opposition to Aristotle and to the whole school of the Peripatetics, he eagerly seized the occasion to back up his opinion by the authority of an illustrious name.

REMARK. The problem of compound syllogisms raised by Galen has considerable interest from the systematic point of view. Investigating the number of valid moods of the syllogisms consisting of three premisses, I have found that there are forty-four valid moods, the figures F1, F2, F4, F5, F6, and F7 having six moods each, and figure F8 eight. Figure F3 is empty. It has no valid moods, for it is not possible to find premisses of the form *A–B*, *C–B*, *C–D* such that a conclusion of the form *A–D* would follow from them. This result, if known, would certainly be startling for students of the traditional logic. Mr. C. A. Meredith, who attended my lectures delivered on this subject in 1949 at University College, Dublin, has found some general formulae concerning the number of figures and valid moods for syllogisms of *n* terms, including expressions of 1 and 2 terms. I publish these formulae here with his kind permission:

Number of terms	n
Number of figures	2^{n-1}
Number of figures with valid moods .	$\frac{1}{2}(n^2-n+2)$
Number of valid moods	$n(3n-1)$

For all *n* every non-empty figure has 6 valid moods, except one that has 2*n* valid moods.

Examples:

Number of terms	1, 2, 3, 4,..., 10
Number of figures	1, 2, 4, 8,..., 512
Number of figures with valid moods .	1, 2, 4, 7,..., 46
Number of valid moods	2, 10, 24, 44,..., 290

It is obvious that for large *n*'s the number of figures with valid moods is comparatively small against the number of all figures. For $n = 10$ we have 46 against 512 respectively, i.e. 466 figures are empty. For $n = 1$ there is only 1 figure, *A–A*, with 2 valid moods, i.e. the laws of identity. For $n = 2$ there are 2 figures:

	Premiss	*Conclusion*
F1	*A–B*	*A–B*
F2	*B–A*	*A–B*

with 10 valid moods, 6 in F1 (viz. four substitutions of the propositional law of identity, e.g. 'if all *A* is *B*, then all *A* is *B*', and two laws of subordination), and 4 moods in F2 (viz. four laws of conversion).

CHAPTER III

THE SYSTEM

§ 15. *Perfect and imperfect syllogisms*

IN the introductory chapter to the syllogistic Aristotle divides all syllogisms into perfect and imperfect. 'I call that a perfect syllogism', he says, 'which needs nothing other than what has been stated to make the necessity evident; a syllogism is imperfect, if it needs either one or more components which are necessary by the terms set down, but have not been stated by the premisses.'[1] This passage needs translation into logical terminology. Every Aristotelian syllogism is a true implication, the antecedent of which is the joint premisses and the consequent the conclusion. What Aristotle says means, therefore, that in a perfect syllogism the connexion between the antecedent and the consequent is evident of itself without an additional proposition. Perfect syllogisms are self-evident statements which do not possess and do not need a demonstration; they are indemonstrable, ἀναπόδεικτοι.[2] Indemonstrable true statements of a deductive system are now called axioms. The perfect syllogisms, therefore, are the axioms of the syllogistic. On the other hand, the imperfect syllogisms are not self-evident; they must be proved by means of one or more propositions which result from the premisses, but are different from them.

Aristotle knows that not all true propositions are demonstrable.[3] He says that a proposition of the form '*A* belongs to *B*' is demonstrable if there exists a middle term, i.e. a term which forms with *A* and *B* true premisses of a valid syllogism having the above proposition as the conclusion. If such a middle term does

[1] *An. pr.* i. 1, 24^{b}22 τέλειον μὲν οὖν καλῶ συλλογισμὸν τὸν μηδενὸς ἄλλου προσδεόμενον παρὰ τὰ εἰλημμένα πρὸς τὸ φανῆναι τὸ ἀναγκαῖον, ἀτελῆ δὲ τὸν προσδεόμενον ἢ ἑνὸς ἢ πλειόνων, ἃ ἔστι μὲν ἀναγκαῖα διὰ τῶν ὑποκειμένων ὅρων, οὐ μὴν εἴληπται διὰ πρότασεων.

[2] Commenting upon the above passage Alexander uses the expression ἀναπόδεικτος, 24. 2: ἑνὸς μὲν οὖν προσδέονται οἱ ἀτελεῖς συλλογισμοὶ οἱ μιᾶς ἀντιστροφῆς δεόμενοι πρὸς τὸ ἀναχθῆναι εἴς τινα τῶν ἐν τῷ πρώτῳ σχήματι τῶν τελείων καὶ ἀναποδείκτων, πλειόνων δὲ ὅσοι διὰ δύο ἀντιστροφῶν εἰς ἐκείνων τινὰ ἀνάγονται. Cf. also p. 27, n. 2.

