *TEba* by *reductio ad absurdum.* This attempt reads: if we suppose that it is contingent that no *b* should be an *a*, then it is contingent that no *a* should be a *b*. For if the latter proposition were false, then it would be necessary that some *a* should be a *b*, and hence it would be necessary that some *b* should be an *a* which is contrary to our supposition.[1] In symbols: If *TEba* is supposed to be true, then *TEab* also must be true. For from *NTEab* would result *LIab*, and consequently *LIba*, which is incompatible with the supposition *TEba.*

Refuting this argument Aristotle rightly points out that *LIab* does not follow from *NTEab*.[2] We have, indeed, according to 48 the equivalence:

141. *QTEabKNLEabNLNEab* or
142. *QTEabKNLEabNLIab.*

Thus for *NTEab*, applying *QNKNpNqHpq*, i.e. one of the so-called 'De Morgan's laws',[3] we have the formula:

143. *QNTEabHLEabLIab.*

It can be seen that by means of 143 and the thesis *CCHpqrCqr* we can derive *NTEab* from *LIab*, but the converse implication does not hold, since from *NTEab* we can derive only the alternation *HLEabLIab* from which, of course, *LIab* does not follow. The attempted proof is wrong, but it does not follow that the conclusion which was to be proved is false.

One point in this reduction deserves our attention: it is apparent that instead of 143 Aristotle accepts the formula:

(λ) *QNTEabHLOabLIab*

which is not justified by definition 48. Similarly for the case of *NTAab* he adopts the formula:[4]

[1] *An. pr.* i. 17, $37^{a}9$ *ἀλλὰ μὴν οὐδ' ἐκ τοῦ ἀδυνάτου δειχθήσεται ἀντιστρέφον, οἷον εἴ τις ἀξιώσειεν, ἐπεὶ ψεῦδος τὸ ἐνδέχεσθαι τὸ B τῷ A μηδενὶ ὑπάρχειν, ἀληθὲς τὸ μὴ ἐνδέχεσθαι μηδενί (φάσις γὰρ καὶ ἀπόφασις), εἰ δὲ τοῦτ', ἀληθὲς ἐξ ἀνάγκης τινὶ τῷ A ὑπάρχειν· ὥστε καὶ τὸ A τινὶ τῷ B· τοῦτο δ' ἀδύνατον.*

[2] Ibid. $37^{a}14$ (continuation of the foregoing note) *οὐ γὰρ εἰ μὴ ἐνδέχεται μηδενὶ τὸ B τῷ A, ἀνάγκη τινὶ ὑπάρχειν. τὸ γὰρ μὴ ἐνδέχεσθαι μηδενὶ διχῶς λέγεται, τὸ μὲν εἰ ἐξ ἀνάγκης τινὶ ὑπάρχει, τὸ δ' εἰ ἐξ ἀνάγκης τινὶ μὴ ὑπάρχει.*

[3] These should properly be called Ockham's Laws, for so far as we know, Ockham was the first to state them. See Ph. Boehner, 'Bemerkungen zur Geschichte der De Morganschen Gesetze in der Scholastik', *Archiv für Philosophie* (September 1951), p. 115, n.

[4] *An. pr.* i. 17, $37^{a}24$ *τῷ ἐνδέχεσθαι παντὶ ὑπάρχειν τό τ' ἐξ ἀνάγκης τινὶ ὑπάρχειν ἀντίκειται καὶ τὸ ἐξ ἀνάγκης τινὶ μὴ ὑπάρχειν.*

(μ) *QNTAabHLOabLIab*

which, again, is not justified by 48, whereas the correct formula runs:

144. *QNTAabHLOabLAab.*

From (λ) and (μ) Aristotle may have deduced the equivalence *QNTAabNTEab*, and then (ι), which is not justified by his definition of contingency.

