CHAPTER V

THE PROBLEM OF DECISION

§ 29. *The number of undecidable expressions*

I TAKE as the basis of my present investigation the following fundamental elements of the syllogistic:

(1) The four asserted axioms 1–4.
(2) The rule (*a*) of substitution and the rule (*b*) of detachment for the asserted expressions.
(3) The two rejected axioms *59 and *59*a*.
(4) The rule (*c*) of detachment and the rule (*d*) of substitution for the rejected expressions.

To this system of axioms and rules the theory of deduction must be added as the auxiliary theory. From the axioms and rules of assertion there can be derived all the known theses of the Aristotelian logic, i.e. the laws of the square of opposition, the laws of conversion, and all the valid syllogistic moods; on the basis of the axioms and rules of rejection all the invalid syllogistical forms can be rejected. But, as we have already seen, this system of axioms and rules does not suffice to describe the Aristotelian syllogistic adequately, because there exist significant expressions, for instance *CIabCNAabAba*, which can neither be proved by our axioms and rules of assertion nor disproved by our axioms and rules of rejection. I call such expressions undecidable with respect to our basis. Undecidable expressions may be either true in the Aristotelian logic or false. The expression *CIabCNAabAba* is, of course, false.

There are two questions we have to settle on this basis in order to solve the problem of decision. The first question is, Is the number of undecidable expressions finite or not? If it is finite, the problem of decision is easily solved: we may accept true expressions as new asserted axioms, and reject false expressions axiomatically. This method, however, is not practicable if the number of undecidable expressions is not finite. We cannot assert or reject an infinity of axioms. A second question arises in this case: Is it possible to complete our system of axioms and rules so that we could decide whether a given expression had to

be asserted or rejected? Both these questions were solved by Słupecki: the first negatively by showing that the number of undecidable expressions on our basis is not finite, the second affirmatively by the addition of a new rule of rejection.[1]

I begin with the first question. Every student of the traditional logic is familiar with the interpretation of syllogisms by means of Eulerian circles: according to this interpretation the term-variables *a*, *b*, *c* are represented by circles, the premiss *Aab* being true when and only when the circle *a* is either identical with the circle *b* or is included in *b*, and the premiss *Iab* being true when and only when the circles *a* and *b* have a common area. Consequently the premiss *Eab*, as the negation of *Iab*, is true when and only when the circles *a* and *b* have no common area, i.e. when they exclude each other. If, therefore, *a* and *b* are identical, *Iab* is true and *Eab* is false.

I shall now investigate various suppositions concerning the number of circles assumed as our 'universe of discourse', i.e. as the field of our interpretation. It is obvious that the rules of our basis remain valid throughout all the interpretations. If our universe of discourse consists of three circles or more, the four axioms of assertion are of course verified, and the axiomatically rejected expression

*59. *CKAcbAabIac*

is rejected, as it is possible to draw two circles *c* and *a* excluding each other and both included in the third circle *b*. The premisses *Acb* and *Aab* are then true, and the conclusion *Iac* is false. The expression

*59*a*. *CKEcbEabIac*

also is rejected, as we can draw three circles each excluding the two others, so that the premisses *Ecb* and *Eab* are true and the conclusion *Iac* is false. This interpretation therefore satisfies the conditions of our basis, and so do all our other interpretations.

Let us now suppose that our universe of discourse consists of

[1] See the paper of Słupecki quoted in p. 76, n. I have tried to simplify the author's arguments in order to make them comprehensible to readers not trained in mathematical thinking. I am, of course, alone responsible for the following exposition of Słupecki's ideas.

only three circles, but no more, and let us consider the following expression:

(F3) *CEabCEacCEadCEbcCEbdIcd.*

This expression contains four different variables, but each of them can assume only three different values, as we can only draw three different circles. Whatever be the way to substitute these three values for the variables, two variables must always receive the same value, i.e. must be identified. But if some one of the pairs of variables, *a* and *b*, or *a* and *c*, or *a* and *d*, or *b* and *c*, or *b* and *d*, consists of identical elements, the corresponding *E*-premiss becomes false, and the whole implication, i.e. the expression (F3), is verified; and if the last pair of variables, *c* and *d*, has identical elements, the conclusion *Icd* becomes true, and the whole implication is again verified. Under the condition that only three circles can be drawn, the expression (F3) is true and cannot be disproved by our axioms and rules of rejection. If we suppose, however, that our universe of discourse consists of more than three circles, we can draw four circles, each of them excluding the three others, and (F3) becomes false. (F3) therefore cannot be proved by our axioms and rules of assertion. As (F3) can neither be proved nor disproved by the system of our axioms and rules, it is an undecidable expression.

Let us now consider an expression of the form

(F4) $C\alpha_1C\alpha_2C\alpha_3...C\alpha_n\beta$,

containing *n* different variables:

$$a_1, a_2, a_3, ..., a_n,$$

and let us suppose that: (1) every antecedent of (F4) is of the type Ea_ia_j, a_i differing from a_j; (2) the consequent β is of the type Ia_ka_l, a_k differing from a_l; (3) all the possible pairs of different variables occur in (F4). If our universe of discourse consists of only $(n-1)$ circles, (F4) is verified, because some two variables must be identified, and either one of the antecedents becomes false or the consequent is true. But if our universe of discourse consists of more than $(n-1)$ circles, (F4) is not verified, for *n* circles may be drawn each excluding the remainder, so that all the antecedents become true and the consequent is false. (F4), therefore, is an undecidable expression.

Such undecidable expressions are infinite in number, as *n* may be any integer whatever. It is obvious that they are all false in the Aristotelian logic, and must be rejected, for we cannot restrict the Aristotelian logic to a finite number of terms, and expressions of the form (F4) are disproved when the number of terms is infinite. This infinite number of undecidable expressions cannot be rejected otherwise than axiomatically, as results from the following consideration: (F3) cannot be disproved by the system of our axioms and rules, and therefore must be rejected axiomatically. The next undecidable expression of the form (F4) containing five different terms cannot be disproved by our system of axioms and rules together with the already rejected expression (F3), and must again be rejected axiomatically. The same argument may be repeated with respect to every other undecidable expression of the form (F4). Since it is impossible to reject axiomatically an infinity of expressions, we must look for another device if we want to solve the problem of decision affirmatively.

§ 30. *Słupecki's rule of rejection*

I start from two terminological remarks: Expressions of the type *Aab*, *Iab*, *Eab*, and *Oab* I call simple expressions; the first two are simple affirmative expressions, and the third and fourth simple negative expressions. Simple expressions as well as expressions of the type:

$$C\alpha_1C\alpha_2C\alpha_3...C\alpha_{n-1}\alpha_n,$$

where all the α's are simple expressions, I call elementary expressions. With the help of this terminology Słupecki's rule of rejection may be formulated as follows:

> If α and β are simple negative expressions and γ is an elementary expression, then if $C\alpha\gamma$ and $C\beta\gamma$ are rejected, $C\alpha C\beta\gamma$ must be rejected too.

Słupecki's rule of rejection has a close connexion with the following metalogical principle of traditional logic: 'utraque si praemissa neget, nil inde sequetur.' This principle, however, is not general enough, as it refers only to simple syllogisms of three terms. Another formulation of the same principle, 'ex mere negativis nihil sequitur', is apparently more general, but it is false when applied not only to syllogisms but also to other expressions of the syllogistic. Such theses as *CEabEba* or *CEabOab* show clearly that something does follow from merely negative premisses. Słupecki's rule is a general rule and avoids the awkwardness of traditional formulations.

Let us explain this point more fully in order to make Słupecki's rule clear. The proposition *Aac* does not follow either from the premiss *Aab* or from the premiss *Abc*; but when we conjoin these premisses, saying '*Aab* and *Abc*', we get the conclusion *Aac* by the mood Barbara. *Eac* does not follow from *Ebc*, or from *Aab* either: but from the conjunction of these premisses '*Ebc* and *Aab*' we get the conclusion *Eac* by the mood Celarent. In both cases we obtain from the conjunction of premisses some new proposition which does not result from either of them separately. If we have, however, two negative premisses, like *Ecb* and *Eab*, we can of course obtain from the first the conclusion *Ocb* and from the second *Oab*, but from the conjunction of these premisses no new proposition can be drawn except those that follow from each of them separately. This is the meaning of Słupecki's rule of rejection: if γ does not follow either from α or from β, it cannot follow from their conjunction, as nothing can be drawn from two negative premisses that does not follow from them separately. Słupecki's rule is as plain as the corresponding principle of traditional logic.

I shall now show how this rule can be applied in the rejection of undecidable expressions. For this purpose I use the rule in a symbolic form, denoted by RS (Rule of Słupecki):

RS. $*C\alpha\gamma,\ *C\beta\gamma \rightarrow *C\alpha C\beta\gamma$.

Here as everywhere I employ Greek letters to denote variable expressions satisfying certain conditions: thus, α and β must be simple negative expressions of the syllogistic, γ must be an elementary expression as explained above, and all three expressions must be such that $C\alpha\gamma$ and $C\beta\gamma$ may be rejected. The arrow ($\rightarrow$) means 'therefore'. I want to lay stress on the fact that RS is a peculiar rule, valid only for negative expressions α and β of the Aristotelian logic, and, as we have already seen, cannot be applied to affirmative expressions of the syllogistic. Nor can it be applied to the theory of deduction. This results from the following example: the expressions *CNCpqr* and *CNCqpr* are both not true and would be rejected, if rejection were introduced into this theory, but *CNCpqCNCqpr* is a thesis. Also in algebra the proposition '*a* equals *b*' does not follow either from the premiss '*a* is not less than *b*' or from the premiss '*b* is not less than *a*', but it follows from the conjunction of these premisses.

As the first application of the new rule I shall show that the expression

*59*a*. *CKEcbEabIac*,

which was rejected axiomatically, can now be disproved. This results from the following deduction:

9. $p/Eac,\ a/c,\ b/a \times 79$

79. *CCEacIcaCEacIac*

$79 \times C*80\text{–}*64$

*80. *CEacIca*

$*80 \times *81.\ c/a,\ b/c,\ a/c$

*81. *CEcbIac*

$*64 \times *82.\ b/c$

*82. *CEabIac*

RS. $\alpha/Ecb,\ \beta/Eab,\ \gamma/Iac \times *81,\ *82 \rightarrow *83$

*83. *CEcbCEabIac*.

The rule RS is here applied for the first time; α and β are simple negative expressions, and γ is also a simple expression. From *83 we get by the law of exportation VII the formula *59*a*:

VII. $p/Ecb,\ q/Eab,\ r/Iac \times 84$

84. *CCKEcbEabIacCEcbCEabIac*

$84 \times C*59a\text{–}*83$

*59*a*. *CKEcbEabIac*.

It follows from the above that Słupecki's rule is stronger than our axiomatically rejected expression *59*a*. Since *59*a* has to be cancelled, formula *59, i.e. *CKAcbAabIac*, remains the sole expression axiomatically rejected.

In the second place I shall apply the rule RS repeatedly to disprove the formula (F3):

$*64 \times *85.\ d/c,\ c/a$

*85. *CEadIcd*

$*85 \times *86.\ b/a$

*86. *CEbdIcd*

RS. $\alpha/Ead,\ \beta/Ebd,\ \gamma/Icd \times$ *85, *86 → *87

*87. $CEadCEbdIcd$

*80 × *88. $b/a,\ d/a$

*88. $CEbcIcd$

RS. $\alpha/Ebc,\ \beta/Ebd,\ \gamma/Icd \times$ *88, *86 → *89

*89. $CEbcCEbdIcd$

RS. $\alpha/Ead,\ \beta/Ebc,\ \gamma/CEbdIcd \times$ *87, *89 → *90

*90. $CEadCEbcCEbdIcd$

*88 × *91. a/b

*91. $CEacIcd$

RS. $\alpha/Eac,\ \beta/Ebd,\ \gamma/Icd \times$ *91, *86 → *92

*92. $CEacCEbdIcd$

RS. $\alpha/Eac,\ \beta/Ebc,\ \gamma/CEbdIcd \times$ *92, *89 → *93

*93. $CEacCEbcCEbdIcd$

RS. $\alpha/Eac,\ \beta/Ead,\ \gamma/CEbcCEbdIcd \times$ *93; *90 → *94

*94. $CEacCEadCEbcCEbdIcd$

*85 × *95. b/d

*95. $CEabIcd$

RS. $\alpha/Eab,\ \beta/Ebd,\ \gamma/Icd \times$ *95, *86 → *96

*96. $CEabCEbdIcd$

RS. $\alpha/Eab,\ \beta/Ebc,\ \gamma/CEbdIcd \times$ *96, *89 → *97

*97. $CEabCEbcCEbdIcd$

RS. $\alpha/Eab,\ \beta/Ead,\ \gamma/CEbcCEbdIcd \times$ *97, *90 → *98

*98. $CEabCEadCEbcCEbdIcd$

RS. $\alpha/Eab,\ \beta/Eac,\ \gamma/CEadCEbcCEbdIcd \times$ *98, *94 → *99

*99. $CEabCEacCEadCEbcCEbdIcd$

The rule RS is used in this deduction ten times; α and β are always simple negative expressions, and γ is everywhere an elementary expression. In the same manner we could disprove other formulae of the form (F4), and also the formula (F1) of section 28. It is needless, however, to perform these deductions, since we can now set forth the general problem of decision.

§ 31. *Deductive equivalence*

We need for our proof of decision the concept of deductive or inferential equivalence. Since there are, in my opinion, some misunderstandings in the treatment of this concept, its meaning must be carefully defined. I shall do this on the basis of the theory of deduction.

It is usually said that two expressions, α and β, are deductively equivalent to each other when it is possible to deduce β from α if α is asserted, and conversely α from β if β is asserted. The rules of inference are always supposed as given. But they are seldom sufficient. They suffice, for instance, in the following example. From the asserted law of commutation $CCpCqrCqCpr$ we can deduce the thesis $CqCCpCqrCpr$:

(1) $CCpCqrCqCpr$

(1) $p/CpCqr,\ r/Cpr \times C(1)-(2)$

(2) $CqCCpCqrCpr$,

and again from this thesis we can deduce the law of commutation:

(2) $q/CqCCpCqrCpr,\ p/s,\ r/t \times C(2)-(3)$

(3) $CCsCCqCCpCqrCprtCst$

(2) $q/CpCqr,\ p/q,\ r/Cpr \times (4)$

(4) $CCpCqrCCqCCpCqrCprCqCpr$

(3) $s/CpCqr,\ t/CqCpr \times C(4)-(1)$

(1) $CCpCqrCqCpr$.[1]

But we cannot in this simple way deduce from the asserted expression $CNpCpq$ the law of Duns Scotus $CpCNpq$, because from the first expression we can derive new propositions only by substitution, and all the substitutions of $CNpCpq$ begin with CN, none with Cp. To deduce one of those expressions from another we must have further assistance. Speaking generally, the relation of deductive equivalence is seldom absolute, but in most cases it is relative to a certain basis of theses. In our case this basis is the law of commutation. Starting from

(5) $CNpCpq$

we get by commutation the law of Duns Scotus:

(1) $p/Np,\ q/p,\ r/q \times C(5)-(6)$

(6) $CpCNpq$,

and starting from (6) we get again by commutation (5):

(1) $q/Np,\ r/q \times C(6)-(5)$

(5) $CNpCpq$.

[1] This neat deduction was given by A. Tarski in Warsaw.