[3] *An. post.* i. 3, 72^{b}18 ἡμεῖς δέ φαμεν οὔτε πᾶσαν ἐπιστήμην ἀποδεικτικὴν εἶναι, ἀλλὰ τὴν τῶν ἀμέσων ἀναπόδεικτον.

not exist, the proposition is called 'immediate', ἄμεσος, i.e. without a middle term. Immediate propositions are indemonstrable; they are basic truths, ἀρχαί.[1] To these statements of the *Posterior Analytics* may be added a passage of the *Prior Analytics* which states that every demonstration and every syllogism must be formed by means of the three syllogistical figures.[2]

This Aristotelian theory of proof has a fundamental flaw: it supposes that all problems can be expressed by the four kinds of syllogistic premiss and that therefore the categorical syllogism is the only instrument of proof. Aristotle did not realize that his own theory of the syllogism is an instance against this conception. The syllogistic moods, being implications, are propositions of another kind than the syllogistic premisses, but nevertheless they are true propositions, and if any of them is not self-evident and indemonstrable it requires a proof to establish its truth. The proof, however, cannot be done by means of a categorical syllogism, because an implication does not have either a subject or a predicate, and it would be useless to look for a middle term between non-existent extremes. This is perhaps a subconscious cause of the special terminology Aristotle uses in the doctrine of the syllogistic figures. He does not speak of 'axioms' or 'basic truths' but of 'perfect syllogisms', and does not 'demonstrate' or 'prove' the imperfect syllogisms but 'reduces' them (ἀνάγει or ἀναλύει) to the perfect. The effects of this improper terminology persist till today. Keynes devotes to this matter a whole section of his *Formal Logic*, entitled 'Is Reduction an essential part of the Doctrine of the Syllogism?', and comes to the conclusion 'that reduction is not a necessary part of the doctrine of the syllogism, so far as the establishment of the validity of the different moods is concerned'.[3] This conclusion cannot be applied to the Aristotelian theory of the syllogism, as this theory is an axiomatized deductive system, and the reduction of the other syllogistic moods to those of the first figure, i.e. their proof as theorems by means of the axioms, is an indispensable part of the system.

Aristotle accepts as perfect syllogisms the moods of the first

[1] *An. post.* i. 23, $84^{b}19$ φανερὸν δὲ καὶ ὅτι, ὅταν τὸ *A* τῷ *B* ὑπάρχῃ, εἰ μὲν ἔστι τι μέσον, ἔστι δεῖξαι ὅτι τὸ *A* τῷ *B* ὑπάρχει . . ., εἰ δὲ μή ἐστιν, οὐκέτι ἔστιν ἀπόδειξις, ἀλλ' ἡ ἐπὶ τὰς ἀρχὰς ὁδὸς αὕτη ἐστίν.

[2] *An. pr.* i. 23, $41^{b}1$ πᾶσαν ἀπόδειξιν καὶ πάντα συλλογισμὸν ἀνάγκη γίνεσθαι διὰ τριῶν τῶν προειρημένων σχημάτων.

[3] Op. cit., pp. 325–7.

figure, called Barbara, Celarent, Darii, and Ferio.[1] Yet in the last chapter of his systematic exposition he reduces the third and fourth moods to the first two, and takes therefore as axioms of his theory the most clearly evident syllogisms, Barbara and Celarent.[2] This detail is of no little interest. Modern formal logic tends to reduce the number of axioms in a deductive theory to a minimum, and this is a tendency which has its first exponent in Aristotle.

Aristotle is right when he says that only two syllogisms are needed as axioms to build up the whole theory of the syllogism. He forgets, however, that the laws of conversion, which he uses to reduce the imperfect moods to the perfect ones, also belong to his theory and cannot be proved by means of the syllogisms. There are three laws of conversion mentioned in the *Prior Analytics*: the conversion of the *E*-premiss, of the *A*-premiss, and of the *I*-premiss. Aristotle proves the first of these laws by what he calls ecthesis, which requires, as we shall see later, a logical process lying outside the limits of the syllogistic. As it cannot be proved otherwise, it must be stated as a new axiom of the system. The conversion of the *A*-premiss is proved by a thesis belonging to the square of opposition of which there is no mention in the *Prior Analytics*. We must therefore accept as a fourth axiom either this law of conversion or the thesis of the square of opposition, from which this law follows. Only the law of conversion of the *I*-premisses can be proved without a new axiom.

There are still two theses that have to be taken into account, although neither of them is explicitly stated by Aristotle, viz. the laws of identity: '*A* belongs to all *A*' and '*A* belongs to some *A*'. The first of these laws is independent of all other theses of the syllogistic. If we want to have this law in the system, we must accept it axiomatically. The second law of identity can be derived from the first.

Modern formal logic distinguishes in a deductive system not only between primitive and derivative propositions, but also between primitive and defined terms. The constants of the Aristotelian syllogistic are the four relations: 'to belong to all'

[1] At the end of chapter 4, containing the moods of the first figure, Aristotle says, *An. Pr.* i. 4, $26^{b}29$ δῆλον δὲ καὶ ὅτι πάντες οἱ ἐν αὐτῷ συλλογισμοὶ τέλειοί εἰσιν.