§ 60. *Rectification of Aristotle's mistakes*

Aristotle's theory of contingent syllogisms is full of grave mistakes. He does not draw the right consequences from his definition of contingency, and denies the convertibility of universally-negative contingent propositions, though it is obviously admissible. Nevertheless his authority is still so strong that very able logicians have in the past failed to see these mistakes. It is obvious that if somebody, Albrecht Becker for example, accepts the definition

48. *QTpKNLpNLNp*

with *p* as propositional variable, then he must also accept the formula:

141. *QTEabKNLEabNLNEab*

which is derived from 48 by the substitution *p*/*Eab*. And since by valid logical transformations formula 141 yields the thesis

143. *QNTEabHLEabLIab*,

he must also accept 143. Yet Becker rejects this thesis in favour of 'structural formulae'—a product of his imagination.[1]

The remarks of the foregoing section were written from the standpoint of basic modal logic which is an incomplete system. Let us now discuss our problem from the point of view of four-valued modal logic.

From the Aristotelian definition of contingency we obtained the consequence 138, *QTpTNp*, from which we may deduce the implication:

[1] See A. Becker, loc. cit., p. 14, where formula T11 = 48 written in another symbolism, but with the propositional variable *p*, is accepted, and p. 27 where formula 143 is rejected.

145. *CTpTNp.*

Now we get from the premisses:

51. *CδpCδNpδq* (axiom of the *C*–*N*–δ–*p*-system)
146. *CCpCqrCCpqCpr* (principle of Frege)

the consequences:

51. δ/*T*'$\times$147
147. *CTpCTNpTq*
146. *p*/*Tp*, *q*/*TNp*, *r*/*Tq*$\times$*C*147–*C*145–148
148. *CTpTq*,

and as the converse implication *CTqTp* is also true, as may be proved by the substitutions *p*/*q* and *q*/*p* in 148, we have the equivalence:

149. *QTpTq*.

From 149 we get by substitution first the law of conversion 136 *QTEbaTEab*, then formula (ι) *QTAbaTEba* which Aristotle asserts, and formula (κ) *QTAbaTAab* which he rejects. We can now determine where the flaw in Aristotle's disproof of the law of conversion is: Aristotle is wrong in rejecting (κ).

Formula *QTpTq* shows that the truth-value of the function *Tp* is independent of the argument *p*, which means that *Tp* is a constant. We know, in fact, from § 52 that *KMpMNp* which is the *definiens* of *Tp* has the constant value *3*, and therefore *Tp* also has the constant value *3* and is never true. For this reason *Tp* is not suitable to denote a contingent proposition in Aristotle's sense, since he believes that some contingent propositions are true. *Tp* must be replaced by *Xp* or *Yp*, i.e. by the function '*p* is *X*-contingent' or its twin '*p* is *Y*-contingent'. I shall take into consideration merely *X*-contingency, as what is true of *X*-contingency will also be true of *Y*-contingency.

First, I should like to state that the convertibility of universally-negative contingent propositions is independent of any definition of contingency. As *Eba* is equivalent to *Eab*, we must accept the formula

150. *CδEbaδEab*

according to the principle of extensionality *CQpqCδpδq*, which results from our axiom 51. From 150 we get a true statement for any value of δ, hence also for δ/*X*':

151. $CXEbaXEab$.

Alexander reports that Theophrastus and Eudemus, unlike Aristotle, accepted the convertibility of universally-negative contingent propositions,[1] but says in another passage that in proving this law they used *reductio ad absurdum*.[2] This seems doubtful, for the only correct thing Aristotle had done in this matter was to refute the proof of convertibility by *reductio*, a refutation which cannot have been unknown to his pupils. *Reductio* can be used to prove, from $CLIbaLIab$, the convertibility of universally-negative propositions when they are possible (that is, to prove $CMEbaMEab$), but not when they are contingent. Another proof is given by Alexander, continuing the former passage, but he scarcely formulates it clearly enough. We know that Theophrastus and Eudemus interpreted universally-negative premisses, Eba as well as Eab, as denoting a symmetric relation of disconnexion between b and a,[3] and they may have argued accordingly that if it is contingent for b to be disconnected from a, it is also contingent for a to be disconnected from b.[4] This proof would conform with the principle of extensionality. At any rate, Theophrastus and Eudemus have corrected the gravest mistake in Aristotle's theory of contingency.