I say therefore that *CNpCpq* and *CpCNpq* are deductively equivalent with respect to the law of commutation, and I write:

CNpCpq $\sim$ *CpCNpq* with respect to (1).

The sign $\sim$ denotes the relation of deductive equivalence. This relation is different from the ordinary relation of equivalence, denoted here by *Q*, which is defined by the conjunction of two implications each converse to the other,

Qpq = *KCpqCqp*,

and requires no basis. If an ordinary equivalence $Q\alpha\beta$ is asserted, and α, or a substitution of α, is asserted too, then we can assert β, or the corresponding substitution of β, and conversely. An asserted ordinary equivalence $Q\alpha\beta$ is therefore a sufficient basis for the deductive equivalence $\alpha \sim \beta$; but it is not a necessary one. This is just the point where explanation is needed.

Not only asserted or true expressions may be deductively equivalent, but also false ones. In order to solve the problem of decision for the *C–N*-system we have to transform an arbitrary significant expression α into the expression $CN\alpha\pi$, where π is a propositional variable not occurring in α. This can be done by means of two theses:

S1. *CpCNpq*
S2. *CCNppp*.

I say that α is deductively equivalent to $CN\alpha\pi$ with respect to S1 and S2, and I write:

I. $\alpha \sim CN\alpha\pi$ with respect to S1 and S2.

All goes easily when α is asserted. Take as example *NNCpp*. This is a thesis easily verified by the *0–1* method. I state according to formula I that

NNCpp $\sim$ *CNNNCppq* with respect to S1 and S2.

Starting from

(7) *NNCpp*

we get by S1:

S1. *p*/*NNCpp* × *C*(7)–(8)

(8) *CNNNCppq*,

and starting again from (8) we get by substitution and S2:

(8) *q*/*NNCpp* × (9)

(9) *CNNNCppNNCpp*

S2. *p*/*NNCpp* × *C*(9)–(7)

(7) *NNCpp*.

But α is an arbitrary expression; it may be false, e.g. *Cpq*. In this case formula I reads:

Cpq $\sim$ *CNCpqr* with respect to S1 and S2.

Here the difficulty begins: we can get the thesis *CCpqCNCpqr* from S1 by the substitution *p*/*Cpq*, *q*/*r*, but we cannot derive from this thesis the consequent *CNCpqr*, for *Cpq* is not a thesis and cannot be asserted. Therefore *CNCpqr* cannot be detached. A still greater difficulty arises in the other direction: we can get from S2 by the substitution *p*/*Cpq* the thesis *CCNCpqCpqCpq*, but *CNCpqCpq* is not asserted, nor can we get *CNCpqCpq* from *CNCpqr* by substitution, because *CNCpqr* is not a thesis. We cannot say: Suppose that *Cpq* be asserted; then *CNCpqr* would follow. The assertion of a false expression is an error, and we cannot expect to prove anything by an error. It seems therefore that formula I is valid not for all expressions but only for those that are asserted.

There exists, in my opinion, only one way to avoid these difficulties: it is the introduction of rejection into the theory of deduction. We reject axiomatically the variable *p*, and accept the clear rules of rejection, (*c*) and (*d*). It can easily be shown on this basis that *Cpq* must be rejected. For we get from the axiom

(*10) *p*

and the thesis

(11) *CCCpppp*

by the rules of rejection:

(11) × *C*(*12)–(*10)

(*12) *CCppp*

(*12) × (*13) *p*/*Cpp*, *q*/*p*

(*13) *Cpq*.

Now we are able to prove that if *Cpq* is rejected, *CNCpqr* must be rejected too; and conversely, if *CNCpqr* is rejected, *Cpq* must be rejected too. Starting from

(*13) *Cpq*

we get by S2 and the rules of rejection:

S2. $p/Cpq \times (14)$

(14) $CCNCpqCpqCpq$

$(14) \times C(*15)-(*13)$

(*15) $CNCpqCpq$

$(*15) \times (*16)\ r/Cpq$

(*16) $CNCpqr$.

In the other direction we easily get Cpq from (*16) by S1:

S1. $p/Cpq,\ q/r \times (17)$

(17) $CCpqCNCpqr$

$(17) \times C(*13)-(*16)$

(*13) Cpq.

Formula I is now fully justified. We have, however, to correct our previous definition of deductive equivalence, saying:

> Two expressions are deductively equivalent to each other with respect to certain theses when and only when we can prove by means of these theses and of the rules of inference that if one of those expressions is asserted, the other must be asserted too, or if one of them is rejected, the other must be rejected too.

It follows from this definition that ordinary equivalence is not a necessary basis of deductive equivalence. If $Q\alpha\beta$ is a thesis, it is true that α is deductively equivalent to β with respect to $Q\alpha\beta$; but if α is deductively equivalent to β with respect to certain theses, it is not always true that $Q\alpha\beta$ is a thesis. Take as example the deductive equivalence just considered:

$Cpq \sim CNCpqr$ with respect to S1 and S2.

The corresponding ordinary equivalence $QCpqCNCpqr$ is not a thesis, for it is false for $p/1,\ q/0,\ r/1$.

It is obvious that the relation of deductive equivalence is reflexive, symmetrical, and transitive. There are cases where α is deductively equivalent to two expressions β and γ with respect to certain theses. That means: if α is asserted, then β is asserted and γ is asserted, and consequently their conjunction 'β and γ' is asserted; and conversely, if both β and γ, or their conjunction 'β and γ', is asserted, then α is asserted too. Again, if α is rejected, then the conjunction 'β and γ' must be rejected, and in this case it is sufficient that only one of them, β or γ, should be rejected; and conversely, if only one of them is rejected, α must be rejected too.

§ 32. *Reduction to elementary expressions*

Our proof of decision is based on the following theorem:

> (TA) Every significant expression of the Aristotelian syllogistic can be reduced in a deductively equivalent way, with respect to theses of the theory of deduction, to a set of elementary expressions, i.e. expressions of the form
>
> $$C\alpha_1 C\alpha_2 C\alpha_3 \ldots C\alpha_{n-1}\alpha_n,$$
>
> where all the α's are simple expressions of the syllogistic, i.e. expressions of the type Aab, Iab, Eab, or Oab.

All known theses of the syllogistic either are elementary expressions or can easily be transformed into elementary expressions. The laws of conversion, e.g. $CIabIba$ or $CAabIba$, are elementary expressions. All the syllogisms are of the form $CK\alpha\beta\gamma$, and expressions of this kind are deductively equivalent to elementary expressions of the form $C\alpha C\beta\gamma$ with respect to the laws of exportation and importation. But there are other significant expressions of the syllogistic, some of them true, some false, that are not elementary. We have already met such an expression: it was thesis 78, $CCNAabAbaIab$, the antecedent of which is not a simple expression but an implication. There exists, of course, an infinity of such expressions, and they must all be taken into account in the proof of decision.

Theorem (TA) can easily be proved on the basis of an analogous theorem for the theory of deduction:

> (TB) Every significant expression of the theory of deduction with C and N as primitive terms can be reduced in a deductively equivalent way with respect to a finite number of theses to a set of elementary expressions of the form
>
> $$C\alpha_1 C\alpha_2 C\alpha_3 \ldots C\alpha_{n-1}\alpha_n,$$
>
> where all the α's are simple expressions, i.e. either variables or their negations.

The proof of this theorem is not easy, but since it is essential for the problem of decision it cannot be omitted. The proof of (TB) given below is intended for readers interested in formal logic; those not trained in mathematical logic may take both theorems, (TA) and (TB), for granted.

Let α be an arbitrary significant expression of the theory of deduction other than a variable (which may, but need not, be transformed): every such expression can be transformed, as we already know, in a deductively equivalent way with respect to the theses S1 and S2:

S1. *CpCNpq*
S2. *CCNppp*

into the expression $CN\alpha\pi$, where π is a variable not occurring in α. We have therefore as transformation I:

I. $\alpha \sim CN\alpha\pi$ with respect to S1 and S2.

Transformation I allows us to reduce all significant expressions to implications that have a variable as their last term. Now we must try to transform $N\alpha$, the antecedent of $CN\alpha\pi$, into a variable or its negation. For this purpose we employ the following three transformations:

II.	$CNN\alpha\beta \sim C\alpha\beta$	with respect to	S3 and S4,
III.	$CNC\alpha\beta\gamma \sim C\alpha CN\beta\gamma$	,, ,,	S5 and S6,
IV.	$CC\alpha\beta\gamma \sim CN\alpha\gamma, C\beta\gamma$	,, ,,	S7, S8, and S9.

The respective theses are: for transformation II:

S3. *CCNNpqCpq*
S4. *CCpqCNNpq*;

for transformation III:

S5. *CCNCpqrCpCNqr*
S6. *CCpCNqrCNCpqr*;

for transformation IV:

S7. *CCCpqrCNpr*
S8. *CCCpqrCqr*
S9. *CCNprCCqrCCpqr*.

Let us now explain how we can get by these transformations a variable or its negation in the antecedent of $CN\alpha\pi$. The expression α occurring in $CN\alpha\pi$ may, like every significant expression of the *C*–*N*-system, be either a variable, or a negation, or an implication. If α is a variable, no transformation is needed; if it is a negation, we get $CNN\alpha\beta$, and two negations annul each other according to transformation II; if it is an implication, we get from $CNC\alpha\beta\gamma$ the equivalent expression $C\alpha CN\beta\gamma$, the antecedent of which, α, is simpler than the initial antecedent $NC\alpha\beta$. This new α may again be a variable—no transformation is then needed—or a negation—this case has already been settled—or an implication. In this last case we get from $CC\alpha\beta\gamma$ two expressions, $CN\alpha\gamma$ and $C\beta\gamma$, with simpler antecedents than the initial antecedent $C\alpha\beta$. By repeated applications of II, III, and IV we must finally reach in the antecedent a variable or its negation.

Let us now see by examples how these transformations work.

First example: *NNCpp*.

NNCpp	$\sim$ *CNNNCppq*	by	I;
CNNNCppq	$\sim$ *CNCppq*	,,	II;
CNCppq	$\sim$ *CpCNpq*	,,	III.

NNCpp is thus reduced to the expression *CpCNpq* with the variable *p* in the antecedent. *CpCNpq* is an elementary expression.

Second example: *CCCpqpp*.

CCCpqpp	$\sim$ *CNCCCpqppr*	by	I;
CNCCCpqppr	$\sim$ *CCCpqpCNpr*	,,	III;
CCCpqpCNpr	$\sim$ *CNCpqCNpr*, *CpCNpr*	by	IV;
CNCpqCNpr	$\sim$ *CpCNqCNpr*	by	III.

CCCpqpp is thus reduced to two expressions: *CpCNqCNpr* and *CpCNpr*, both with the variable *p* in the antecedent; both are elementary expressions.

Third example: *CCCpqqCCqpp*.

CCCpqqCCqpp	$\sim$ *CNCCCpqqCCqppr*	by	I;
CNCCCpqqCCqppr	$\sim$ *CCCpqqCNCCqppr*	,,	III;
CCCpqqCNCCqppr	$\sim$ *CNCpqCNCCqppr*, *CqCNCCqppr*	by	IV;
CNCpqCNCCqppr	$\sim$ *CpCNqCNCCqppr*	by	III.

CCCpqqCCqpp is reduced to two expressions *CpCNqCNCCqppr* and *CqCNCCqppr*, both with a variable in the first antecedent. Neither of them, however, is elementary, since the first has the compound expression *NCCqpp* as its third antecedent and the second has the same compound expression as its second antecedent.

As we can see from this last example, our task is not yet finished. By transformations I–IV we obtain implications with a variable in the first antecedent, and also expressions of the form:

$$C\alpha_1 C\alpha_2 C\alpha_3 \ldots C\alpha_{n-1}\alpha_n,$$

but not all antecedents of this form, apart from α_1, need be simple expressions. In order to get rid of such compound antecedents we need three further transformations:

V. $C\alpha C\beta\gamma \sim C\beta C\alpha\gamma$ with respect to S10,
VI. $C\alpha C\beta C\gamma\delta \sim C\alpha C\gamma C\beta\delta$,, ,, S11,
VII. $C\alpha C\beta\gamma \sim CNC\alpha N\beta\gamma$,, ,, S12 and S13.

The respective theses are: for transformation V:

S10. *CCpCqrCqCpr*;

for transformation VI:

S11. *CCpCqCrsCpCrCqs*;

for transformation VII:

S12. *CCpCqrCNCpNqr*
S13. *CCNCpNqrCpCqr*.

By S10 we can move a compound antecedent from the second place to the first, and by S11 from the third place to the second. Applying these transformations to the expressions *CpCNqCNCCqppr* and *CqCNCCqppr* of our third example we get:

(α) *CpCNqCNCCqppr* ∼ *CpCNCCqppCNqr* by VI;
CpCNCCqppCNqr ∼ *CNCCqppCpCNqr* ,, V;
CNCCqppCpCNqr ∼ *CCqpCNpCpCNqr* ,, III;
CCqpCNpCpCNqr ∼ *CNqCNpCpCNqr*, *CpCNpCpCNqr* by IV.

(β) *CqCNCCqppr* ∼ *CNCCqppCqr* by V;
CNCCqppCqr ∼ *CCqpCNpCqr* ,, III;
CCqpCNpCqr ∼ *CNqCNpCqr*, *CpCNpCqr* by IV.

CCCpqqCCqpp is thus reduced to four elementary expressions: *CNqCNpCpCNqr*, *CpCNpCpCNqr*, *CNqCNpCqr*, and *CpCNpCqr*.

Transformation VII is used in all those cases where the compound antecedent occurs in the fourth place or farther. This transformation allows us to reduce the number of antecedents; in fact, *NCpNq* means the same as *Kpq*, and S12 and S13 are other forms of the laws of importation and exportation respectively. Now $CNC\alpha N\beta\gamma$, like $CK\alpha\beta\gamma$, has only one antecedent, whereas the equivalent expression $C\alpha C\beta\gamma$ has two antecedents. If, therefore, a compound expression occurs in the fourth place, as δ in $C\alpha C\beta C\gamma C\delta\epsilon$, we can move it to the third place, applying VII and then VI:

$C\alpha C\beta C\gamma C\delta\epsilon \sim CNC\alpha N\beta C\gamma C\delta\epsilon$ by VII;
$CNC\alpha N\beta C\gamma C\delta\epsilon \sim CNC\alpha N\beta C\delta C\gamma\epsilon$,, VI.

From this last expression we get by the converse application of VII the formula:

$CNC\alpha N\beta C\delta C\gamma\epsilon \sim C\alpha C\beta C\delta C\gamma\epsilon$ by VII.

It is now easy to bring δ to the first place by VI and V:

$C\alpha C\beta C\delta C\gamma\epsilon \sim C\alpha C\delta C\beta C\gamma\epsilon$ by VI,
$C\alpha C\delta C\beta C\gamma\epsilon \sim C\delta C\alpha C\beta C\gamma\epsilon$,, V.

Applying transformation VII repeatedly in both directions we can move any antecedent from the *n*th place to the first, and transform it, if it is compound, by II, III, and IV into a simple expression.

The proof of theorem (TB) is thus completed. It is now easy to show that this theorem entails the proof of decision for the *C–N*-system of the theory of deduction. If all the elementary expressions to which a given expression α has been reduced are true, i.e. if they have among their antecedents two expressions of the type *p* and *Np*, then α is a thesis and must be asserted. On the other hand, if among the elementary expressions to which α has been reduced there exists at least one expression such that no two antecedents in it are of the type *p* and *Np*, then α must be rejected. In the first case we can prove α by means of the theses S1–S13, in the second we can disprove it, adding to the above theses two new ones:

S14. *CpCCpqq*
S15. *NNCpp*,

and the axiom of rejection:

*S16. *p*.