[2] Ibid. 7, $29^{b}1$ ἔστι δὲ καὶ ἀναγαγεῖν πάντας τοὺς συλλογισμοὺς εἰς τοὺς ἐν τῷ πρώτῳ σχήματι καθόλου συλλογισμούς.

or *A*, 'to belong to none' or *E*, 'to belong to some' or *I*, and 'to not-belong to some' or *O*. Two of them may be defined by the other two by means of propositional negation in the following way: '*A* does not belong to some *B*' means the same as 'It is not true that *A* belongs to all *B*', and '*A* belongs to no *B*' means the same as 'It is not true that *A* belongs to some *B*'. In the same manner *A* could be defined by *O*, and *I* by *E*. Aristotle does not introduce these definitions into his system, but he uses them intuitively as arguments of his proofs. Let us quote as only one example the proof of conversion of the *I*-premiss. It runs as follows: 'If *A* belongs to some *B*, then *B* must belong to some *A*. For if *B* should belong to no *A*, *A* would belong to no *B*.'[1] It is obvious that in this indirect proof Aristotle treats the negation of '*B* belongs to some *A*' as equivalent to '*B* belongs to no *A*'. As to the other pair, *A* and *O*, Alexander says explicitly that the phrases 'to not-belong to some' and 'to not-belong to all' are different only in words, but have equivalent meanings.[2]

If we accept as primitive terms of the system the relations *A* and *I*, defining *E* and *O* by means of them, we may, as I stated many years ago,[3] build up the whole theory of the Aristotelian syllogism on the following four axioms:

1. *A* belongs to all *A*.
2. *A* belongs to some *A*.
3. If *A* belongs to all *B* and *B* belongs to all *C*, then *A* belongs to all *C*. — Barbara
4. If *A* belongs to all *B* and *C* belongs to some *B*, then *A* belongs to some *C*. — Datisi

It is impossible to reduce the number of these axioms. In particular they cannot be derived from the so-called *dictum de omni et nullo*. This principle is differently formulated in different text-books of logic, and always very vaguely. The classic formulation, 'quidquid de omnibus valet, valet etiam de quibusdam et de singulis' and 'quidquid de nullo valet, nec de quibusdam nec de

[1] *An. pr.* i. 2, 25ª20 *εἰ γὰρ τὸ A τινὶ τῷ B, καὶ τὸ B τινὶ τῷ A ἀνάγκη ὑπάρχειν. εἰ γὰρ μηδενί, οὐδὲ τὸ A οὐδενὶ τῷ B.* [Corr. by W. D. Ross.]

[2] Alexander 84. 6 *τὸ τινὶ μὴ ὑπάρχειν ἴσον δυνάμενον τῷ μὴ παντὶ κατὰ τὴν λέξιν διαφέρει.*

[3] J. Łukasiewicz, *Elementy logiki matematycznej* (Elements of Mathematical Logic), edited by M. Presburger (mimeographed), Warsaw (1929), p. 172; 'Znaczenie analizy logicznej dla poznania' (Importance of Logical Analysis for Knowledge), *Przegl. Filoz.* (*Philosophical Review*), vol. xxxvii, Warsaw (1934), p. 373.

singulis valet', cannot be strictly applied to the Aristotelian logic, as singular terms and propositions do not belong to it. Besides, I do not see how it would be possible to deduce from this principle the laws of identity and the mood Datisi, if anything at all can be deduced from it. Moreover, it is evident that it is not one single principle but two. It must be emphasized that Aristotle is by no means responsible for this obscure principle. It is not true that the *dictum de omni et nullo* was given by Aristotle as the axiom on which all syllogistic inference is based, as Keynes asserts.[1] It is nowhere formulated in the *Prior Analytics* as a principle of syllogistic. What is sometimes quoted as a formulation of this principle is only an explanation of the words 'to be predicated of all' and 'of none'.[2]

It is a vain attempt to look for the principle of the Aristotelian logic, if 'principle' means the same as 'axiom'. If it has another meaning, I do not understand the problem at all. Maier, who has devoted to this subject another obscure chapter of his book,[3] spins out philosophic speculations that neither have a basis in themselves nor are supported by texts of the *Prior Analytics*. From the standpoint of logic they are useless.

§ 16. *The logic of terms and the logic of propositions*

To this day there exists no exact logical analysis of the proofs Aristotle gives to reduce the imperfect syllogisms to the perfect. The old historians of logic, like Prantl and Maier, were philosophers and knew only the 'philosophical logic' which in the nineteenth century, with very few exceptions, was below a scientific level. Prantl and Maier are now dead, but perhaps it would not be impossible to persuade living philosophers that they should cease to write about logic or its history before having acquired a solid knowledge of what is called 'mathematical logic'. It would otherwise be a waste of time for them as well as for their readers. It seems to me that this point is of no small practical importance.