Secondly, it follows from the definition of X-contingency:

82. $C\delta KMpWNp\delta Xp$

that the so-called 'complementary conversion' cannot be admitted. $QTpTNp$ is true, but $QXpXNp$ must be rejected, because its negation, i.e.:

152. $NQXpXNp$

is asserted in our system as can be verified by the matrix method. It is therefore not right in our system to convert the proposition

[1] Alexander 220. 9 *Θεόφραστος μέντοι καὶ Εὔδημος . . . ἀντιστρέφειν φασὶ καὶ τὴν καθόλου ἀποφατικὴν* (scil. *ἐνδεχομένην*) *αὐτῇ, ὥσπερ ἀντέστρεφε καὶ ἡ ὑπάρχουσα καθόλου ἀποφατικὴ καὶ ἡ ἀναγκαία.*

[2] Ibid. 223. 3 *δόξει τισὶ διά γε τῆς εἰς ἀδύνατον ἀπαγωγῆς δύνασθαι δείκνυσθαι ἡ καθόλου ἀποφατικὴ ἐνδεχομένη ἀντιστρέφουσα. τῇ αὐτῇ δείξει καὶ οἱ ἑταῖροι αὐτοῦ κέχρηνται.*

[3] See ibid. 31. 4–10.

[4] Ibid. 220. 12 *ὅτι δὲ ἀντιστρέφει, δεικνῦσιν οὕτως· εἰ τὸ A τῷ B ἐνδέχεται μηδενί, καὶ τὸ B τῷ A ἐνδέχεται μηδενί. ἐπεὶ γὰρ ἐνδέχεται τὸ A τῷ B μηδενί, ὅτε ἐνδέχεται μηδενί, τότε ἐνδέχεται ἀπεζεῦχθαι τὸ A πάντων τῶν τοῦ B· εἰ δὲ τοῦτ', ἔσται τότε καὶ τὸ B τοῦ A ἀπεζευγμένον· εἰ δὲ τοῦτο, καὶ τὸ B τῷ A ἐνδέχεται μηδενί.*

'It is contingent that every b should be an a' into the proposition 'It is contingent that some b should not be an a', or into the proposition 'It is contingent that no b should be an a', conversions which Aristotle accepts without any justification.[1] I think that Aristotle was led to a wrong conception of 'complementary conversion' by the ambiguity of the term 'contingent' (*ἐνδεχόμενον*). He uses this term in the *De Interpretatione* as a synonym of the term 'possible' (*δυνατόν*),[2] and continues to use it thus in the *Prior Analytics*, although the phrase 'It is contingent that p' has there got another meaning, viz. 'It is possible that p and it is possible that not p'. If we replace in the last phrase the term 'possible' by the term 'contingent', as Aristotle apparently does, we get the nonsense that 'It is contingent that p' means the same as 'It is contingent that p and it is contingent that not p'. So far as I know, this nonsense has hitherto not been observed by anybody.

Thirdly, it follows from definition 82 that Xp is stronger than Mp, because we have the thesis:

153. $CXpMp$,

but not conversely. This thesis is important, because it enables us to retain, with a little correction, a large number of syllogisms with contingent premisses, in spite of the serious mistakes made by Aristotle.

§ 61. *Moods with contingent premisses*

There is no need to enter into a detailed description of the syllogistic moods with contingent premisses, as Aristotle's definition of contingency is wrong and his syllogistic should be rebuilt according to the correct definition. This, however, does not seem to be worth while, for it is very doubtful whether a syllogistic with contingent premisses will ever find a useful application. I think that the following general remarks will be sufficient.

First, it may be shown that all the Aristotelian moods with a contingent conclusion are wrong. Let us take as an example the mood Barbara with contingent premisses and conclusion, i.e. the mood

*154. $CXAbaCXAcbXAca$.

[1] See n. 1, p. 195. [2] See n. 1, p. 134.