Two examples will clarify this.

First example: Proof of the thesis $CpCCpqq$.

This thesis must first be reduced to elementary expressions. This is done by the following analysis (L):

$CpCCpqq \sim CNCpCCpqqr$ by I;
$CNCpCCpqqr \sim CpCNCCpqqr$ „ III;
$CpCNCCpqqr \sim CNCCpqqCpr$ „ V;
$CNCCpqqCpr \sim CCpqCNqCpr$ „ III;
$CCpqCNqCpr \sim CNpCNqCpr,\ CqCNqCpr$ by IV.

The elementary expressions to which $CpCCpqq$ is reduced are $CNpCNqCpr$ and $CqCNqCpr$. Both, like all expressions to which transformation I has been applied, have as their last term a variable not occurring in the antecedents. Such expressions can be true only on condition that they have two antecedents of the type p and Np, and any expression of this kind can be reduced by transformations V, VI, or VII to a substitution of S1 from which the proof of a thesis must always begin. Here are the required deductions:

S1. $q/CNqr \times (1)$
(1) $CpCNpCNqr$

S10. $q/Np,\ r/CNqr \times C(1)-(2)$
(2) $CNpCpCNqr$

S11. $p/Np,\ q/p,\ r/Nq,\ s/r \times C(2)-(3)$
(3) $CNpCNqCpr$

S1. $p/q,\ q/Cpr \times (4)$
(4) $CqCNqCpr$.

Having got in (3) and (4) the same elementary expressions as we reached at the end of our analysis (L), we now proceed from them to their equivalents on the left, by applying theses on which the successive transformations were based. Thus, step by step, we get our original thesis by means of S9, S6, S10, and S2:

S9. $r/CNqCpr \times C(3)-C(4)-(5)$
(5) $CCpqCNqCpr$

S6. $p/Cpq,\ r/Cpr \times C(5)-(6)$
(6) $CNCCpqqCpr$

S10. $p/NCCpqq,\ q/p \times C(6)-(7)$
(7) $CpCNCCpqqr$

S6. $q/CCpqq \times C(7)-(8)$
(8) $CNCpCCpqqr$

(8) $r/CpCCpqq \times (9)$
(9) $CNCpCCpqqCpCCpqq$

S2. $p/CpCCpqq \times C(9)-(10)$
(10) $CpCCpqq$.

Upon this model we can prove any thesis we want.

Second example: Disproof of the expression $CCNpqq$.

We first reduce this expression to elementary expressions on the basis of the following analysis:

$CCNpqq \sim CNCCNpqqr$ by I;
$CNCCNpqqr \sim CCNpqCNqr$ „ III;
$CCNpqCNqr \sim CNNpCNqr,\ CqCNqr$ by IV;
$CNNpCNqr \sim CpCNqr$ by II.

The expression $CCNpqq$ is thus reduced to two elementary expressions, $CqCNqr$ and $CpCNqr$. The first of these is a thesis, but the second is not true, for it has no two antecedents of the type p and Np. The expression $CCNpqq$ therefore, which leads to this not-true consequence, must be rejected. We begin the disproof from the top, successively applying according to the given transformations the theses S1, S5, S7, and S3:

S1. $p/CCNpqq,\ q/r \times (11)$
(11) $CCCNpqqCNCCNpqqr$

S5. $p/CNpq \times (12)$
(12) $CCNCCNpqqrCCNpqCNqr$

S7. $p/Np,\ r/CNqr \times (13)$
(13) $CCCNpqCNqrCNNpCNqr$

S3. $q/CNqr \times (14)$
(14) $CCNNpCNqrCpCNqr$.

Now we must disprove the expression $CpCNqr$; we need for this purpose the new theses S14 and S15 and the axiom of rejection.

S14. $p/NNCpp,\ q/p \times CS15-(15)$
(15) $CCNNCpppp$

(15) $\times C(*16)-*S16$
(*16) $CNNCppp$

S14. $p/CpCNpq$, $q/CNNCppp \times CS1-(17)$

(17) *CCCpCNpqCNNCpppCNNCppp*

$(17) \times C(*18)-(*16)$

(*18) *CCpCNpqCNNCppp*

$(*18) \times (*19)$ $p/CpCNpq$, $q/NCpp$, r/p

(*19) *CpCNqr*

Having rejected *CpCNqr*, we can now successively reject its antecedents till we reach the original expression *CCNpqq*.

$(14) \times C(*20)-(*19)$

(*20) *CNNpCNqr*

$(13) \times C(*21)-(*20)$

(*21) *CCNpqCNqr*

$(12) \times C(*22)-(*21)$

(*22) *CNCCNpqqr*

$(11) \times C(*23)-(*22)$

(*23) *CCNpqq*

In this way you can disprove any not-true expression of the *C–N*-system. All these deductions could have been made shorter, but I was anxious to show the method implied in the proof of decision. This method enables us to decide effectively, on the basis of only fifteen fundamental theses, S1–S15, and the axiom of rejection, whether a given significant expression of the *C–N*-system should be asserted or rejected. As all the other functors of the theory of deduction may be defined by *C* and *N*, all significant expressions of the theory of deduction are decidable on an axiomatic basis. A system of axioms from which the fifteen fundamental theses can be drawn is complete in this sense, that all true expressions of the system can be deduced in it. Of this kind is the system of three axioms set out in section 23, and also the system of those three axioms on which transformation IV is based, viz. *CCCpqrCNpr*, *CCCpqrCqr*, and *CCNprCCqrCCpqr*.

The proof of theorem (TA), according to which every significant expression of the Aristotelian logic can be reduced to elementary expressions, is implicitly contained in the proof of the analogous theorem for the theory of deduction. If we take instead of the Greek letters used in our transformations I–VII (except the final variable in transformation I) propositional

expressions of the Aristotelian logic, we can apply those transformations to them in the same way as to expressions of the theory of deduction. This can easily be seen in the example of *CCNAabAbaIab*. We get:

CCNAabAbaIab $\sim$ *CNCCNAabAbaIabp* by I;
CNCCNAabAbaIabp $\sim$ *CCNAabAbaCNIabp* „ III;
CCNAabAbaCNIabp $\sim$ *CNNAabCNIabp*, *CAbaCNIabp* by IV;
CNNAabCNIabp $\sim$ *CAabCNIabp* by II.

Instead of *NAab* we can always write *Oab*, and *Eab* instead of *NIab*. In what follows, however, it will be more convenient to employ forms with *N*.

Both elementary expressions, *CAabCNIabp* and *CAbaCNIabp*, to which *CCNAabAbaIab* has been reduced, have a propositional variable as their last term. This variable is introduced by transformation I. We can get rid of it by the following deductively equivalent transformations where π is a propositional variable not occurring in either α or β:

VIII. $C\alpha C\beta\pi \sim C\alpha N\beta$ with respect to S17 and S18,
IX. $C\alpha CN\beta\pi \sim C\alpha\beta$ „ „ „ S19 and S20.

Theses for transformation VIII:

S17. *CCpCqNqCpNq*
S18. *CCpNqCpCqr*.

Theses for transformation IX:

S19. *CCpCNqqCpq*
S20. *CCpqCpCNqr*.

When $C\alpha C\beta\pi$ is asserted, we get from it by substituting $N\beta$ for π the expression $C\alpha C\beta N\beta$, and then $C\alpha N\beta$ by S17; and conversely from $C\alpha N\beta$ the expression $C\alpha C\beta\pi$ by S18. When $C\alpha C\beta\pi$ is rejected, we get by S18 $CC\alpha N\beta C\alpha C\beta\pi$, therefore $C\alpha N\beta$ must be rejected; and conversely, when $C\alpha N\beta$ is rejected, we get by S17 $CC\alpha C\beta N\beta C\alpha N\beta$, therefore $C\alpha C\beta N\beta$ must be rejected and consequently $C\alpha C\beta\pi$. Transformation IX can be explained in the same way. This we can apply directly to our example. Take *Aab* for α, *Iab* for β, and *p* for π; you get *CAabIab*. In the same way from *CAbaCNIabp* results *CAbaIab*. If we have an expression with more antecedents than two, e.g. with *n* antecedents, we must first reduce by repeated application of transformation VII the $n-1$ antecedents to one antecedent, and then apply transformation VIII or IX. Take, for instance, the following example:

CNIabCAcbCAdcCIadp ~ *CNCNIabNAcbCAdcCIadp* by VII;

CNCNIabNAcbCAdcCIadp ~ *CNCNCNIabNAcbNAdcCIadp* by VII;

CNCNCNIabNAcbNAdcCIadp ~ *CNCNCNIabNAcbNAdcNIad* by VIII;

CNCNCNIabNAcbNAdcNIad ~ *CNCNIabNAcbCAdcNIad* by VII;

CNCNIabNAcbCAdcNIad ~ *CNIabCAcbCAdcNIad* „ VII.

Theorem (TA) is now fully proved; we can proceed therefore to our main subject, the proof of decision of the Aristotelian syllogistic.

§ 33. *Elementary expressions of the syllogistic*

According to theorem (TA), every significant expression of the Aristotelian syllogistic can be reduced in a deductively equivalent way to a set of elementary expressions, i.e. expressions of the form

$$C\alpha_1 C\alpha_2 C\alpha_3 \ldots C\alpha_{n-1}\alpha_n,$$

where all the α's are simple expressions of the syllogistic, i.e. expressions of the type *Aab*, *Iab*, *Eab* or *NIab*, and *Oab* or *NAab*. Now I shall show that every elementary expression of the syllogistic is decidable, i.e. either asserted or rejected. I shall first prove that all the simple expressions, except expressions of the type *Aaa* and *Iaa*, are rejected. We have already seen (section 27, formula *61) that *Iac* is rejected. Here are the proofs of rejection of the other expressions:

*100 × *61. *c*/*b*

*100. *Iab*

8 × *C**101–*100 (8. *CAabIab*)

*101. *Aab*

IV. *p*/*Aaa*, *q*/*Iab* × *C*1–102 (IV. *CpCNpq*)

102. *CNAaaIab*

102 × *C**103–*100

*103. *NAaa* (= *Oaa*)

*103 × *104. *b*/*a*

*104. *NAab* (= *Oab*)

IV. *p*/*Iaa*, *q*/*Iab* × *C*2–105

105. *CNIaaIab*

105 × *C**106–*100

*106. *NIaa* (= *Eaa*)

*106 × *107. *b*/*a*

*107. *NIab* (= *Eab*).

Turning now to compound elementary expressions I shall successively investigate all the possible cases, omitting the formal proofs where it is possible, and giving only hints how they could be done. Six cases have to be investigated.

First case: The consequent α_n is negative, and all the antecedents are affirmative. Such expressions are rejected.

Proof: By identifying all the variables occurring in the expression with *a*, all the antecedents become true, being laws of identity *Aaa* or *Iaa*, and the consequent becomes false. We see that for the solution of this case the laws of identity are essential.

Second case: The consequent is negative, and only one of the antecedents is negative. This case may be reduced to the case with only affirmative elements, and such cases, as we shall see later, are always decidable.

Proof: Expressions of the form $C\alpha CN\beta N\gamma$ are deductively equivalent to expressions of the form $C\alpha C\gamma\beta$ with respect to the theses *CCpCNrNqCpCqr* and *CCpCqrCpCNrNq*. This is true not only for one affirmative antecedent α, but for any number of them.

Third case: The consequent is negative, and more than one antecedent is negative. Expressions of this kind can be reduced to simpler expressions, and eventually to the second case. The solution of this case requires Słupecki's rule of rejection.

Proof: Let us suppose that the original expression is of the form $CN\alpha CN\beta C\gamma \ldots N\rho$. This supposition can always be made, as any antecedent may be moved to any place whatever. We reduce this expression to two simpler expressions $CN\alpha C\gamma \ldots N\rho$ and $CN\beta C\gamma \ldots N\rho$, omitting the second or the first antecedent respectively. If these expressions have more negative antecedents than one we repeat the same procedure till we get formulae with only one negative antecedent. As such formulae according to the second case are deductively equivalent to decidable affirmative expressions, they are always either asserted or rejected. If only one of them is asserted, the original expression must be asserted too, for by the law of simplification we can add to this asserted formula all the other negative antecedents which were previously omitted. If, however, all the formulae with one negative antecedent are rejected, we gather from them by repeated application of Słupecki's rule of rejection that the original expression must be rejected. Two examples will explain the matter thoroughly.

First example: *CNAabCNAbcCNIbdCIbcNAcd*, a thesis.

We reduce this expression to (1) and (2):

(1) *CNAabCNIbdCIbcNAcd*, (2) *CNAbcCNIbdCIbcNAcd*.

In the same way we reduce (1) to (3) and (4):

(3) *CNAabCIbcNAcd*, (4) *CNIbdCIbcNAcd*,

and (2) to (5) and (6):

(5) *CNAbcCIbcNAcd*, (6) *CNIbdCIbcNAcd*.

Now the last expression is a thesis; it is the mood Ferison of the third figure. Putting in *CpCqp* (6) for *p*, and *NAbc* for *q*, we get (2), and applying *CpCqp* once more by putting (2) for *p*, and *NAab* for *q*, we reach the original thesis.

Second example: *CNAabCNAbcCNIcdCIbdNAad*, not a thesis. We reduce this expression as in the foregoing example:

(1) *CNAabCNIcdCIbdNAad*, (2) *CNAbcCNIcdCIbdNAad*;

then we reduce (1) to (3) and (4), and (2) to (5) and (6):

(3) *CNAabCIbdNAad*, (4) *CNIcdCIbdNAad*,
(5) *CNAbcCIbdNAad*, (6) *CNIcdCIbdNAad*.

None of the above formulae with one negative antecedent is a thesis, as can be proved by reducing them to the case with only affirmative elements. Expressions (3), (4), (5), and (6) are rejected. Applying the rule of Słupecki, we gather from the rejected expressions (5) and (6) that (2) must be rejected, and from the rejected expressions (3) and (4) that (1) must be rejected. But if (1) and (2) are rejected, then the original expression must be rejected too.

Fourth case: The consequent is affirmative, and some (or all)

antecedents are negative. This case can be reduced to the third.

Proof: Expressions of the form $C\alpha CN\beta\gamma$ are deductively equivalent to expressions of the form $C\alpha CN\beta CN\gamma NAaa$ on the ground of the theses *CCpCNqrCpCNqCNrNAaa* and *CCpCNqCNrNAaaCpCNqr*, as *NAaa* is always false.

All the cases with negative elements are thus exhausted.

Fifth case: All the antecedents are affirmative, and the consequent is a universal affirmative proposition. Several sub-cases have to be distinguished.

(*a*) The consequent is *Aaa*; this expression is asserted, for its consequent is true.

(*b*) The consequent is *Aab*, and *Aab* is also one of the antecedents. The expression is of course asserted.

In what follows it is supposed that *Aab* does not occur as antecedent.

(*c*) The consequent is *Aab*, but no antecedent is of the type *Aaf* with *f* different from *a* (and from *b*, of course). Such expressions are rejected.

Proof: By identifying all variables different from *a* and *b* with *b*, we can only get the following antecedents:

Aaa, *Aba*, *Abb*, *Iaa*, *Iab*, *Iba*, *Ibb*.