No one can fully understand Aristotle's proofs who does not know that there exists besides the Aristotelian system another system of logic more fundamental than the theory of the syllogism.

[1] Op. cit., p. 301.

[2] *An. pr.* i. 1, 24ᵇ28 *λέγομεν δὲ τὸ κατὰ παντὸς κατηγορεῖσθαι, ὅταν μηδὲν ᾖ λαβεῖν [τοῦ ὑποκειμένου* (secl. W. D. Ross)], *καθ' οὗ θάτερον οὐ λεχθήσεται· καὶ τὸ κατὰ μηδενὸς ὡσαύτως.*

[3] Op. cit., vol. ii *b*, p. 149.

It is the logic of propositions. Let us explain by an example the difference between the logic of terms, of which the Aristotelian logic is only a part, and the logic of propositions. Besides the Aristotelian law of identity '*A* belongs to all *A*' or 'All *A* is *A*', we have still another law of identity of the form 'If *p*, then *p*'. Let us compare these two, which are the simplest logical formulae:

All *A* is *A* and If *p*, then *p*.

They differ in their constants, which I call functors: in the first formula the functor reads 'all—is', in the second 'if—then'. Both are functors of two arguments which are here identical. But the main difference lies in the arguments. In both formulae the arguments are variables, but of a different kind: the values which may be substituted for the variable *A* are terms, like 'man' or 'plant'. From the first formula we get thus the propositions 'All men are men' or 'All plants are plants'. The values of the variable *p* are not terms but propositions, like 'Dublin lies on the Liffey' or 'Today is Friday'; we get, therefore, from the second formula the propositions: 'If Dublin lies on the Liffey, then Dublin lies on the Liffey' or 'If today is Friday, then today is Friday'. This difference between term-variables and proposition-variables is the primary difference between the two formulae and consequently between the two systems of logic, and, as propositions and terms belong to different semantical categories, the difference is a fundamental one.

The first system of propositional logic was invented about half a century after Aristotle: it was the logic of the Stoics. This logic is not a system of theses but of rules of inference. The so-called *modus ponens*, now called the rule of detachment: 'If α, then β; but α; therefore β' is one of the most important primitive rules of the Stoic logic. The variables α and β are propositional variables, as only propositions can be significantly substituted for them.[1] The modern system of the logic of propositions was created only in 1879 by the great German logician Gottlob Frege. Another outstanding logician of the nineteenth century, the American Charles Sanders Peirce, made important contributions to this logic by his discovery of logical matrices (1885). The authors of *Principia Mathematica*, Whitehead and Russell, later put this

[1] Cf. Łukasiewicz, 'Zur Geschichte des Aussagenkalküls', *Erkenntnis*, vol. v, Leipzig (1935), pp. 111–31.

system of logic at the head of all mathematics under the title 'Theory of Deduction'. All this was entirely unknown to philosophers of the nineteenth century. To this day they seem to have no idea of the logic of propositions. Maier says that the Stoic logic, which in fact is a masterpiece equal to the logic of Aristotle, yields a poor and barren picture of formalistic-grammatical unsteadiness and lack of principle, and adds in a footnote that the unfavourable judgement of Prantl and Zeller on this logic must be maintained.[1] The *Encyclopaedia Britannica* of 1911 says briefly of the logic of the Stoics that 'their corrections and fancied improvements of the Aristotelian logic are mostly useless and pedantic'.[2]

It seems that Aristotle did not suspect the existence of another system of logic besides his theory of the syllogism. Yet he uses intuitively the laws of propositional logic in his proofs of imperfect syllogisms, and even sets forth explicitly three statements belonging to this logic in Book II of the *Prior Analytics*. The first of these is a law of transposition: 'When two things', he says, 'are so related to one another, that if the one is, the other necessarily is, then if the latter is not, the former will not be either.'[3] That means, in terms of modern logic, that whenever an implication of the form 'If α, then β' is true, then there must also be true another implication of the form 'If not-β, then not-α'. The second is the law of the hypothetical syllogism. Aristotle explains it by an example: 'Whenever if *A* is white, then *B* should be necessarily great, and if *B* is great, then *C* should not be white, then it is necessary if *A* is white that *C* should not be white.'[4] That means: whenever two implications of the form 'If α, then β' and 'If β, then γ' are true, then there must also be true a third implication 'If α, then γ'. The third statement is an application of the two foregoing laws to a new example and, curiously enough, it is false. This very interesting passage runs thus:

'It is impossible that the same thing should be necessitated by the being and by the not-being of the same thing. I mean, for example,

[1] Maier, op. cit., vol. ii *b*, p. 384: 'In der Hauptsache jedoch bietet die Logik der Stoiker . . . ein dürftiges, ödes Bild formalistisch-grammatischer Prinzip- und Haltlosigkeit.' Ibid., n. 1: 'In der Hauptsache wird es bei dem ungünstigen Urteil, das Prantl und Zeller über die stoische Logik fällen, bleiben müssen.'