This mood though accepted by Aristotle[1] must be rejected. Take *Aba* and *Acb* as false, and *Aca* as true. These conditions fulfil the assertoric mood Barbara, but from 154, applying the matrices M9 and M15, we get the following equations: $CXoCXoX1 = C3C32 = C32 = 2$. Similarly mood

*155. *CXAbaCAcbXAca*

also accepted by Aristotle[2] must be rejected, since, for $Aba = o$, and $Acb = Aca = 1$, we have: $CXoC1X1 = C3C12 = C32 = 2$. It was just these two moods that I was referring to when I said at the end of § 58 that formulae 131 and 132, which Aristotle asserts, became false, if we interpreted ἐνδέχεσθαι as 'contingent'. It may be said too that formulae 154 and 155 become true, if for *X* is put *T*, but *T*-contingency is a useless concept.

Secondly, all the moods got by complementary conversion should be rejected. I shall show by an example how Aristotle deals with this sort of mood. He applies to 154 the formula

*156. *QXAbaXEba*

which should be rejected (take $Aba = 1$, and $Eba = o$), and gets the following moods:

*157. *CXAbaCXEcbXAca*
*158. *CXEbaCXEcbXAca*

which must be rejected too.[3] To show this, it suffices to choose the terms *a*, *b*, and *c* of 157 in such a way that $Aba = Ecb = o$, and $Aca = 1$, and those of 158 in such a way that $Eba = Ecb = o$, and $Aca = 1$. We then have in both cases: $CXoCXoX1 = C3C32 = C32 = 2$.

It seems that Aristotle does not put much trust in these moods,

[1] *An. pr.* i. 14, 32^b38 ὅταν οὖν τὸ *Α* παντὶ τῷ *Β* ἐνδέχηται καὶ τὸ *Β* παντὶ τῷ *Γ*, συλλογισμὸς ἔσται τέλειος ὅτι τὸ *Α* παντὶ τῷ *Γ* ἐνδέχεται ὑπάρχειν. τοῦτο δὲ φανερὸν ἐκ τοῦ ὁρισμοῦ· τὸ γὰρ ἐνδέχεσθαι παντὶ ὑπάρχειν οὕτως ἐλέγομεν.

[2] Ibid. 15, 33^b25 ἐὰν δ' ἡ μὲν ὑπάρχειν ἡ δ' ἐνδέχεσθαί λαμβάνηται τῶν προτάσεων, ὅταν μὲν ἡ πρὸς τὸ μεῖζον ἄκρον ἐνδέχεσθαι σημαίνῃ, τέλειοί τ' ἔσονται πάντες οἱ συλλογισμοὶ καὶ τοῦ ἐνδέχεσθαι κατὰ τὸν εἰρημένον διορισμόν.

[3] Ibid. 14, 33^a5 ὅταν δὲ τὸ *Α* παντὶ τῷ *Β* ἐνδέχηται, τὸ δὲ *Β* ἐνδέχηται μηδενὶ τῷ *Γ*, διὰ μὲν τῶν εἰλημμένων προτάσεων οὐδεὶς γίνεται συλλογισμός, ἀντιστραφείσης δὲ τῆς *ΒΓ* κατὰ τὸ ἐνδέχεσθαι γίνεται ὁ αὐτὸς ὅσπερ πρότερον. —33^a12 ὁμοίως δὲ καὶ εἰ πρὸς ἀμφοτέρας τὰς προτάσεις ἡ ἀπόφασις τεθείη μετὰ τοῦ ἐνδέχεσθαι. λέγω δ' οἷον εἰ τὸ *Α* ἐνδέχεται μηδενὶ τῷ *Β* καὶ τὸ *Β* μηδενὶ τῷ *Γ*· διὰ μὲν γὰρ τῶν εἰλημμένων προτάσεων οὐδεὶς γίνεται συλλογισμός, ἀντιστρεφομένων δὲ πάλιν ὁ αὐτὸς ἔσται ὅσπερ καὶ πρότερον.

because he does not call them syllogisms at all. He merely says that they can be reduced to syllogisms by means of complementary conversion. But moods reduced by the ordinary conversion are called by him syllogisms; why does he make a difference between ordinary and complementary conversion, if both kinds of conversion are equally valid?