(We cannot get *Aab*, for no antecedent is of the type *Aaf*, *f* being different from *a*.) Premisses *Aaa*, *Abb*, *Iaa*, *Ibb* can be omitted as true. (If there are no other premisses, the expression is rejected, as in the first case.) If there is *Iba* besides *Iab*, one of them may be omitted, as they are equivalent to each other. If there is *Aba*, both *Iab* and *Iba* may be omitted, as *Aba* implies them both. After these reductions only *Aba* or *Iab* can remain as antecedents. Now it can be shown that both implications,

CAbaAab and *CIabAab*,

are rejected on the ground of our axiom of rejection:

X. *p*/*Acb*, *q*/*Aba*, *r*/*Iac*, *s*/*Aab* × *C*27–108

108. *CCAabAbaCKAcbAabIac* (X. *CCKpqrCCsqCKpsr*; 27. *CKAcbAbaIac*)

108 × *C**109–*59

*109. *CAabAba*

*109 × *110. *b*/*a*, *a*/*b*

*110. *CAbaAab*.

If *CAbaAab* is rejected, then *CIabAab* must be rejected too, for *Iab* is a weaker premiss than *Aba*.

(*d*) The consequent is *Aab*, and there are antecedents of the type *Aaf* with *f* different from *a*. If there is a chain leading from *a* to *b*, the expression is asserted on the ground of axiom 3, the mood Barbara; if there is no such chain, the expression is rejected.

Proof: By a chain leading from *a* to *b* I understand an ordered series of universal affirmative premisses:

$$Aac_1,\ Ac_1c_2,\ \ldots,\ Ac_{n-1}c_n,\ Ac_nb,$$

where the first term of the series has *a* as its first argument, the last term *b* as its second argument, and the second argument of every other term is identical with the first argument of its successor. It is evident that from a series of such expressions *Aab* results by repeated application of the mood Barbara. If, therefore, there is a chain leading from *a* to *b*, the expression is asserted; if there is no such chain, we can get rid of antecedents of the type *Aaf*, identifying their second argument with *a*. The expression is reduced in this way to the sub-case (*c*), which was rejected.

Sixth case: All the antecedents are affirmative, and the consequent is a particular affirmative proposition. Here also we have to distinguish several sub-cases.

(*a*) The consequent is *Iaa*; the expression is asserted, for its consequent is true.

(*b*) The consequent is *Iab*, and as antecedent occurs either *Aab*, or *Aba*, or *Iab*, or *Iba*; it is obvious that in all these cases the expression must be asserted.

In what follows it is supposed that none of the above four premisses occurs as antecedent.

(*c*) The consequent is *Iab*, and no antecedent is of the type *Afa*, *f* different from *a*, or of the type *Agb*, *g* different from *b*. The expression is rejected.

Proof: We identify all variables different from *a* and *b* with *c*; then we get, besides true premisses of the type *Acc* or *Icc*, only the following antecedents:

$$Aac,\ Abc,\ Iac,\ Ibc.$$

Aac implies *Iac*, and *Abc* implies *Ibc*. The strongest combination

of premisses is therefore *Aac* and *Abc*. From this combination, however, *Iab* does not result, as the formula

$$CAacCAbcIab$$

is equivalent to our axiom of rejection.

(*d*) The consequent is *Iab*, and among the antecedents there are expressions of the type *Afa* (*f* different from *a*), but not of the type *Agb* (*g* different from *b*). If there is *Abe* or *Ibe* (*Ieb*), and a chain leading from *e* to *a*:

(α) $Abe;\ Aee_1,\ Ae_1e_2,\ \ldots,\ Ae_na,$

(β) $Ibe;\ Aee_1,\ Ae_1e_2,\ \ldots,\ Ae_na,$

we get from (α) *Abe* and *Aea*, and therefore *Iab* by the mood Bramantip, and from (β) *Ibe* and *Aea*, and therefore *Iab* by the mood Dimaris. In both cases the expression is asserted. If, however, the conditions (α) and (β) are not fulfilled, we can get rid of antecedents of the type *Afa* by identifying their first arguments with *a*, and the expression must be rejected according to sub-case (*c*).

(*e*) The consequent is *Iab*, and among the antecedents there are expressions of the type *Agb* (*g* different from *b*), but not of the type *Afa* (*f* different from *a*). This case can be reduced to sub-case (*d*), as *a* and *b* are symmetrical with respect to the consequent *Iab*.

(*f*) The consequent is *Iab*, and among the antecedents there are expressions of the type *Afa* (*f* different from *a*), and expressions of the type *Agb* (*g* different from *b*). We may suppose that the conditions (α) and (β) are not fulfilled for *Afa*, or the analogous conditions for *Agb* either; otherwise, as we already know, the original expression would be asserted. Now, if there is *Aca* and a chain leading from *c* to *b*:

(γ) $Aca;\ Acc_1,\ Ac_1c_2,\ \ldots,\ Ac_nb,$

or *Adb* and a chain leading from *d* to *a*:

(δ) $Adb;\ Add_1,\ Ad_1d_2,\ \ldots,\ Ad_na,$

we get from (γ) *Aca* and *Acb*, from (δ) *Adb* and *Ada*, and therefore in both cases *Iab* by the mood Darapti. Further, if there is an antecedent *Icd* (or *Idc*) and two chains, one leading from *c* to *a*, and another from *d* to *b*:

$$(\epsilon)\ \begin{cases} Icd;\ Acc_1,\ Ac_1c_2,\ \ldots,\ Ac_na, \\ Icd;\ Add_1,\ Ad_1d_2,\ \ldots,\ Ad_nb, \end{cases}$$

we get by the first chain the premiss *Aca*, by the second chain the premiss *Adb*, and both premisses yield together with *Icd* the conclusion *Iab* on the basis of the polysyllogism:

CIcdCAcaCAdbIab.

We prove the polysyllogism by deducing *Iad* from *Icd* and *Aca* by the mood Disamis, and then *Iab* from *Iad* and *Adb* by the mood Darii. In all these cases the original expression must be asserted. If, however, none of the conditions (γ), (δ), or (ϵ) is satisfied, we can get rid of expressions of the type *Afa* and *Agb* by identifying their first arguments with *a* or with *b* respectively, and the original expression must be rejected according to sub-case (c). All possible cases are now exhausted, and it is proved that every significant expression of the Aristotelian syllogistic is either asserted or rejected on the basis of our axioms and rules of inference.

§ 34. *An arithmetical interpretation of the syllogistic*

In 1679 Leibniz discovered an arithmetical interpretation of the Aristotelian syllogistic which deserves our attention from the historical as well as from the systematic point of view.[1] It is an isomorphic interpretation. Leibniz did not know that the Aristotelian syllogistic could be axiomatized, and he knew nothing about rejection and its rules. He only tested some laws of conversion and some syllogistic moods in order to be sure that his interpretation was not wrong. It seems, therefore, to be a mere coincidence that his interpretation satisfies our asserted axioms 1–4, the axiom of rejection *59, and the rule of Słupecki. In any case it is strange that his philosophic intuitions, which guided him in his research, yielded such a sound result.

Leibniz's arithmetical interpretation is based on a correlation of variables of the syllogistic with ordered pairs of natural numbers prime to each other. To the variable *a*, for instance, correspond two numbers, say a_1 and a_2, prime to each other; to the variable *b* correspond two other numbers, say b_1 and b_2, also prime to each other. The premiss *Aab* is true when and only when a_1 is divisible by b_1, and a_2 is divisible by b_2. If one of these conditions is not satisfied, *Aab* is false, and therefore *NAab* is

[1] See L. Couturat, *Opuscules et fragments inédits de Leibniz*, Paris (1903), pp. 77 seq. Cf. also J. Łukasiewicz, 'O sylogistyce Arystotelesa' (On Aristotle's Syllogistic), *Comptes Rendus de l'Acad. des Sciences de Cracovie*, xliv, No. 6 (1939), p. 220.

true. The premiss *Iab* is true when and only when a_1 is prime to b_2, and a_2 is prime to b_1. If one of these conditions is not satisfied, *Iab* is false, and therefore *NIab* is true.

It can easily be seen that our asserted axioms 1–4 are verified. Axiom 1, *Aaa*, is verified, for every number is divisible by itself. Axiom 2, *Iaa*, is verified, for it is supposed that the two numbers corresponding to *a*, a_1 and a_2, are prime to each other. Axiom 3, the mood Barbara *CKAbcAabAac*, is also verified, since the relation of divisibility is transitive. Axiom 4, the mood Datisi *CKAbcIbaIac*, is verified too; for if b_1 is divisible by c_1, b_2 is divisible by c_2, b_1 is prime to a_2, and b_2 is prime to a_1, then a_1 must be prime to c_2, and a_2 must be prime to c_1. For if a_1 and c_2 had a common factor greater than 1, a_1 and b_2 would also have the same common factor, since b_2 contains c_2. But this is against the supposition that a_1 is prime to b_2. In the same way we prove that a_2 must be prime to c_1.

It is also easy to show that the axiom *59 *CKAcbAabIac* must be rejected. Take as examples the following numbers:

$$a_1 = 15,\ b_1 = 3,\ c_1 = 12,$$
$$a_2 = 14,\ b_2 = 7,\ c_2 = 35.$$

Acb is true, for c_1 is divisible by b_1 and c_2 is divisible by b_2; *Aab* is also true, for a_1 is divisible by b_1 and a_2 is divisible by b_2; but the conclusion *Iac* is not true, for a_1 and c_2 are not prime to each other.

The verification of Słupecki's rule of rejection is more complicated. I shall explain the matter with the help of an example. Let us take as the rejected expressions,

(*1) *CNAabCNIcdCIbdNAad* and (*2) *CNIbcCNIcdCIbdNAad.*

From them we get, by the rule of Słupecki,

$$*CN\alpha\gamma,\ *CN\beta\gamma \rightarrow *CN\alpha CN\beta\gamma,$$

a third rejected expression,

(*3) *CNAabCNIbcCNIcdCIbdNAad.*

Expression (1) is disproved, for instance by the following set of numbers:

$$(4)\ \begin{cases} a_1 = 4,\ b_1 = 7,\ c_1 = 3,\ d_1 = 4, \\ a_2 = 9,\ b_2 = 5,\ c_2 = 8,\ d_2 = 3. \end{cases}$$

It can easily be proved that according to this interpretation *Aab* is false (since 4 is not divisible by 7), and therefore *NAab* is true; *Icd* is false (since c_2 is not prime to d_1), and therefore *NIcd* is true; *Ibd* is true (for both pairs of numbers, b_1 and d_2, b_2 and d_1, are prime to each other); but *NAad* is false, because *Aad* is true (a_1 being divisible by d_1, and a_2 by d_2). All the antecedents are true, the consequent is false; therefore expression (1) is disproved.

The same set of numbers does not disprove expression (2), because *Ibc* is true (as both pairs of numbers, b_1 and c_2, and b_2 and c_1, are prime to each other), and therefore *NIbc* is false. But if the antecedent of an implication is false, the implication is true. In order to disprove expression (2) we must take another set of numbers, for instance the following:

$$(5)\ \begin{cases} a_1 = 9,\ b_1 = 3,\ c_1 = 8,\ d_1 = 3, \\ a_2 = 2,\ b_2 = 2,\ c_2 = 5,\ d_2 = 2. \end{cases}$$

According to this interpretation all the antecedents of expression (2) are true, and the consequent is false; the expression is therefore disproved. But this second set of numbers does not disprove expression (1), because *Aab* is true, and therefore *NAab* is false, and a false antecedent yields a true implication. Neither, therefore, of the sets (4) and (5) disproves expression (3), which contains *NAab* as well as *NIbc*.

There is a general method that enables us to disprove expression (3) when expressions (1) and (2) are disproved.[1] First, we write down all the prime numbers which make up the sets of numbers disproving (1) and (2). We get for (1) the series 2, 3, 5, and 7, and for (2) the series 2, 3, and 5. Secondly, we replace the numbers of the second series by new primes, all different from the primes of the first series, for instance: 2 by 11, 3 by 13, and 5 by 17. We get thus a new set of numbers:

$$(6)\ \begin{cases} a_1 = 13.13,\ b_1 = 13,\ c_1 = 11.11.11,\ d_1 = 13, \\ a_2 = 11,\quad b_2 = 11,\ c_2 = 17,\qquad d_2 = 11. \end{cases}$$

This set also disproves (2), since the relations of divisibility and primeness remain the same as they were before the replacement.

[1] This method was discovered by Słupecki, op. cit., pp. 28–30.

Thirdly, we multiply the numbers of corresponding variables occurring in the sets (4) and (6). We thus get a new set:

$$(7)\ \begin{cases} a_1 = 4.13.13,\ b_1 = 7.13,\ c_1 = 3.11.11.11,\ d_1 = 4.13, \\ a_2 = 9.11,\qquad b_2 = 5.11,\ c_2 = 8.17,\qquad d_2 = 3.11. \end{cases}$$[1]

This set disproves (3). For it is evident, first, that if to the premiss *Aef* or *Ief* there corresponds the set of numbers

$$e_1, e_2, f_1, f_2, \quad e_1 \text{ prime to } e_2, f_1 \text{ prime to } f_2,$$

and there is another set of numbers

$$e_1', e_2', f_1', f_2', \quad e_1' \text{ prime to } e_2', f_1' \text{ prime to } f_2',$$

all of them composed of different primes from the numbers of the first set, then the product of e_1 and e_1', i.e. $e_1.e_1'$, must be prime to the product of e_2 and e_2', i.e. $e_2.e_2'$, and $f_1.f_1'$ prime to $f_2.f_2'$. Secondly, if *Aef* is verified by the first set, i.e. if e_1 is divisible by f_1, and e_2 by f_2, and the same is true of the second set, so that e_1' is divisible by f_1', and e_2' by f_2', then $e_1.e_1'$ must be divisible by $f_1.f_1'$, and $e_2.e_2'$ by $f_2.f_2'$. Again, if *Ief* is verified by the first set, i.e. e_1 is prime to f_2, and e_2 is prime to f_1, and the same is true of the second set, so that e_1' is prime to f_2', and e_2' is prime to f_1', then $e_1.e_1'$ must be prime to $f_2.f_2'$ and $e_2.e_2'$ prime to $f_1.f_1'$, since all the numbers of the second set are prime to the numbers of the first set. On the contrary, if only one of the conditions for divisibility or primeness is not satisfied, the respective premisses must be false. It can be seen in our example that *Aad* and *Icd* are verified by (7), for they are verified by (4) and (6), and *Ibc* is disproved both by (4) and (6), and therefore also by (7). *Aab* is disproved only by (4) (but this suffices to disprove it by (7)), and *Ibc* is disproved only by (6) (but this also suffices to disprove it by (7)). This procedure may be applied to any case of the kind, and therefore Słupecki's rule is verified by the Leibnizian interpretation.

Leibniz once said that scientific and philosophic controversies could always be settled by a calculus. It seems to me that his famous 'calculemus' is connected with the above arithmetical interpretation of the syllogistic rather than with his ideas on mathematical logic.

[1] If there is a variable occurring in one of the disproved expressions but not in the other, we simply take its corresponding numbers after eventual replacement.