[2] 11th ed., Cambridge (1911), vol. xxv, p. 946 (s.v. 'Stoics').

[3] *An. pr.* ii. 4, 57^{b}1 *ὅταν δύο ἔχῃ οὕτω πρὸς ἄλληλα ὥστε θατέρου ὄντος ἐξ ἀνάγκης εἶναι θάτερον, τούτου μὴ ὄντος μὲν οὐδὲ θάτερον ἔσται.*

[4] Ibid. 6 *ὅταν γὰρ τουδὶ ὄντος λευκοῦ τοῦ A τοδὶ ἀνάγκη μέγα εἶναι τὸ B, μεγάλου δὲ τοῦ B ὄντος τὸ Γ μὴ λευκόν, ἀνάγκη, εἰ τὸ A λευκόν, τὸ Γ μὴ εἶναι λευκόν.*

that it is impossible that *B* should necessarily be great if *A* is white, and that *B* should necessarily be great if *A* is not white. For if *B* is not great *A* cannot be white. But if, when *A* is not white, it is necessary that *B* should be great, it necessarily results that if *B* is not great, *B* itself is great. But this is impossible.'[1]

Although the example chosen by Aristotle is unfortunate, the sense of his argument is clear. In terms of modern logic it can be stated thus: Two implications of the form 'If α, then β' and 'If not-α, then β' cannot be together true. For by the law of transposition we get from the first implication the premiss 'If not-β, then not-α', and this premiss yields together with the second implication the conclusion 'If not-β, then β' by the law of the hypothetical syllogism. According to Aristotle this conclusion is impossible.

Aristotle's final remark is erroneous. The implication 'If not-β, then β', the antecedent of which is the negation of the consequent, is not impossible; it may be true, and yields as conclusion the consequent β, according to the law of the logic of propositions: 'If (if not-*p*, then *p*), then *p*.'[2] Commenting upon this passage, Maier says that there would here result a connexion contrary to the law of contradiction and therefore absurd.[3] This comment again reveals Maier's ignorance of logic. It is not the implication 'If not-β, then β' that is contrary to the law of contradiction, but only the conjunction 'β and not-β'.

A few years after Aristotle, the mathematician Euclid gave a proof of a mathematical theorem which implies the thesis 'If (if not-*p*, then *p*), then *p*'.[4] He states first that 'If the product of two

[1] *An. pr.* ii. 4, 57^b3 *τοῦ δ' αὐτοῦ ὄντος καὶ μὴ ὄντος, ἀδύνατον ἐξ ἀνάγκης εἶναι τὸ αὐτό. λέγω δ' οἷον τοῦ A ὄντος λευκοῦ τὸ B εἶναι μέγα ἐξ ἀνάγκης, καὶ μὴ ὄντος λευκοῦ τοῦ A τὸ B εἶναι μέγα ἐξ ἀνάγκης.* Here follows the example of the hypothetical syllogism quoted in p. 49, n. 4, and a second formulation of the law of transposition. The conclusion reads, 11 *τοῦ δὴ B μὴ ὄντος μεγάλου τὸ A οὐχ οἷόν τε λευκὸν εἶναι. τοῦ δὲ A μὴ ὄντος λευκοῦ, εἰ ἀνάγκη τὸ B μέγα εἶναι, συμβαίνει ἐξ ἀνάγκης τοῦ B μεγάλου μὴ ὄντος αὐτὸ τὸ B εἶναι μέγα. τοῦτο δ' ἀδύνατον.*

[2] See A. N. Whitehead and B. Russell, *Principia Mathematica*, vol. i, Cambridge (1910), p. 108, thesis *2·18.

[3] Op. cit., vol. ii *a*, p. 331: 'Es ergäbe sich also ein Zusammenhang, der dem Gesetze des Widerspruchs entgegenstünde und darum absurd wäre.'

[4] See *Scritti di G. Vailati*, Leipzig–Firenze, cxv. 'A proposito d'un passo del Teeteto e di una dimostrazione di Euclide', pp. 516–27; cf. Łukasiewicz, 'Philosophische Bemerkungen zu mehrwertigen Systemen des Aussagenkalküls', *Comptes Rendus des séances de la Société des Sciences et des Lettres de Varsovie*, xxiii (1930), Cl. III, p. 67.

integers, *a* and *b*, is divisible by a prime number *n*, then if *a* is not divisible by *n*, *b* should be divisible by *n*'. Let us now suppose that $a = b$ and the product $a \times a$ (a^2) is divisible by *n*. It results from this supposition that 'If *a* is not divisible by *n*, then *a* is divisible by *n*'. Here we have an example of a true implication the antecedent of which is the negation of the consequent. From this implication Euclid derives the theorem: 'If a^2 is divisible by a prime number *n*, then *a* is divisible by *n*.'