Light upon this question is thrown by Alexander who, commenting on this passage, refers to a very important remark of his master on two ontological meanings of contingency: 'In one sense "contingent" means "usual (ἐπὶ τὸ πολύ) but not necessary" or "natural", e.g. it is contingent that men should go grey; in another sense it is used of the indefinite, which is capable of being thus and of not being thus, or in general of that which is by chance. In either sense contingent propositions are convertible with respect to their contradictory arguments, but not for the same reason: "natural" propositions because they do not express something necessary, "indefinite" propositions because there is not, in their case, a greater tendency to be more thus than not thus. About the indefinite there is no science or syllogistic demonstration, because the middle term is only accidentally connected with the extremes; only about the "natural" are there such things, and most arguments and inquiries are concerned with what is contingent in this sense.'[1]

Alexander discusses this passage: his idea seems to be that, if we take any scientifically useful syllogism the premisses of which are contingent in the sense of 'usual' (ἐπὶ τὸ πολύ) or even 'most usual' (ἐπὶ τὸ πλεῖστον), then we get premisses and a conclusion which are indeed contingent but are very seldom (ἐπ' ἔλαττον) realized: such a syllogism is useless (ἄχρηστος). Perhaps this is why Aristotle refuses to call what is so obtained a syllogism.[2]

[1] *An. pr.* i. 13, 32^b4–21 τὸ ἐνδέχεσθαι κατὰ δύο λέγεται τρόπους, ἕνα μὲν τὸ ὡς ἐπὶ τὸ πολὺ γίνεσθαι καὶ διαλείπειν τὸ ἀναγκαῖον, οἷον τὸ πολιοῦσθαι ἄνθρωπον . . ., ἢ ὅλως τὸ πεφυκὸς ὑπάρχειν . . ., ἄλλον δὲ τὸ ἀόριστον, ὃ καὶ οὕτως καὶ μὴ οὕτως δυνατόν, . . . ἢ ὅλως τὸ ἀπὸ τύχης γινόμενον. —($^b13$) ἀντιστρέφει μὲν οὖν καὶ κατὰ τὰς ἀντικειμένας προτάσεις ἑκάτερον τῶν ἐνδεχομένων, οὐ μὴν τὸν αὐτόν γε τρόπον, ἀλλὰ τὸ μὲν πεφυκὸς εἶναι τῷ μὴ ἐξ ἀνάγκης ὑπάρχειν . . ., τὸ δ' ἀόριστον τῷ μηδὲν μᾶλλον οὕτως ἢ ἐκείνως. ἐπιστήμη δὲ καὶ συλλογισμὸς ἀποδεικτικὸς τῶν μὲν ἀορίστων οὐκ ἔστι διὰ τὸ ἄτακτον εἶναι τὸ μέσον, τῶν δὲ πεφυκότων ἔστι, καὶ σχεδὸν οἱ λόγοι καὶ αἱ σκέψεις γίνονται περὶ τῶν οὕτως ἐνδεχομένων.

[2] Alexander 169. 1 τῷ γὰρ ὡς ἐπὶ τὸ πλεῖστον ἀποφατικῷ ἐνδεχομένῳ τὸ ἐπ' ἔλαττον καταφατικὸν ἀντιστρέφει. —5 τούτου δὲ κειμένου συλλογισμὸς μὲν ἔσται, οὐ μὴν χρήσιμόν τι ἔχων, ὡς αὐτὸς προεῖπε. διὸ καὶ ἐροῦμεν ταύτας τὰς συζυγίας . . .

This point, more than any other, reveals a capital error in Aristotle's syllogistic, viz. his disregard of singular propositions. It is possible that an individual, *Z*, should be going grey while growing older, indeed this is probable, though not necessary, since it is the natural tendency to do so. It is also possible, though rather improbable, that *Z* should not be going grey. What Alexander says about the different degrees of possibility is true when applied to singular propositions but becomes false when applied to universal or particular propositions. If there is no general law that every old man should go grey, because this is merely 'usual' and some old men do not go grey, then, of course, the latter proposition is true and therefore possible, but the former is simply false, and from our point of view a false proposition is neither possibly nor contingently true.