§ 35. *Conclusion*

The results we have reached on the basis of an historical and systematic investigation of the Aristotelian syllogistic are at more than one point different from the usual presentation. Aristotle's logic was not only misrepresented by logicians who came from philosophy, since they wrongly identified it with the traditional syllogistic, but also by logicians who came from mathematics. In text-books of mathematical logic one can read again and again that the law of conversion of the *A*-premiss and some syllogistical moods derived by this law, like Darapti or Felapton, are wrong. This criticism is based on the mistaken notion that the Aristotelian universal affirmative premiss 'All *a* is *b*' means the same as the quantified implication 'For all *c*, if *c* is *a*, then *c* is *b*', where *c* is a singular term, and that the particular affirmative premiss 'Some *a* is *b*' means the same as the quantified conjunction 'For some *c*, *c* is *a* and *c* is *b*', where *c* is again a singular term. If one accepts such an interpretation, one can say of course that the law *CAabIba* is wrong, because *a* may be an empty term, so that no *c* is *a*, and the above quantified implication becomes true (for its antecedent is false), and the above quantified conjunction becomes false (for one of its factors is false). But all this is an imprecise misunderstanding of the Aristotelian logic. There is no passage in the *Analytics* that would justify such an interpretation. Aristotle does not introduce into his logic singular or empty terms or quantifiers. He applies his logic only to universal terms, like 'man' or 'animal'. And even these terms belong only to the application of the system, not to the system itself. In the system we have only expressions with variable arguments, like *Aab* or *Iab*, and their negations, and two of these expressions are primitive terms and cannot be defined; they have only those properties that are stated by the axioms. For the same reason such a controversy as whether the Aristotelian syllogistic is a theory of classes or not is in my opinion futile. The syllogistic of Aristotle is a theory neither of classes nor of predicates; it exists apart from other deductive systems, having its own axiomatic and its own problems.

I have tried to set forth this system free from foreign elements. I do not introduce into it singular, empty, or negative terms, as Aristotle has not introduced them. I do not introduce quantifiers either; I have only tried to explain some ideas of Aristotle by the help of quantifiers. In formal proofs I employ theses of the theory of deduction, since Aristotle uses them intuitively in his proofs, and I employ rejection, because Aristotle himself rejects some formulae and even states a rule of rejection. Wherever in Aristotle's exposition there was something not completely correct, I have been anxious to correct the flaws of his exposition, e.g. some unsatisfactory proofs by *reductio per impossibile*, or the rejection through concrete terms. It has been my intention to build up the original system of the Aristotelian syllogistic on the lines laid down by the author himself, and in accordance with the requirements of modern formal logic. The crown of the system is the solution of the problem of decision, and that was made possible by Słupecki's rule of rejection, not known to Aristotle or to any other logician.

The syllogistic of Aristotle is a system the exactness of which surpasses even the exactness of a mathematical theory, and this is its everlasting merit. But it is a narrow system and cannot be applied to all kinds of reasoning, for instance to mathematical arguments. Perhaps Aristotle himself felt that his system was not fitted for every purpose, for he added later to the theory of assertoric syllogisms a theory of modal syllogisms.[1] This was of course an extension of logic, but probably not in the right direction. The logic of the Stoics, the inventors of the ancient form of the propositional calculus, was much more important than all the syllogisms of Aristotle. We realize today that the theory of deduction and the theory of quantifiers are the most fundamental branches of logic.

Aristotle is not responsible for the fact that for many centuries his syllogistic, or rather a corrupt form of his syllogistic, was the sole logic known to philosophers. He is not responsible either for the fact that the influence of his logic on philosophy was, as it seems to me, disastrous. At the bottom of this disastrous influence there lies, in my opinion, the prejudice that every proposition has a subject and a predicate, like the premisses of Aristotelian logic. This prejudice, together with the criterion of truth known as *adaequatio rei et intellectus*, is the basis

[1] I take it that the theory of modal syllogisms expounded by Aristotle in Chapters 8–22 of Book I of the *Prior Analytics* was inserted later, since Chapter 23 is obviously an immediate continuation of Chapter 7.

of some famous but fantastic philosophical speculations. Kant divided all propositions (he calls them 'judgements') into analytic and synthetic according to the relation of the predicate of a proposition to its subject. His *Critique of Pure Reason* is chiefly an attempt to explain the problem how true synthetic *a priori* propositions are possible. Now some Peripatetics, for instance Alexander, were apparently already aware that there exists a large class of propositions having no subject and no predicate, such as implications, disjunctions, conjunctions, and so on.[1] All these may be called functorial propositions, since in all of them there occurs a propositional functor, like 'if—then', 'or', 'and'. These functorial propositions are the main stock of every scientific theory, and to them neither Kant's distinction of analytic and synthetic judgements nor the usual criterion of truth is applicable, for propositions without a subject or predicate cannot be immediately compared with facts. Kant's problem loses its importance and must be replaced by a much more important problem: How are true functorial propositions possible? It seems to me that here lies the starting-point for a new philosophy as well as for a new logic.

[1] In connexion with Aristotle's definition of the *πρότασις* Alexander writes, 11. 17: *εἰσὶ δὲ οὗτοι οἱ ὅροι προτάσεως οὐ πάσης ἀλλὰ τῆς ἁπλῆς τε καὶ καλουμένης κατηγορικῆς· τὸ γάρ τι κατά τινος ἔχειν καὶ τὸ καθόλου ἢ ἐν μέρει ἢ ἀδιόριστον ἴδια ταύτης· ἡ γὰρ ὑποθετικὴ οὐκ ἐν τῷ τι κατά τινος λέγεσθαι ἀλλ' ἐν ἀκολουθίᾳ ἢ μάχῃ τὸ ἀληθὲς ἢ τὸ ψεῦδος ἔχει.*

CHAPTER VI

ARISTOTLE'S MODAL LOGIC OF PROPOSITIONS

§ 36. *Introduction*

THERE are two reasons why Aristotle's modal logic is so little known. The first is due to the author himself: in contrast to the assertoric syllogistic which is perfectly clear and nearly free of errors, Aristotle's modal syllogistic is almost incomprehensible because of its many faults and inconsistencies. He devoted to this subject some interesting chapters of *De Interpretatione*, but the system of his modal syllogistic is expounded in Book I, chapters 3 and 8–22 of the *Prior Analytics*. Gohlke[1] suggested that these chapters were probably later insertions, because chapter 23 was obviously an immediate continuation of chapter 7. If he is right, the modal syllogistic was Aristotle's last logical work and should be regarded as a first version not finally elaborated by the author. This would explain the faults of the system as well as the corrections of Theophrastus and Eudemus, made perhaps in the light of hints given by the master himself.

The second reason is that modern logicians have not as yet been able to construct a universally acceptable system of modal logic which would yield a solid basis for the interpretation and appreciation of Aristotle's work. I have tried to construct such a system, different from those hitherto known, and built up upon Aristotle's ideas.[2] The present monograph on Aristotle's modal logic is written from the standpoint of this system.

A modal logic of terms presupposes a modal logic of propositions. This was not clearly seen by Aristotle whose modal syllogistic is a logic of terms; nevertheless it is possible to speak of an Aristotelian modal logic of propositions, as some of his theorems are general enough to comprise all kinds of proposition, and some others are expressly formulated by him with propositional variables. I shall begin with Aristotle's modal logic of propositions,

[1] Paul Gohlke, *Die Entstehung der Aristotelischen Logik*, Berlin (1936), pp. 88–94.

[2] Jan Łukasiewicz, 'A System of Modal Logic', *The Journal of Computing Systems*, vol. i, St. Paul (1953), pp. 111–49. A summary of this paper appeared under the same title in the *Proceedings of the XIth International Congress of Philosophy*, vol. xiv, Brussels (1953), pp. 82–87. A short description of the system is given below in § 49.

which is logically and philosophically far more important than his modal syllogistic of terms.

§ 37. *Modal functions and their interrelations*

There are four modal terms used by Aristotle: ἀναγκαῖον—'necessary', ἀδύνατον—'impossible', δυνατόν—'possible', and ἐνδεχόμενον—'contingent'. This last term is ambiguous: in the *De Interpretatione* it means the same as δυνατόν, in the *Prior Analytics* it has besides a more complicated meaning which I shall discuss later.

According to Aristotle, only propositions are necessary, impossible, possible, or contingent. Instead of saying: 'The proposition "*p*" is necessary', where "*p*" is the name of the proposition *p*, I shall use the expression: 'It is necessary that *p*', where *p* is a proposition. So, for instance, instead of saying: 'The proposition "man is an animal" is necessary', I shall say: 'It is necessary that man should be an animal.' I shall express the other modalities in a similar way. Expressions like: 'It is necessary that *p*', denoted here by *Lp*, or 'It is possible that *p*', denoted by *Mp*, I call 'modal functions'; *L* and *M*, which respectively correspond to the words 'it is necessary that' and 'it is possible that', are 'modal functors', *p* is their 'argument'. As modal functions are propositions, I say that *L* and *M* are proposition-forming functors of one propositional argument. Propositions beginning with *L* or their equivalents are called 'apodeictic', those beginning with *M* or their equivalents 'problematic'. Non-modal propositions are called 'assertoric'. This modern terminology and symbolism will help us to give a clear exposition of Aristotle's propositional modal logic.

Two of the modal terms, 'necessary' and 'possible', and their interrelations, are of fundamental importance. In the *De Interpretatione* Aristotle mistakenly asserts that possibility implies non-necessity, i.e. in our terminology:

(*a*) *If it is possible that p, it is not necessary that p.*[1] He later sees that this cannot be right, because he accepts that necessity implies possibility, i.e.:

(*b*) *If it is necessary that p, it is possible that p*, and from (*b*) and (*a*) there would follow by the hypothetical syllogism that

[1] *De int.* 13, 22ᵃ15 τῷ μὲν γὰρ δυνατῷ εἶναι τὸ ἐνδέχεσθαι εἶναι (ἀκολουθεῖ), καὶ τοῦτο ἐκείνῳ ἀντιστρέφει, καὶ τὸ μὴ ἀδύνατον εἶναι καὶ τὸ μὴ ἀναγκαῖον εἶναι.

(*c*) *If it is necessary that p, it is not necessary that p*, which is absurd.[1] After a further examination of the problem Aristotle rightly states that

(*d*) *If it is possible that p, it is not necessary that not p*,[2] but does not correct his former mistake in the text of *De Interpretatione*. This correction is given in the *Prior Analytics* where the relation of possibility to necessity has the form of an equivalence:

(*e*) *It is possible that p—if and only if—it is not necessary that not p.*[3]

I gather from this that the other relation, that of necessity to possibility, which is stated in the *De Interpretatione* as an implication,[4] is also meant as an equivalence and should be given the form:

(*f*) *It is necessary that p—if and only if—it is not possible that not p.*

If we denote the functor 'if and only if' by *Q*,[5] putting it before its arguments, and 'not' by *N*, we can symbolically express the relations (*e*) and (*f*) thus:

1. *QMpNLNp*, i.e. *Mp—if and only if—NLNp*,
2. *QLpNMNp*, i.e. *Lp—if and only if—NMNp*.

The above formulae are fundamental to any system of modal logic.

§ 38. *Basic modal logic*

Two famous scholastic principles of modal logic: *Ab oportere ad esse valet consequentia*, and *Ab esse ad posse valet consequentia*, were known to Aristotle without being formulated by him explicitly. The first principle runs in our symbolic notation (*C* is the sign of the functor 'if–then'):

3. *CLpp*, i.e. *If it is necessary that p, then p.*

The second reads:

[1] Ibid. 22ᵇ11 τὸ μὲν γὰρ ἀναγκαῖον εἶναι δυνατὸν εἶναι . . . 14 ἀλλὰ μὴν τῷ γε δυνατὸν εἶναι τὸ οὐκ ἀδύνατον εἶναι ἀκολουθεῖ, τούτῳ δὲ τὸ μὴ ἀναγκαῖον εἶναι· ὥστε συμβαίνει τὸ ἀναγκαῖον εἶναι μὴ ἀναγκαῖον εἶναι, ὅπερ ἄτοπον.

[2] Ibid. 22ᵇ22 λείπεται τοίνυν τὸ οὐκ ἀναγκαῖον μὴ εἶναι ἀκολουθεῖν τῷ δυνατὸν εἶναι.

[3] *An. pr.* i. 13, 32ᵃ25 τὸ 'ἐνδέχεται ὑπάρχειν' καὶ 'οὐκ ἀδύνατον ὑπάρχειν' καὶ 'οὐκ ἀνάγκη μὴ ὑπάρχειν', ἤτοι ταὐτὰ ἔσται ἢ ἀκολουθοῦντα ἀλλήλοις.

[4] *De int.* 13, 22ᵃ20 τῷ δὲ μὴ δυνατῷ μὴ εἶναι καὶ μὴ ἐνδεχομένῳ μὴ εἶναι τὸ ἀναγκαῖον εἶναι καὶ τὸ ἀδύνατον μὴ εἶναι (ἀκολουθεῖ).

[5] I usually denote equivalence by *E*, but as this letter has already another meaning in the syllogistic, I have introduced (p. 108) the letter *Q* for equivalence.

4. $CpMp$, i.e. *If p, it is possible that p.*

According to a passage of the *Prior Analytics*[1] Aristotle knows that from the assertoric negative conclusion 'Not p', i.e. Np, there results the problematic consequence 'It is possible that not p', i.e. MNp. We have therefore $CNpMNp$. Alexander, commenting on this passage, states as a general rule that existence implies possibility, i.e. $CpMp$, but not conversely, i.e. $CMpp$ should be rejected.[2] If we denote rejected expressions by an asterisk, we get the formula:[3]

*5. $CMpp$, i.e. *If it is possible that p, then p*—rejected.

The corresponding formulae for necessity are also stated by Alexander who says that necessity implies existence, i.e. $CLpp$, but not conversely, i.e. $CpLp$ should be rejected.[4] We get thus another rejected expression:

*6. $CpLp$, i.e. *If p, it is necessary that p*—rejected.

Formulae 1–6 are accepted by the traditional logic, and so far as I know, by all the modern logicians. They are, however, insufficient to characterize Mp and Lp as modal functions, because all the above formulae are satisfied if we interpret Mp as always true, i.e. as '*verum* of p', and Lp as always false, i.e. as '*falsum* of p'. With this interpretation a system built up on the formulae 1–6 would cease to be a modal logic. We cannot therefore assert Mp, i.e. accept that all problematic propositions are true, or assert NLp, i.e. accept that all apodeictic propositions are false; both expressions should be rejected, for any expression which cannot be asserted should be rejected. We get thus two additional rejected formulae:

*7. Mp, i.e. *It is possible that p*—rejected, and
*8. NLp, i.e. *It is not necessary that p*—rejected.

Both formulae may be called Aristotelian, as they are consequences of the presumption admitted by Aristotle that there exist

[1] *An. pr.* i. 16, 36ª15 φανερὸν δ' ὅτι καὶ τοῦ ἐνδέχεσθαι μὴ ὑπάρχειν γίγνεται συλλογισμός, εἴπερ καὶ τοῦ μὴ ὑπάρχειν. — ἐνδέχεσθαι means here the 'possible', not the 'contingent'.

[2] Alexander 209. 2 τὸ μὲν γὰρ ὑπάρχον καὶ ἐνδεχόμενον ἀληθὲς εἰπεῖν, τὸ δ' ἐνδεχόμενον οὐ πάντως καὶ ὑπάρχον.

[3] Asserted expressions are marked throughout the Chapters VI–VIII by arabic numerals without asterisks.