§ 17. *The proofs by conversion*

The proofs of imperfect syllogisms by conversion of a premiss are both the simplest and those most frequently employed by Aristotle. Let us analyse two examples. The proof of the mood Festino of the second figure runs thus: 'If *M* belongs to no *N*, but to some *X*, then it is necessary that *N* should not belong to some *X*. For since the negative premiss is convertible, *N* will belong to no *M*; but *M* was admitted to belong to some *X*; therefore *N* will not belong to some *X*. The conclusion is reached by means of the first figure.'[1]

The proof is based on two premisses: one of them is the law of conversion of the *E*-propositions:

(1) If *M* belongs to no *N*, then *N* belongs to no *M*,

and the other is the mood Ferio of the first figure:

(2) If *N* belongs to no *M* and *M* belongs to some *X*, then *N* does not belong to some *X*.

From these premisses we have to derive the mood Festino:

(3) If *M* belongs to no *N* and *M* belongs to some *X*, then *N* does not belong to some *X*.

Aristotle performs the proof intuitively. Analysing his intuitions we find two theses of the propositional calculus: one of them is the above-mentioned law of the hypothetical syllogism, which may be stated in the following form:

(4) If (if *p*, then *q*), then [if (if *q*, then *r*), then (if *p*, then *r*)];[2]

[1] *An. pr.* i. 5, 27^a32 *εἰ γὰρ τὸ M τῷ μὲν N μηδενὶ τῷ δὲ Ξ τινὶ ὑπάρχει, ἀνάγκη τὸ N τινὶ τῷ Ξ μὴ ὑπάρχειν. ἐπεὶ γὰρ ἀντιστρέφει τὸ στερητικόν, οὐδενὶ τῷ M ὑπάρξει τὸ N· τὸ δέ γε M ὑπέκειτο τινὶ τῷ Ξ ὑπάρχειν· ὥστε τὸ N τινὶ τῷ Ξ οὐχ ὑπάρξει. γίνεται γὰρ συλλογισμὸς διὰ τοῦ πρώτου σχήματος.*

[2] See *Principia Mathematica*, p. 104, thesis *2·06.

The other thesis reads:

(5) If (if p, then q), then (if p and r, then q and r).

This thesis is called in *Principia Mathematica*, following Peano, the principle of the factor. It shows that we may 'multiply' both sides of an implication by a common factor, i.e. we may add, by means of the word 'and', to p and to q a new proposition r.[1]

We start with thesis (5). As p, q, and r are propositional variables, we may substitute for them premisses of the Aristotelian logic. Putting 'M belongs to no N' for p, 'N belongs to no M' for q, and 'M belongs to some X' for r, we get from the antecedent of (5) the law of conversion (1), and we may detach the consequent of (5) as a new thesis. This new thesis has the form:

(6) If M belongs to no N and M belongs to some X, then N belongs to no M and M belongs to some X.

The consequent of this thesis is identical with the antecedent of thesis (2). Therefore we may apply to (6) and (2) the law of the hypothetical syllogism, substituting for p the conjunction 'M belongs to no N and M belongs to some X', for q the conjunction 'N belongs to no M and M belongs to some X', and for r the proposition 'N does not belong to some X'. By applying the rule of detachment twice we get from this new thesis the mood Festino.

The second example I want to analyse is somewhat different. It is the above-mentioned proof of the mood Disamis.[2] We have to prove the following imperfect syllogism:

(7) If R belongs to all S and P belongs to some S, then P belongs to some R.

The proof is based on the mood Darii of the first figure:

(8) If R belongs to all S and S belongs to some P, then R belongs to some P,

and on the law of conversion of the I-propositions applied twice, once in the form:

(9) If P belongs to some S, then S belongs to some P,

and for the second time in the form:

(10) If R belongs to some P, then P belongs to some R.

As auxiliary theses of the propositional logic we have the law of

[1] See *Principia Mathematica*, p. 119, thesis *3·45. The conjunction 'p and r' is called in the *Principia* 'logical product'.

[2] See the Greek text in p. 25, n. 1.

the hypothetical syllogism, and the following thesis, which is slightly different from thesis (5), but also may be called the principle of the factor:

(11) If (if p, then q), then (if r and p, then r and q).

The difference between (5) and (11) consists in this, that the common factor r is not in the second place, as in (5), but in the first. As conjunction is commutable and 'p and r' is equivalent to 'r and p', this difference does not affect the validity of the thesis.

The proof given by Aristotle begins with the conversion of the premiss 'P belongs to some S'. Following this procedure, let us substitute for p in (11) the premiss 'P belongs to some S', for q the premiss 'S belongs to some P', and for r the premiss 'R belongs to all S'. By this substitution we get from the antecedent of (11) the law of conversion (9), and therefore we may detach the consequent of (11) which reads:

(12) If R belongs to all S and P belongs to some S, then R belongs to all S and S belongs to some P.

The consequent of (12) is identical with the antecedent of (8). By applying the law of the hypothetical syllogism we can get from (12) and (8) the syllogism:

(13) If R belongs to all S and P belongs to some S, then R belongs to some P.