Thirdly, from a valid mood with possible premisses we can get other valid moods by replacing a possible premiss by the corresponding contingent one. This rule is based on formula 153 which states that *Xp* is stronger than *Mp*, and it is obvious that any implication will remain true, if one or more of its antecedents is replaced by a stronger antecedent. So we get, for instance, from

126. *CMAbaCMAcbMAca* the mood 159. *CXAbaCXAcbMAca*

and from

128. *CMAbaCAcbMAca* the mood 160. *CXAbaCAcbMAca*.

Comparing the rejected moods 154 and 155 with the asserted moods 159 and 160, we see that they differ only by the substitution of *M* for *X* in the conclusion. If we examine the table of Aristotelian syllogistic moods with problematic premisses, given by Sir David Ross,[1] we shall find it a useful rule that by this small correction, *M* in the conclusion, instead of *X*, all those moods become valid. Only the moods obtained by complementary conversion cannot be corrected, and must be definitively rejected.

ἀχρήστους τε καὶ ἀσυλλογίστους εἶναι. —10 ἴσως δὲ καὶ αὐτὸς τοῦτο ὑφορώμενος εἶπε τὸ 'ἢ οὐ γίνεται συλλογισμός'. Cf. W. D. Ross's paraphrase of this passage, loc. cit., p. 326.

[1] W. D. Ross, loc. cit., facing p. 286; in the conclusion the index *c* should everywhere be replaced by *p*.

§ 62. *Philosophical implications of modal logic*

It may seem that the Aristotelian modal syllogistic, even when corrected, has no useful application to scientific or philosophic problems. But in reality, Aristotle's propositional modal logic is historically and systematically of the greatest importance for philosophy. All elements required for a complete system of modal logic are to be found in his works: basic modal logic and the theorems of extensionality. But Aristotle was not able to combine those elements in the right way. He did not know the logic of propositions which was created after him by the Stoics; he tacitly accepted the logical principle of bivalence, i.e. the principle that every proposition is either true or false, whereas modal logic cannot be a two-valued system. Discussing the contingency of a future sea-fight he comes very near to the conception of a many-valued logic, but he lays no stress on this great idea, and for many centuries his suggestion remained fruitless. Owing to Aristotle I was able to discover this idea in 1920 and to construct the first many-valued system of logic in opposition to the logic, hitherto known, which I called 'two-valued logic' thus introducing a term now commonly accepted by logicians.[1]

Under the influence of Plato's theory of ideas Aristotle developed a logic of universal terms and set forth views on necessity which were, in my opinion, disastrous for philosophy. Propositions which ascribe essential properties to objects are according to him not only factually, but also necessarily true. This erroneous distinction was the beginning of a long evolution which led to the division of sciences into two groups: the *a priori* sciences consisting of apodeictic theorems, such as logic and mathematics, and the *a posteriori* or empirical sciences consisting chiefly of assertoric statements based on experience. This distinction is, in my opinion, false. There are no true apodeictic propositions, and from the standpoint of logic there is no difference between a mathematical and an empirical truth. Modal logic can be described as an extension of the customary logic by the introduction of a 'stronger'

[1] See J. Łukasiewicz, 'Logika dwuwartościowa' (Two-valued Logic), *Przegląd Filozoficzny*, 23, Warszawa (1921). A passage of this paper concerning the principle of bivalence was translated into French by W. Sierpiński, 'Algèbre des ensembles', *Monografie Matematyczne*, 23, p. 2, Warszawa-Wrocław (1951). An appendix of my German paper quoted in n. 1, p. 166, is devoted to the history of this principle in antiquity.

and a 'weaker' affirmation; the apodeictic affirmation Lp is stronger, and the problematic Mp weaker than the assertoric affirmation p. If we use the non-committal expressions 'stronger' and 'weaker' instead of 'necessary' and 'contingent', we get rid of some dangerous associations connected with modal terms. Necessity implies compulsion, contingency implies chance. We assert the necessary, for we feel compelled to do so. But if $L\alpha$ is merely a stronger affirmation than α, and α is true, why should we assert $L\alpha$? Truth is strong enough, there is no need to have a 'supertruth' stronger than truth.