[4] Alexander 152. 32 τὸ γὰρ ἀναγκαῖον καὶ ὑπάρχον, οὐκέτι δὲ τὸ ὑπάρχον ἀναγκαῖον.

asserted apodeictic propositions. For, if $L\alpha$ is asserted, then $LNN\alpha$ must be asserted too, and from the principle of Duns Scotus $CpCNpq$ we get by substitution and detachment the asserted formulae $CNL\alpha p$ and $CNLNN\alpha p$. As p is rejected, $NL\alpha$ and $NLNN\alpha$ are rejected too, and consequently NLp and $NLNp$, i.e. Mp, must be rejected.

I call a system 'basic modal logic' if and only if it satisfies the formulae 1–8. I have shown that basic modal logic can be axiomatized on the basis of the classical calculus of propositions.[1] Of the two modal functors, M and L, one may be taken as the primitive term, and the other can be defined. Taking M as the primitive term and formula 2 as the definition of L, we get the following independent set of axioms of the basic modal logic:

4. $CpMp$ *5. $CMpp$ *7. Mp 9. $QMpMNNp$,

where 9 is deductively equivalent to formula 1 on the ground of the definition 2 and the calculus of propositions. Taking L as the primitive term and formula 1 as the definition of M, we get a corresponding set of axioms:

3. $CLpp$ *6. $CpLp$ *8. NLp 10. $QLpLNNp$,

where 10 is deductively equivalent to formula 2 on the ground of the definition 1 and the calculus of propositions. The derived formulae 9 and 10 are indispensable as axioms.

Basic modal logic is the foundation of any system of modal logic and must always be included in any such system. Formulae 1–8 agree with Aristotle's intuitions and are at the roots of our concepts of necessity and possibility; but they do not exhaust the whole stock of accepted modal laws. For instance, we believe that if a conjunction is possible, each of its factors should be possible, i.e. in symbols:

11. $CMKpqMp$ and 12. $CMKpqMq$,

and if a conjunction is necessary, each of its factors should be necessary, i.e. in symbols:

13. $CLKpqLp$ and 14. $CLKpqLq$.

None of these formulae can be deduced from the laws 1–8. Basic modal logic is an incomplete modal system and requires the addition of some new axioms. Let us see how it was supplemented by Aristotle himself.

[1] See pp. 114–17 of my paper on modal logic.

§ 39. *Laws of extensionality*

Aristotle's most important and—as I see it—most successful attempt to go beyond basic modal logic consisted in his accepting certain principles which may be called 'laws of extensionality for modal functors'. These principles are to be found in Book I, chapter 15 of the *Prior Analytics*, and are formulated in three passages. We read at the beginning of the chapter:

'First it has to be said that if (if α is, β must be), then (if α is possible, β must be possible too).'[1]

A few lines further Aristotle says referring to his syllogisms:

'If one should denote the premisses by α, and the conclusion by β, it would not only result that if α is necessary, then β is necessary, but also that if α is possible, then β is possible.'[2]

And at the end of the section he repeats:

'It has been proved that if (if α is, β is), then (if α is possible, then β is possible).'[3]

Let us first analyse these modal laws beginning with the second passage, which refers to syllogisms.

All Aristotelian syllogisms are implications of the form $C\alpha\beta$ where α is the conjunction of the two premisses and β the conclusion. Take as example the mood Barbara:

15. $\underbrace{CKAbaAcb}_{\alpha}\underbrace{Aca}_{\beta}$.

According to the second passage we get two modal theorems, in the form of implications taking $C\alpha\beta$ as the antecedent and $CL\alpha L\beta$ or $CM\alpha M\beta$ as the consequent, in symbols:

16. $C C\alpha\beta CL\alpha L\beta$ and 17. $CC\alpha\beta CM\alpha M\beta$.

The letters α and β stand here for the premisses and the conclusion of an Aristotelian syllogism. As in the final passage there is

[1] *An. pr.* i. 15, 34^a5 πρῶτον δὲ λεκτέον ὅτι εἰ τοῦ *A* ὄντος ἀνάγκη τὸ *B* εἶναι, καὶ δυνατοῦ ὄντος τοῦ *A* δυνατὸν ἔσται καὶ τὸ *B* ἐξ ἀνάγκης.

[2] Ibid. 34^a22 εἴ τις θείη τὸ μὲν *A* τὰς προτάσεις, τὸ δὲ *B* τὸ συμπέρασμα, συμβαίνοι ἂν οὐ μόνον ἀναγκαίου τοῦ *A* ὄντος ἅμα καὶ τὸ *B* εἶναι ἀναγκαῖον, ἀλλὰ καὶ δυνατοῦ δυνατόν.

[3] Ibid. 34^a29 δέδεικται ὅτι εἰ τοῦ *A* ὄντος τὸ *B* ἔστι, καὶ δυνατοῦ ὄντος τοῦ *A* ἔσται τὸ *B* δυνατόν.

no reference to syllogisms, we may treat these theorems as special cases of general principles which we get by replacing the Greek letters by propositional variables:

18. $CCpqCLpLq$ and 19. $CCpqCMpMq$.

Both formulae may be called in a wider sense 'laws of extensionality', the first for L, the second for M. The words 'in a wider sense' require an explanation.

The general law of extensionality, taken *sensu stricto*, is a formula of the classical calculus of propositions enlarged by the introduction of variable functors, and has the form:

20. $CQpqC\delta p\delta q$.

This means roughly speaking: If p is equivalent to q, then if δ of p, δ of q, where δ is any proposition-forming functor of one propositional argument, e.g. N. Accordingly, the strict laws of extensionality for L and M will have the form:

21. $CQpqCLpLq$ and 22. $CQpqCMpMq$.

These two formulae have stronger antecedents than formulae 18 and 19, and are easily deducible from them, 21 from 18, and 22 from 19, by means of the thesis $CQpqCpq$ and the principle of the hypothetical syllogism. It can be proved, however, on the ground of the calculus of propositions and the basic modal logic that conversely 18 is deducible from 21, and 19 from 22. I give here the full deduction of the L-formula:

The premisses:

23. $CCQpqrCpCCpqr$
24. $CCpqCCqrCpr$
25. $CCpCqCprCqCpr$
3. $CLpp$.

The deduction:

23. $r/CLpLq \times C21\text{–}26$
26. $CpCCpqCLpLq$
24. $p/Lp,\ q/p,\ r/CCpqCLpLq \times C3\text{–}C26\text{–}27$
27. $CLpCCpqCLpLq$
25. $p/Lp,\ q/Cpq,\ r/Lq \times C27\text{–}18$
18. $CCpqCLpLq$.

In a similar way 19 is deducible from 22 by means of the premisses *CCQpqrCNqCCpqr*, *CCpqCCqrCpr*, *CCNpCqCrpCqCrp*, and the transposition *CNMpNp* of the modal thesis *CpMp*.

We see from the above that, given the calculus of propositions and basic modal logic, formula 18 is deductively equivalent to the strict law of extensionality 21, and formula 19 to the strict law of extensionality 22. We are right, therefore, to call those formulae 'laws of extensionality in a wider sense'. Logically, of course, it makes no difference whether we complete the *L*-system of basic modal logic by the addition of *CCpqCLpLq* or by the addition of *CQpqCLpLq*; the same holds for the alternative additions to the *M*-system of *CCpqCMpMq* or *CQpqCMpMq*. Intuitively, however, the difference is great. Formulae 18 and 19 are not so evident as formulae 21 and 22. If *p* implies *q* but is not equivalent to it, it is not always true that if δ of *p*, δ of *q*; e.g. *CNpNq* does not follow from *Cpq*. But if *p* is equivalent to *q*, then always if δ of *p*, δ of *q*, i.e. if *p* is true, *q* is true, and if *p* is false, *q* is false; similarly if *p* is necessary, *q* is necessary, and if *p* is possible, *q* is possible. This seems to be perfectly evident, unless modal functions are regarded as intensional functions, i.e. as functions whose truth-values do *not* depend solely on the truth-values of their arguments. But what in this case the necessary and the possible would mean, is for me a mystery as yet.

§ 40. *Aristotle's proof of the M-law of extensionality*

In the last passage quoted above Aristotle says that he has proved the law of extensionality for possibility. He argues in substance thus: If α is possible and β impossible, then when α came to be, β would not come to be, and therefore α would be without β, which is against the premiss that if α is, β is.[1] It is difficult to recast this argument into a logical formula, as the term 'to come to be' has an ontological rather than a logical meaning. The comment, however, given on this argument by Alexander deserves a careful examination.

Aristotle defines the contingent as that which is not necessary and the supposed existence of which implies nothing impossible.[2]

[1] *An. pr.* i. 15, 34ᵃ8 εἰ οὖν τὸ μὲν δυνατόν, ὅτε δυνατὸν εἶναι, γένοιτ' ἄν, τὸ δ' ἀδύνατον, ὅτ' ἀδύνατον, οὐκ ἂν γένοιτο, ἅμα δ' εἰ τὸ *A* δυνατὸν καὶ τὸ *B* ἀδύνατον, ἐνδέχοιτ' ἂν τὸ *A* γενέσθαι ἄνευ τοῦ *B*, εἰ δὲ γενέσθαι, καὶ εἶναι.

[2] See below, p. 154, n. 3.

Alexander assimilates this Aristotelian definition of contingency to that of possibility by omitting the words 'which is not necessary'. He says 'that a β which is impossible cannot follow from an α which is possible may also be proved from the definition of possibility: that is possible, the supposed existence of which implies nothing impossible'.[1] The words 'impossible' and 'nothing' here require a cautious interpretation. We cannot interpret 'impossible' as 'not possible', because the definition would be circular; we must either take 'impossible' as a primitive term or, taking 'necessary' as primitive, define the expression 'impossible that *p*' by 'necessary that not *p*'. I prefer the second way and shall discuss the new definition on the ground of the *L*-basic modal logic. The word 'nothing' should be rendered by a universal quantifier, as otherwise the definition would not be correct. We get thus the equivalence:

28. *QMpΠqCCpqNLNq*.

That means in words: 'It is possible that *p*—if and only if—for all *q*, if (if *p*, then *q*), it is not necessary that not *q*.' This equivalence has to be added to the *L*-basic modal logic as the definition of *Mp* instead of the equivalence 1 which must now be proved as a theorem.

The equivalence 28 consists of two implications:

29. *CMpΠqCCpqNLNq* and 30. *CΠqCCpqNLNqMp*.

From 29 we get by the theorem *CΠqCCpqNLNqCCpqNLNq* and the hypothetical syllogism the consequence:

31. *CMpCCpqNLNq*,

and from 31 there easily results by the substitution *q*/*p*, *Cpp*, commutation and detachment the implication *CMpNLNp*. The converse implication *CNLNpMp* which, when combined with the original implication, would give the equivalence 1, cannot be proved otherwise than by means of the law of extensionality for *L*: *CCpqCLpLq*. As this proof is rather complicated, I shall give it in full.

[1] Alexander 177. 11 δεικνύοιτο δ' ἄν, ὅτι μὴ οἷόν τε δυνατῷ ὄντι τῷ *A* ἀδύνατον ἕπεσθαι τὸ *B*, καὶ ἐκ τοῦ ὁρισμοῦ τοῦ δυνατοῦ . . . δυνατόν ἐστιν, οὗ ὑποτεθέντος εἶναι οὐδὲν ἀδύνατον συμβαίνει διὰ τοῦτο.

The premisses:

18. *CCpqCLpLq*
24. *CCpqCCqrCpr*
30. *CΠqCCpqNLNqMp*
32. *CCpqCNqNp*
33. *CCpCqrCqCpr.*

The deduction:

18. *p/Nq, q/Np* × 34
34. *CCNqNpCLNqLNp*
    24. *p/Cpq, q/CNqNp, r/CLNqLNp* × C32–C34–35
35. *CCpqCLNqLNp*
    32. *p/LNq, q/LNp* × 36
36. *CCLNqLNpCNLNpNLNq*
    24. *p/Cpq, q/CLNqLNp, r/CNLNpNLNq* × C35–C36–37
37. *CCpqCNLNpNLNq*
    33. *p/Cpq, q/NLNp, r/NLNq* × C37–38
38. *CNLNpCCpqNLNq*
    38. *Π2q* × 39
39. *CNLNpΠqCCpqNLNq*
    24. *p/NLNp, q/ΠqCCpqNLNq, r/Mp* × C39–C30–40
40. *CNLNpMp.*

We can now prove the law of extensionality for *M*, which was the purpose of Alexander's argument. This law easily results from the equivalence 1 and thesis 37. We see besides that the proof by means of the definition with quantifiers is unnecessarily complicated. It suffices to retain definition 1 and to add to the *L*-system the *L*-law of extensionality in order to get the *M*-law of extensionality. In the same way we may get the *L*-law of extensionality, if we add the *M*-law of extensionality to the *M*-system and definition 2. The *L*-system is deductively equivalent to the *M*-system with the laws of extensionality as well as without them.

It is, of course, highly improbable that an ancient logician could have invented such an exact proof as that given above. But the fact that the proof is correct throws an interesting light on Aristotle's ideas of possibility. I suppose that he intuitively saw what may be shortly expressed thus: what is possible today, say a sea-fight, may become existent or actual tomorrow; but what is

impossible, can never become actual. This idea seems to lie at the bottom of Aristotle's proof and of Alexander's.

§ 41. *Necessary connexions of propositions*

The *L*-law of extensionality was formulated by Aristotle only once, together with the *M*-law, in the passage where he refers to syllogisms.[1]

According to Aristotle there exists a necessary connexion between the premisses α of a valid syllogism and its conclusion β. It would seem therefore that the laws of extensionality formulated above in the form:

16. *CCαβCLαLβ* and 17. *CCαβCMαMβ*,

should be expressed with necessary antecedents:

41. *CLCαβCLαLβ* and 42. *CLCαβCMαMβ*,

and the corresponding general laws of extensionality should run:

43. *CLCpqCLpLq* and 44. *CLCpqCMpMq.*

This is corroborated for the *M*-law by the first passage quoted above where we read: 'If (if α is, β must be), then (if α is possible, β is possible).'

Formulae 43 and 44 are weaker than the corresponding formulae with assertoric antecedents, 18 and 19, and can be got from them by the axiom *CLpp* and the hypothetical syllogism 24. It is not, however, possible to derive the stronger formulae conversely from the weaker. The problem is whether we should reject the stronger formulae 18 and 19, and replace them by the weaker formulae 43 and 44. To solve this problem we have to inquire into the Aristotelian concept of necessity.

Aristotle accepts that some necessary, i.e. apodeictic, propositions are true and should be asserted. Two kinds of asserted apodeictic proposition can be found in the *Analytics*: to the one kind there belong necessary connexions of propositions, to the other necessary connexions of terms. As example of the first kind any valid syllogism may be taken, for instance the mood Barbara:

(*g*) *If every b is an a, and every c is a b, then it is necessary that every c should be an a.*

Here the 'necessary' does not mean that the conclusion is an

[1] See p. 138, n. 2.

apodeictic proposition, but denotes a necessary connexion between the premisses of the syllogism and its assertoric conclusion. This is the so called 'syllogistic necessity'. Aristotle sees very well that there is a difference between syllogistic necessity and an apodeictic conclusion when he says, discussing a syllogism with an assertoric conclusion, that this conclusion is not 'simply' (ἁπλῶς) necessary, i.e. necessary in itself, but is necessary 'on condition', i.e. with respect to its premisses (τούτων ὄντων).[1] There are passages where he puts two marks of necessity into the conclusion saying, for instance, that from the premisses: 'It is necessary that every *b* should be an *a*, and some *c* is a *b*', there follows the conclusion: 'It is necessary that some *c* should be necessarily an *a*.'[2] The first 'necessary' refers to the syllogistic connexion, the second denotes that the conclusion is an apodeictic proposition.