This syllogism, however, is not the required mood Disamis, but Datisi. Of course, the mood Disamis could be derived from Datisi by converting its consequent according to thesis (10), i.e. by applying the hypothetical syllogism to (13) and (10). It seems, however, that Aristotle took another course: instead of deriving Datisi and converting its conclusion, he converts the conclusion of Darii, getting the syllogism:

(14) If R belongs to all S and S belongs to some P, then P belongs to some R,

and then he applies intuitively the law of the hypothetical syllogism to (12) and (14). The syllogism (14) is a mood of the fourth figure called Dimaris. As we already know, Aristotle mentions this mood at the beginning of Book II of the *Prior Analytics*.

In a similar way we could analyse all the other proofs by conversion. It follows from this analysis that if we add to the perfect syllogisms of the first figure and to the laws of conversion three laws of the logic of propositions, viz. the law of the hypothetical syllogism and two laws of the factor, we get strictly formalized proofs of all imperfect syllogisms except Baroco and Bocardo. These two moods require other theses of the propositional logic.

§ 18. *The proofs by* reductio ad impossibile

The moods Baroco and Bocardo cannot be reduced to the first figure by conversion. The conversion of the *A*-premiss would yield an *I*-proposition, from which together with the *O*-premiss nothing results, and the *O*-premiss cannot be converted. Aristotle tries to prove these two moods by a *reductio ad impossibile*, *ἀπαγωγὴ εἰς τὸ ἀδύνατον*. The proof of Baroco runs thus: 'If *M* belongs to all *N*, but not to some *X*, it is necessary that *N* should not belong to some *X*; for if *N* belongs to all *X*, and *M* is predicated also of all *N*, *M* must belong to all *X*; but it was assumed that *M* does not belong to some *X*.'[1] This proof is very concise and needs an explanation. Usually it is explained in the following way:[2]

We have to prove the syllogism:

(1) If *M* belongs to all *N* and *M* does not belong to some *X*, then *N* does not belong to some *X*.

It is admitted that the premisses '*M* belongs to all *N*' and '*M* does not belong to some *X*' are true; then the conclusion '*N* does not belong to some *X*' must also be true. For if it were false, its contradictory, '*N* belongs to all *X*', would be true. This last proposition is the starting-point of our reduction. As it is admitted that the premiss '*M* belongs to all *N*' is true, we get from this premiss and the propostion '*N* belongs to all *X*' the conclusion '*M* belongs to all *X*' by the mood Barbara. But this conclusion is false, for it is admitted that its contradictory '*M* does not belong to some *X*' is true. Therefore the starting-point of our reduction, '*N* belongs to all *X*', which leads to a false conclusion, must be false, and its contradictory, '*N* does not belong to some *X*', must be true.

This argument is only apparently convincing; in fact it does not prove the above syllogism. It can be applied only to the traditional mood Baroco (I quote this mood in its usual form

[1] *An. pr.* i. 5, 27^a37 *εἰ τῷ μὲν N παντὶ τὸ M, τῷ δὲ Ξ τινὶ μὴ ὑπάρχει, ἀνάγκη τὸ N τινὶ τῷ Ξ μὴ ὑπάρχειν· εἰ γὰρ παντὶ ὑπάρχει, κατηγορεῖται δὲ καὶ τὸ M παντὸς τοῦ N, ἀνάγκη τὸ M παντὶ τῷ Ξ ὑπάρχειν· ὑπέκειτο δὲ τινὶ μὴ ὑπάρχειν.*

[2] Cf., for instance, Maier, op. cit., vol. ii *a*, p. 84.

with the verb 'to be', and not in the Aristotelian form with 'to belong'):

(2) All *N* is *M*,
Some *X* is not *M*,
therefore
Some *X* is not *N*.

This is a rule of inference and allows us to assert the conclusion provided the premisses are true. It does not say what happens when the premisses are not true. This is irrelevant for a rule of inference, as it is evident that an inference based on false premisses cannot be valid. But Aristotelian syllogisms are not rules of inference, they are propositions. The syllogism (1) is an implication which is true for all values of the variables *M*, *N*, and *X*, and not only for those values that verify the premisses. If we apply this mood Baroco to the terms *M* = 'bird', *N* = 'animal', and *X* = 'owl', we get a true syllogism (I use forms with 'to be', as does Aristotle in examples):

(3) If all animals are birds
and some owls are not birds,
then some owls are not animals.

This is an example of the mood Baroco, because it results from it by substitution. The above argument, however, cannot be applied to this syllogism. We cannot admit that the premisses are true, because the propositions 'All animals are birds' and 'Some owls are not birds' are certainly false. We need not suppose that the conclusion is false; it is false whether we suppose its falsity or not. But the main point is that the contradictory of the conclusion, i.e. the proposition 'All owls are animals', yields together with the first premiss 'All animals are birds' not a false conclusion, but a true one: 'All owls are birds'. The *reductio ad impossibile* is in this case impossible.