The Aristotelian *a priori* is analytic, based on definitions, and definitions may occur in any science. Aristotle's example 'Man is necessarily an animal', based on the definition of 'man' as a 'two-footed animal', belongs to an empirical science. Every science, of course, must have at its disposal an exactly constructed language and for this purpose well-formed definitions are indispensable, as they explain the meaning of words, but they cannot replace experience. The analytic statement 'I am an animal' made by a man—analytic because 'animal' belongs to the essence of man—conveys no useful information, and can be seen to be silly by comparison with the empirical statement 'I was born the 21st December 1878'. If we want to know what the 'essence' of man is—if there is such a thing as 'essence' at all—we cannot rely on the meanings of words but must investigate human individuals themselves, their anatomy, histology, physiology, psychology, and so on, and this is an endless task. It is not a paradox to say even today that man is an unknown being.

The same is true for the deductive sciences. No deductive system can be based on definitions as its ultimate fundamentals. Every definition supposes some primitive terms, by which other terms may be defined, but the meaning of primitive terms must be explained by examples, axioms or rules, based on experience. The true *a priori* is always synthetic. It does not arise, however, from some mysterious faculty of the mind, but from very simple experiments which can be repeated at any time. If I know by inspection that a certain ballot box contains only white balls, I can say *a priori* that only a white ball will be drawn from it. And if the box contains white and black balls, and two drawings are made, I can foretell *a priori* that only four combinations can possibly occur: white-white, white-black, black-white, and black-

black. On such experiments the axioms of logic and mathematics are based; there is no fundamental difference between *a priori* and *a posteriori* sciences.

While Aristotle's treatment of necessity is in my opinion a failure, his concept of ambivalent possibility or contingency is an important and fruitful idea. I think that it may be successfully applied to refute determinism.

By determinism I understand a theory which states that if an event E happens at the moment t, then it is true at any moment earlier than t that E happens at the moment t. The strongest argument in defence of this theory is based on the law of causality which states that every event has a cause in some earlier event. If so, it seems to be evident that all future events have causes which exist today, and existed from eternity, and therefore all are predetermined.

The law of causality, however, understood in its full generality should be regarded as merely a hypothesis. It is true, of course, that astronomers, relying on some laws known to govern the universe, are able to predict for years in advance the positions and motions of heavenly bodies with considerable accuracy. Just at the moment I finished writing the previous sentence a bee flew humming past my ear. Am I to believe that this event too has been predetermined from all eternity and by some unknown laws governing the universe? To accept this would look more like indulging in whimsical speculation than relying on scientifically verifiable assertions.

But even if we accept the law of causality as generally true, the argument given above is not conclusive. We may assume that every event has a cause, and nothing happens by chance, yet the chain of causes producing a future event, though infinite, does not reach the present moment. This can be explained by a mathematical analogy. Let us denote the present moment by 0, the moment of the future event by 1, and the moments of its causes by fractions greater than $\frac{1}{2}$. As there exists no smallest fraction greater than $\frac{1}{2}$, every event has a cause in an earlier event, but the whole chain of these causes and effects has a limit at the moment $\frac{1}{2}$, later than 0.

We may therefore assume that the Aristotelian sea-fight of tomorrow, though it will have a cause which itself will have cause and so on, does not have a cause today. Similarly we may assume that nothing exists today which would prevent there being a sea-fight tomorrow. If truth consists in the conformity of thought to reality, we may say that those propositions are true today which conform with today's reality or with future reality in so far as that is predetermined by causes existing today. As the sea-fight of tomorrow is not real today, and its future existence or non-existence has no real cause today, the proposition 'There will be a sea-fight tomorrow' is today neither true nor false. We can only say: 'There may be a sea-fight tomorrow' and 'There may not be a sea-fight tomorrow'. Tomorrow's sea-fight is a contingent event, and if there are such events, determinism is refuted.

INDEX