By the way, a curious mistake of Aristotle should be noted: he says that nothing follows necessarily from a single premiss, but only from at least two, as in the syllogism.[3] In the *Posterior Analytics* he asserts that this has been proved,[4] but not even an attempt of proof is given anywhere. On the contrary, Aristotle himself states that 'If some *b* is an *a*, it is necessary that some *a* should be a *b*', drawing thus a necessary conclusion from only one premiss.[5]

I have shown that syllogistic necessity can be reduced to universal quantifiers.[6] When we say that in a valid syllogism the conclusion necessarily follows from the premisses, we want to state that the syllogism is valid for any matter, i.e. for all values of the variables occurring in it. This explanation, as I have found afterwards, is corroborated by Alexander who asserts that: 'syllogistic combinations are those from which something necessarily follows, and such are those in which for all matter the same comes to be'.[7] Syllogistic necessity reduced to universal quantifiers can

[1] *An. pr.* i. 10, 30^b32 τὸ συμπέρασμα οὐκ ἔστιν ἀναγκαῖον ἁπλῶς, ἀλλὰ τούτων ὄντων ἀναγκαῖον.

[2] Ibid. 9, 30^a37 τὸ μὲν Α παντὶ τῷ Β ὑπαρχέτω ἐξ ἀνάγκης, τὸ δὲ Β τινὶ τῷ Γ ὑπαρχέτω μόνον· ἀνάγκη δὴ τὸ Α τινὶ τῷ Γ ὑπάρχειν ἐξ ἀνάγκης.

[3] Ibid. 15, 34^a17 οὐ γὰρ ἔστιν οὐδὲν ἐξ ἀνάγκης ἑνός τινος ὄντος, ἀλλὰ δυοῖν ἐλαχίστοιν ,οἷον ὅταν αἱ προτάσεις οὕτως ἔχωσιν ὡς ἐλέχθη κατὰ τὸν συλλογισμόν.

[4] *An. post.* i. 3, 73^a7 ἑνὸς μὲν οὖν κειμένου δέδεικται ὅτι οὐδέποτ' ἀνάγκη τι εἶναι ἕτερον (λέγω δ' ἑνός, ὅτι οὔτε ὅρου ἑνὸς οὔτε θέσεως μιᾶς τεθείσης), ἐκ δύο δὲ θέσεων πρώτων καὶ ἐλαχίστων ἐνδέχεται.

[5] *An. pr.* i. 2, 25^a20 εἰ γὰρ τὸ Α τινὶ τῷ Β, καὶ τὸ Β τινὶ τῷ Α ἀνάγκη ὑπάρχειν.

[6] See § 5.

[7] Alexander 208. 16 συλλογιστικαὶ δὲ αἱ συζυγίαι αὗται αἱ ἐξ ἀνάγκης τι συνάγουσαι· τοιαῦται δέ, ἐν αἷς ἐπὶ πάσης ὕλης γίνεται τὸ αὐτό.

be eliminated from syllogistic laws, as will appear from the following consideration.

The syllogism (*g*) correctly translated into symbols would have the form:

(*h*) *LCKAbaAcbAca*,

which means in words:

(*i*) *It is necessary that* (*if every b is an a, and every c is a b, then every c should be an a*).

The sign of necessity in front of the syllogism shows that not the conclusion, but the connexion between the premisses and the conclusion is necessary. Aristotle would have asserted (*h*). Formula

(*j*) *CKAbaAcbLAca*,

which literally corresponds to the verbal expression (*g*), is wrong. Aristotle would have rejected it, as he rejects a formula with stronger premisses, viz.

(*k*) *CKAbaLAcbLAca*,

i.e. '*If every b is an a and it is necessary that every c should be a b, it is necessary that every c should be an a.*'[1]

By the reduction of necessity to universal quantifiers formula (*h*) can be transformed into the expression:

(*l*) *ΠaΠbΠcCKAbaAcbAca*,

i.e. 'For all *a*, for all *b*, for all *c* (if every *b* is an *a* and every *c* is a *b*, then every *c* is an *a*).' This last expression is equivalent to the mood Barbara without quantifiers:

(*m*) *CKAbaAcbAca*,

since a universal quantifier may be omitted when it stands at the head of an asserted formula.

Formulae (*h*) and (*m*) are not equivalent. It is obvious that (*m*) can be deduced from (*h*) by the principle *CLpp*, but the converse deduction is not possible without the reduction of necessity to universal quantifiers. This, however, cannot be done at all, if the above formulae are applied to concrete terms. Put, for instance,

[1] *An. pr.* i. 9, 30^a23 εἰ δὲ τὸ μὲν ΑΒ μὴ ἔστιν ἀναγκαῖον, τὸ δὲ ΒΓ ἀναγκαῖον, οὐκ ἔσται τὸ συμπέρασμα ἀναγκαῖον.

in (*h*) 'bird' for *b*, 'crow' for *a*, and 'animal' for *c*; we get the apodeictic proposition:

(*n*) *It is necessary that (if every bird is a crow and every animal is a bird, then every animal should be a crow).*

From (*n*) results the syllogism (*o*):

(*o*) *If every bird is a crow and every animal is a bird, then every animal is a crow,*

but from (*o*) we cannot get (*n*) by the transformation of necessity into quantifiers, as (*n*) does not contain variables which could be quantified.

And here we meet the first difficulty. It is easy to understand the meaning of necessity when the functor *L* is attached to the front of an asserted proposition containing free variables. In this case we have a general law, and we may say: this law we regard as necessary, because it is true of all objects of a certain kind, and does not allow of exception. But how should we interpret necessity, when we have a necessary proposition without free variables, and in particular, when this proposition is an implication consisting of false antecedents and of a false consequent, as in our example (*n*)? I see only one reasonable answer: we could say that whoever accepts the premisses of this syllogism is necessarily compelled to accept its conclusion. But this would be a kind of psychological necessity which is quite alien from logic. Besides it is extremely doubtful that anybody would accept evidently false propositions as true.

I know no better remedy for removing this difficulty than to drop everywhere the *L*-functor standing in front of an asserted implication. This procedure was already adopted by Aristotle who sometimes omits the sign of necessity in valid syllogistical moods.[1]

§ 42. *'Material' or 'strict' implication?*

According to Philo of Megara the implication 'If *p*, then *q*', i.e. *Cpq*, is true if and only if it does not begin with a true antecedent and end with a false consequent.[2] This is the so-called 'material' implication now universally accepted in the classical calculus of propositions. 'Strict' implication: 'It is necessary that

[1] See p. 10, n. 5.

[2] See p. 83, n. 1.

if *p*, then *q*', i.e. *LCpq*, is a necessary material implication and was introduced into symbolic logic by C. I. Lewis. By means of this terminology the problem we are discussing may be stated thus: Should we interpret the antecedent of the Aristotelian laws of extensionality as material, or as strict implication? In other words, should we accept the stronger formulae 18 and 19 (I call this the 'strong interpretation'), or should we reject them accepting the weaker formulae 43 and 44 (weak interpretation)?

Aristotle was certainly not aware of the difference between these two interpretations and of their importance for modal logic. He could not know Philo's definition of the material implication. But his commentator Alexander was very well acquainted with the logic of the Stoic–Megaric school and with the heated controversies about the meaning of the implication amidst the followers of this school. Let us then see his comments on our problem.

Commenting on the Aristotelian passage 'If (if α is, β must be), then (if α is possible, β must be possible)' Alexander emphasizes the necessary character of the premiss 'If α is, β must be'. It seems therefore that he would accept the weaker interpretation $CLC\alpha\beta CM\alpha M\beta$ and the weaker *M*-law of extensionality *CLCpqCMpMq*. But what he means by a necessary implication is different from strict implication in the sense of Lewis. He says that in a necessary implication the consequent should always, i.e. at any time, follow from the antecedent, so that the proposition 'If Alexander is, he is so and so many years old' is not a true implication, even if Alexander were in fact so many years old at the time when this proposition is uttered.[1] We may say that this proposition is not exactly expressed, and requires the addition of a temporal qualification in order to be always true. A true material implication must be, of course, always true, and if it contains variables, must be true for all values of the variables. Alexander's comment is not incompatible with the strong interpretation; it does not throw light on our problem.

Some more light is thrown on it, if we replace in Alexander's proof of the *M*-law of extensionality expounded in § 40 the

[1] Alexander 176. 2 ἔστι δὲ ἀναγκαία ἀκολουθία οὐχ ἡ πρόσκαιρος, ἀλλ' ἐν ᾗ ἀεὶ τὸ εἰλημμένον ἕπεσθαι ἔστι τῷ τὸ εἰλημμένον ὡς ἡγούμενον εἶναι. οὐ γαρ ἀληθὲς συνημμένον τὸ 'εἰ Ἀλέξανδρος ἔστιν, Ἀλέξανδρος διαλέγεται', ἢ 'εἰ Ἀλέξανδρος ἔστι, τοσῶνδε ἐτῶν ἐστι', καὶ ⟨εἰ⟩ εἴη, ὅτε λέγεται ἡ πρότασις, τοσούτων ἐτῶν.

material implication *Cpq* by the strict implication *LCpq*. Transforming thus the formula

31. *CMpCCpqNLNq*,

we get:

45. *CMpCLCpqNLNq*.

From 31 we can easily derive *CMpNLNp* by the substitution *q*/*p* getting *CMpCCppNLNp*, from which our proposition results by commutation and detachment, for *Cpp* is an asserted implication. The same procedure, however, cannot be applied to 45. We get *CMpCLCppNLNp*, but if we want to detach *CMpNLNp* we must assert the apodeictic implication *LCpp*. And here we encounter the same difficulty, as described in the foregoing section. What is the meaning of *LCpp*? This expression may be interpreted as a general law concerning all propositions, if we transform it into *ΠpCpp*; but such a transformation becomes impossible, if we apply *LCpp* to concrete terms, e.g. to the proposition 'Twice two is five'. The assertoric implication 'If twice two is five, then twice two is five' is comprehensible and true being a consequence of the law of identity *Cpp*; but what is the meaning of the apodeictic implication 'It is necessary that if twice two is five, then twice two should be five'? This queer expression is not a general law concerning all numbers; it may be at most a consequence of an apodeictic law, but it is not true that a consequence of an apodeictic proposition must be apodeictic too. *Cpp* is a consequence of *LCpp* according to *CLCppCpp*, a substitution of *CLpp*, but is not apodeictic.

It follows from the above that it is certainly simpler to interpret Alexander's proof by taking the word συμβαίνει of his text in the sense of material rather than strict implication. Nevertheless our problem is not yet definitively solved. Let us therefore turn to the other kind of asserted apodeictic proposition accepted by Aristotle, that is to necessary connexions of terms.

§ 43. *Analytic propositions*

Aristotle asserts the proposition: 'It is necessary that man should be an animal.'[1] He states here a necessary connexion between the subject 'man' and the predicate 'animal', i.e. a

[1] *An. pr.* i. 9, 30ᵃ30 ζῷον μὲν γὰρ ὁ ἄνθρωπος ἐξ ἀνάγκης ἐστί.

necessary connexion between terms. He apparently regards it as obvious that the proposition 'Man is an animal', or better 'Every man is an animal', must be an apodeictic one, because he defines 'man' as an 'animal', so that the predicate 'animal' is contained in the subject 'man'. Propositions in which the predicate is contained in the subject are called 'analytic', and we shall probably be right in supposing that Aristotle would have regarded all analytic propositions based on definitions as apodeictic, since he says in the *Posterior Analytics* that essential predicates belong to things necessarily,[1] and essential predicates result from definitions.

The most conspicuous examples of analytic propositions are those in which the subject is identical with the predicate. If it is necessary that every man should be an animal, it is, *a fortiori*, necessary that every man should be a man. The law of identity 'Every *a* is an *a*' is an analytic proposition, and consequently an apodeictic one. We get thus the formula:

(*p*) *LAaa*, i.e. *It is necessary that every a should be an a.*

Aristotle does not state the law of identity *Aaa* as a principle of his assertoric syllogistic; there is only one passage, found by Ivo Thomas, where in passing he uses this law in a demonstration.[2] We cannot expect, therefore, that he has known the modal thesis *LAaa*.

The Aristotelian law of identity *Aaa*, where *A* means 'every–is' and *a* is a variable universal term, is different from the principle of identity *Jxx*, where *J* means 'is identical with' and *x* is a variable individual term. The latter principle belongs to the theory of identity which can be established on the following axioms:

(*q*) *Jxx*, i.e. *x is identical with x*,

(*r*) *CJxyCφxφy*, i.e. *If x is identical with y, then if x satisfies φ, y satisfies φ*,

where ϕ is a variable proposition-forming functor of one individual argument. Now, if all analytic propositions are necessary, so also is (*q*), and we get the apodeictic principle:

(*s*) *LJxx*, i.e. *It is necessary that x should be identical with x.*

[1] *An. post.* i. 6, 74ᵇ6 τὰ δὲ καθ'αὑτὰ ὑπάρχοντα ἀναγκαῖα τοῖς πράγμασιν.

[2] Ivo Thomas, O.P., 'Farrago Logica', *Dominican Studies*, vol. iv (1951), p. 71. The passage reads (*An. pr.* ii. 22, 68ᵃ19) κατηγορεῖται δὲ τὸ B καὶ αὐτὸ αὑτοῦ.

It has been observed by W. V. Quine that the principle (*s*), if asserted, leads to awkward consequences.[1] For if *LJxx* is asserted, we can derive (*t*) from (*r*) by the substitution ϕ/LJx'—*LJx* works here like a proposition-forming functor of one argument:

(*t*) *CJxyCLJxxLJxy*,

and by commutation

(*u*) *CLJxxCJxyLJxy*,

from which there follows the proposition:

(*v*) *CJxyLJxy*.

That means, any two individuals are necessarily identical, if they are identical at all.

The relation of equality is usually treated by mathematicians as identity and is based on the same axioms (*q*) and (*r*). We may therefore interpret *J* as equality, *x* and *y* as individual numbers and say that equality holds necessarily if it holds at all.

Formula (*v*) is obviously false. Quine gives an example to show its falsity. Let *x* denote the number of planets, and *y* the number 9. It is a factual truth that the number of (major) planets is equal to 9, but it is not necessary that it should be equal to 9. Quine tries to meet this difficulty by raising objections to the substitution of such singular terms for the variables. In my opinion, however, his objections are without foundation.

There is another awkward consequence of the formula (*v*) not mentioned by Quine. From (*v*) we get by the definition of *L* and the law of transposition the consequence:

(*w*) *CMNJxyNJxy*.

That means: 'If it is possible that *x* is not equal to *y*, then *x* is (actually) not equal to *y*.' The falsity of this consequence may be seen in the following example: Let us suppose that a number *x* has been thrown with a die. It is possible that the number *y* next thrown with the die will be different from *x*. But if it is possible that *x* will be different from *y*, i.e. not equal to *y*, then according to (*w*) *x* will actually be different from *y*. This consequence is obviously wrong, as it is possible to throw the same number twice.

[1] W. V. Quine, 'Three Grades of Modal Involvement', *Proceedings of the XIth International Congress of Philosophy*, vol. xiv, Brussels (1953). For the following argumentation I am alone responsible.

There is, in my opinion, only one way to solve the above difficulties: we must not allow that formula *LJxx* should be asserted, i.e. that the principle of identity *Jxx* is necessary. As *Jxx* is a typical analytic proposition, and as there is no reason to treat this principle in a different way from other analytic propositions, we are compelled to assume that no analytic proposition is necessary.

Before dealing with this important topic let us bring to an end our investigation of Aristotle's concepts of modalities.

§ 44. *An Aristotelian paradox*

There is a principle of necessity set forth by Aristotle which is highly controversial. He says in the *De Interpretatione* that 'anything existent is necessary when it exists, and anything non-existent is impossible when it does not exist'. This does not mean, he adds, that whatever exists is necessary, and whatever does not exist is impossible: for it is not the same to say that anything existent is necessary when it does exist, and to say that it is simply necessary.[1] It should be noted that the temporal 'when' (ὅταν) is used in this passage instead of the conditional 'if'. A similar thesis is set forth by Theophrastus. He says, when defining the kinds of things that are necessary, that the third kind (we do not know what the first two are) is 'the existent, for when it exists, then it is impossible that it should not exist'.[2] Here again we find the temporal particles 'when' (ὅτε) and 'then' (τότε). No doubt an analogous principle occurs in medieval logic and scholars could find it there. There is a formulation quoted by Leibniz in his *Theodicee* running thus: *Unumquodque, quando est, oportet esse*.[3] Note again in this sentence the temporal *quando*.

What does this principle mean? It is, in my opinion, ambiguous. Its first meaning seems to be akin to syllogistic necessity, which is a necessary connexion not of terms, but of propositions. Alexander commenting on the Aristotelian distinction between simple and conditional necessity,[4] says that Aristotle was himself

[1] *De int.* 9. 19ᵃ23 τὸ μὲν οὖν εἶναι τὸ ὄν, ὅταν ᾖ, καὶ τὸ μὴ ὂν μὴ εἶναι, ὅταν μὴ ᾖ, ἀνάγκη· οὐ μὴν οὔτε τὸ ὂν ἅπαν ἀνάγκη εἶναι οὔτε τὸ μὴ ὂν μὴ εἶναι. Οὐ γὰρ ταὐτόν ἐστι τὸ ὂν ἅπαν εἶναι ἐξ ἀνάγκης ὅτε ἔστι, καὶ τὸ ἁπλῶς εἶναι ἐξ ἀνάγκης.

[2] Alexander 156. 29 ὁ γοῦν Θεόφραστος ἐν τῷ πρώτῳ τῶν Προτέρων ἀναλυτικῶν λέγων περὶ τῶν ὑπὸ τοῦ ἀναγκαίου σημαινομένων οὕτως γράφει· 'τρίτον τὸ ὑπάρχον· ὅτε γὰρ ὑπάρχει, τότε οὐχ οἷόν τε μὴ ὑπάρχειν.'

[3] *Philosophische Schriften*, ed. Gerhardt, vol. vi, p. 131.

[4] See p. 144, n. 1.

aware of this distinction, which was explicitly made by his friends (that is, by Theophrastus and Eudemus), and quotes as a further argument the passage of the *De Interpretatione* above referred to. He is aware that this passage is formulated by Aristotle in connexion with singular propositions about future events, and calls the necessity involved 'hypothetical necessity' (ἀναγκαῖον ἐξ ὑποθέσεως).[1]

This hypothetical necessity does not differ from conditional necessity, except that it is applied not to syllogisms, but to singular propositions about events. Such propositions always contain a temporal qualification. But if we include this qualification in the content of the proposition, we can replace the temporal particle by the conditional. So, for instance, instead of saying indefinitely: 'It is necessary that a sea-fight should be, when it is', we may say: 'It is necessary that a sea-fight should be tomorrow, if it will be tomorrow.' Keeping in mind that hypothetical necessity is a necessary connexion of propositions, we may interpret this latter implication as equivalent to the proposition: 'It is necessary that if a sea-fight will be tomorrow, it should be tomorrow' which is a substitution of the formula *LCpp*.

The principle of necessity we are discussing would lead to no controversy, if it had only the meaning explained above. But it may have still another meaning: we may interpret the necessity involved in it as a necessary connexion not of propositions, but of terms. This other meaning seems to be what Aristotle himself has in mind, when he expounds the determinist argument that all future events are necessary. In this connexion a general statement given by him deserves our attention. We read in the *De Interpretatione*: 'If it is true to say that something is white or not white, it is necessary that it should be white or not white.'[2] It seems that here a necessary connexion is stated between a 'thing' as subject and 'white' as predicate. Using a propositional variable instead of the sentence 'Something is white' we get the formula: 'If it is

[1] Alexander 141. 1 ἅμα δὲ καὶ τὴν τοῦ ἀναγκαίου διαίρεσιν ὅτι καὶ αὐτὸς οἶδεν, ἣν οἱ ἑταῖροι αὐτοῦ πεποίηνται, δεδήλωκε διὰ τῆς προσθήκης (*scil.* 'τούτων' ὄντων'), ἣν φθάσας ἤδη καὶ ἐν τῷ Περὶ ἑρμηνείας δέδειχεν, ἐν οἷς περὶ τῆς εἰς τὸν μέλλοντα χρόνον λεγομένης ἀντιφάσεως περὶ τῶν καθ' ἕκαστον εἰρημένων λέγει· 'τὸ μὲν οὖν εἶναι τὸ ὄν, ὅταν ᾖ, καὶ τὸ μὴ ὂν μὴ εἶναι, ὅταν μὴ ᾖ, ἀνάγκη'. τὸ γὰρ ἐξ ὑποθέσεως ἀναγκαῖον τοιοῦτόν ἐστι.

[2] *De int.* 9, 18^{a}39 εἰ γὰρ ἀληθὲς εἰπεῖν ὅτι λευκὸν ἢ ὅτι οὐ λευκόν ἐστιν, ἀνάγκη εἶναι λευκὸν ἢ οὐ λευκόν.

true that *p*, it is necessary that *p*'. I do not know whether Aristotle would have accepted this formula or not, but in any case it is interesting to draw some consequences from it.

In two-valued logic any proposition is either true or false. Hence the expression 'It is true that *p*' is equivalent to '*p*'. Applying this equivalence to our case we see that the formula 'If it is true that *p*, it is necessary that *p*' would be equivalent to this simpler expression: 'If *p*, it is necessary that *p*' which reads in symbols: *CpLp*. We know, however, that this formula has been rejected by Alexander, and certainly by Aristotle himself. It must be rejected, for propositional modal logic would collapse, if it were asserted. Any assertoric proposition *p* would be equivalent to its apodeictic correspondent *Lp*, as both formulae, *CLpp* and *CpLp*, would be valid, and it could be proved that any assertoric proposition *p* was equivalent also to its problematic correspondent *Mp*. Under these conditions it would be useless to construct a propositional modal logic.

But it is possible to express in symbolic form the idea implied by the formula 'If it is true that *p*, it is necessary that *p*': we need only replace the words 'It is true that *p*' by the expression 'α is asserted'. These two expressions do not mean the same. We can put forward for consideration not only true, but also false propositions without being in error. But it would be an error to assert a proposition which was not true. It is therefore not sufficient to say '*p* is true', if we want to impart the idea that *p* is really true; *p* may be false, and '*p* is true' is false with it. We must say 'α is asserted' changing '*p*' into 'α', as '*p*' being a substitution-variable cannot be asserted, whereas 'α' may be interpreted as a true proposition. We can now state, not indeed a theorem, but a rule:

$(x)\quad \alpha \rightarrow L\alpha.$

In words: 'α, therefore it is necessary that α'. The arrow means 'therefore', and the formula (*x*) is a rule of inference valid only when α is asserted. Such a rule restricted to 'tautologous' propositions is accepted by some modern logicians.[1]

From rule (*x*) and the asserted principle of identity *Jxx* there follows the asserted apodeictic formula *LJxx* which leads, as we have seen, to awkward consequences. The rule seems to be doubtful, even if restricted to logical theorems or to analytic proposi-

[1] See, e.g. G. H. von Wright, *An Essay in Modal Logic*, Amsterdam (1951), pp. 14–15.

tions. Without this restriction rule (*x*) would yield, as appears from the example given by Aristotle, apodeictic assertions of merely factual truths, a result contrary to intuition. For this reason this Aristotelian principle fully deserves the name of a paradox.

§ 45. *Contingency in Aristotle*

I have already mentioned that the Aristotelian term *ἐνδεχό-μενον* is ambiguous. In the *De Interpretatione*, and sometimes in the *Prior Analytics*, it means the same as *δυνατόν*, but sometimes it has another more complicated meaning which following Sir David Ross I shall translate by 'contingent'.[1] The merit of having pointed out this ambiguity is due to A. Becker.[2]

Aristotle's definition of contingency runs thus: 'By "contingent" I mean that which is not necessary and the supposed existence of which implies nothing impossible.'[3] We can see at once that Alexander's definition of possibility results from Aristotle's definition of contingency by omission of the words 'which is not necessary'. If we add, therefore, the symbols of these words to our formula 28 and denote the new functor by 'T', we get the following definition:

46. $QTpKNLp\Pi qCCpqNLNq$.

This definition can be abbreviated, as $\Pi qCCpqNLNq$ is equivalent to $NLNp$. The implication:

39. $CNLNp\Pi qCCpqNLNq$

has been already proved; the converse implication

47. $C\Pi qCCpqNLNqNLNp$

easily results from the thesis $C\Pi qCCpqNLNqCCpqNLNq$ by the substitution q/p, commutation, Cpp, and detachment. By putting in 46 the simpler expression $NLNp$ for $\Pi qCCpqNLNq$ we get:

48. $QTpKNLpNLNp$.

This means in words: 'It is contingent that *p*—if and only if—it

[1] W. D. Ross, loc. cit., p. 296.

[2] See A. Becker, *Die Aristotelische Theorie der Möglichkeitsschlüsse*, Berlin (1933). I agree with Sir David Ross (loc. cit., Preface) that Becker's book is 'very acute', but I do not agree with Becker's conclusions.

[3] *An. pr.* i. 13, 32ᵃ18 *λέγω δ' ἐνδέχεσθαι καὶ τὸ ἐνδεχόμενον, οὗ μὴ ὄντος ἀναγκαίου, τεθέντος δ' ὑπάρχειν, οὐδὲν ἔσται διὰ τοῦτ' ἀδύνατον.*

is not necessary that *p* and it is not necessary that not *p*.' As the phrase 'not necessary that not *p*' means the same as 'not impossible that *p*', we may say roughly speaking: 'Something is contingent if and only if it is not necessary and not impossible.' Alexander shortly says: 'The contingent is neither necessary nor impossible.'[1]

We get another definition of Tp, if we transform $NLNp$ according to our definition 1 into Mp, and NLp into MNp:

49. $QTpKMNpMp$ or 50. $QTpKMpMNp$.

Formula 50 reads: 'It is contingent that *p*—if and only if—it is possible that *p* and it is possible that not *p*.' This defines contingency as 'ambivalent possibility', i.e. as a possibility which can indeed be the case, but can also not be the case. We shall see that the consequences of this definition, together with other of Aristotle's assertions about contingency, raise a new major difficulty.

In a famous discussion about future contingent events Aristotle tries to defend the indeterministic point of view. He assumes that things which are not always in act have likewise the possibility of being or not being. For instance, this gown may be cut into pieces, and likewise it may not be cut.[2] Similarly a sea-fight may happen tomorrow, and equally it may not happen. He says that 'Of two contradictory propositions about such things one must be true and the other false, but not this one or that one, only whichever may chance (to be fulfilled), one of them may be more true than the other, but neither of them is as yet true, or as yet false.'[3]

These arguments, though not quite clearly expressed or fully thought out, contain an important and most fruitful idea. Let us take the example of the sea-fight, and suppose that nothing is decided today about this fight. I mean that there is nothing that is real today and that would cause there to be a sea-fight tomorrow, nor yet anything that would cause there not to be one. Hence, if

[1] Alexander 158. 20 *οὔτε γὰρ ἀναγκαῖον οὔτε ἀδύνατον τὸ ἐνδεχόμενον.*

[2] *De int.* 9, 19ᵃ9 *ἔστιν ἐν τοῖς μὴ ἀεὶ ἐνεργοῦσι τὸ δυνατὸν εἶναι καὶ μὴ ὁμοίως . . .* 12 *οἷον ὅτι τουτὶ τὸ ἱμάτιον δυνατόν ἐστι διατμηθῆναι, . . . ὁμοίως δὲ καὶ τὸ μὴ διατμηθῆναι δυνατόν.*

[3] Ibid. 19ᵃ36 *τούτων γὰρ* (i.e. *ἐπὶ τοῖς μὴ ἀεὶ οὖσιν ἢ μὴ ἀεὶ μὴ οὖσιν*) *ἀνάγκη μὲν θάτερον μόριον τῆς ἀντιφάσεως ἀληθὲς εἶναι ἢ ψεῦδος, οὐ μέντοι τόδε ἢ τόδε ἀλλ' ὁπότερ' ἔτυχε, καὶ μᾶλλον μὲν ἀληθῆ τὴν ἑτέραν, οὐ μέντοι ἤδη ἀληθῆ ἢ ψευδῆ.*

truth rests on conformity of thought with reality, the proposition 'The sea-fight will happen tomorrow' is today neither true nor false. It is in this sense that I understand the words 'not yet true or false' in Aristotle. But this would lead to the conclusion that it is today neither necessary nor impossible that there will be a sea-fight tomorrow; in other words, that the propositions 'It is possible that there will be a sea-fight tomorrow' and 'It is possible that there will not be a sea-fight tomorrow' are today both true, and this future event is contingent.

It follows from the above that according to Aristotle there exist true contingent propositions, i.e. that the formula Tp and its equivalent $KMpMNp$ are true for some value of p, say α. For example, if α means 'There will be a sea-fight tomorrow', both $M\alpha$ and $MN\alpha$ would be accepted by Aristotle as true, so that he would have asserted the conjunction:

(A) $KM\alpha MN\alpha$.

There exists, however, in the classical calculus of propositions enlarged by the variable functor δ, the following thesis due to Leśniewski's protothetic:

51. $C\delta pC\delta Np\delta q$.

In words: 'If δ of p, then if δ of not p, δ of q', or roughly speaking: 'If something is true of the proposition p, and also true of the negation of p, it is true of an arbitrary proposition q.' Thesis 51 is equivalent to

52. $CK\delta p\delta Np\delta q$

on the ground of the laws of importation and exportation $CCpCqrCKpqr$ and $CCKpqrCpCqr$. From (A) and 52 we get the consequence:

52. δ/M, p/α, $q/p \times C$(A)–(B)

(B) Mp.

Thus, if there is any contingent proposition that we accept as true, we are bound to admit of any proposition whatever that it is possible. But this would cause a collapse of modal logic; Mp must be rejected, and consequently $KM\alpha MN\alpha$ cannot be asserted.

We are at the end of our analysis of Aristotle's propositional

modal logic. This analysis has led us to two major difficulties: the first difficulty is connected with Aristotle's acceptance of true apodeictic propositions, the second with his acceptance of true contingent propositions. Both difficulties will reappear in Aristotle's modal syllogistic, the first in his theory of syllogisms with one assertoric and one apodeictic premiss, the second in his theory of contingent syllogisms. If we want to meet these difficulties and to explain as well as to appreciate his modal syllogistic, we must first establish a secure and consequent system of modal logic.