The proof given by Aristotle is neither sufficient nor a proof by *reductio ad impossibile*. Aristotle describes indirect proof or the demonstration *per impossibile*, by contrast with direct or ostensive proof, as a proof that posits what it wishes to refute, i.e. to refute by reduction to a statement admitted to be false, whereas ostensive proof starts from propositions admitted to be true.[1] Accordingly,

[1] *An. pr.* ii. 14, 62^b29 διαφέρει δ' ἡ εἰς τὸ ἀδύνατον ἀπόδειξις τῆς δεικτικῆς τῷ τιθέναι ὃ βούλεται ἀναιρεῖν, ἀπάγουσα εἰς ὁμολογούμενον ψεῦδος· ἡ δὲ δεικτικὴ ἄρχεται ἐξ ὁμολογουμένων θέσεων (ἀληθῶν).

if we have to prove a proposition by *reductio ad impossibile*, we must start from its negation and derive thence a statement obviously false. The indirect proof of the mood Baroco should start from the negation of this mood, and not from the negation of its conclusion, and this negation should lead to an unconditionally false statement, and not to a proposition that is admitted to be false only under certain conditions. I shall here give a sketch of such a proof. Let α denote the proposition '*M* belongs to all *N*', β '*N* belongs to all *X*', and γ '*M* belongs to all *X*'. As the negation of an *A*-premiss is an *O*-premiss, 'not-β '[1] will have the meaning '*N* does not belong to some *X*', and 'not-γ ' '*M* does not belong to some *X*'. According to the mood Baroco the implication 'If α and not-γ, then not-β ' is true, or in other words, α and not-γ are not true together with β. The negation, therefore, of this proposition would mean that ' α and β and not-γ ' are together true. But from ' α and β ', ' γ ' results by the mood Barbara; we get therefore ' γ and not-γ ', i.e. a proposition obviously false, being a contradiction *in forma*. It can easily be seen that this genuine proof of the mood Baroco by *reductio ad impossibile* is quite different from that given by Aristotle.

The mood Baroco can be proved from the mood Barbara by a very simple ostensive proof which requires one and only one thesis of the propositional logic. It is the following compound law of transposition:

(4) If (if *p* and *q*, then *r*), then if *p* and it is not true that *r*, then it is not true that *q*.[2]

Put for *p* '*M* belongs to all *N*', for *q* '*N* belongs to all *X*', and for *r* '*M* belongs to all *X*'. By this substitution we get in the antecedent of (4) the mood Barbara, and therefore we can detach the consequent, which reads:

(5) If *M* belongs to all *N* and it is not true that *M* belongs to all *X*, then it is not true that *N* belongs to all *X*.

As the *O*-premiss is the negation of the *A*-premiss, we may replace in (5) the forms 'it is not true that belongs to all' by 'does not belong to some', getting thus the mood Baroco.

There can be no doubt that Aristotle knew the law of transposition referred to in the above proof. This law is closely con-

[1] I am using 'not-' as an abbreviation for the propositional negation 'it is not true that'.

[2] See *Principia Mathematica*, p. 118, thesis *3·37.

nected with the so-called 'conversion' of the syllogism, which he investigated thoroughly.[1] To convert a syllogism means to take the contrary or the contradictory (in proofs *per impossibile* only the contradictory) of the conclusion together with one premiss, thereby destroying the other premiss. 'It is necessary,' Aristotle says, 'if the conclusion has been converted and one of the premisses stands, that the other premiss should be destroyed. For if it should stand, the conclusion must also stand.'[2] This is a description of the compound law of transposition. Aristotle therefore knows this law; moreover, he applies it to obtain from the mood Barbara the moods Baroco and Bocardo. Investigating in the same chapter the conversion of the moods of the first figure, he says: 'Let the syllogism be affirmative (i.e. Barbara), and let it be converted as stated (i.e. by the contradictory denial). Then if *A* does not belong to all *C*, but to all *B*, *B* will not belong to all *C*. And if *A* does not belong to all *C*, but *B* belongs to all *C*, *A* will not belong to all *B*.'[3] The proofs of Baroco and Bocardo are here given in their simplest form.

In the systematic exposition of the syllogistic these valid proofs are replaced by insufficient demonstrations *per impossibile*. The reason is, I suppose, that Aristotle does not recognize arguments ἐξ ὑποθέσεως as instruments of genuine proof. All demonstration is for him proof by categorical syllogisms; he is anxious to show that the proof *per impossibile* is a genuine proof in so far as it contains at least a part that is a categorical syllogism. Analysing the proof of the theorem that the side of a square is incommensurable with its diagonal, he states explicitly: We know by a syllogism that the contradictory of this theorem would lead to an absurd consequence, viz. that odd numbers should be equal to evens, but the theorem itself is proved by an hypothesis, since a falsehood results when it is denied.[4] Of the same kind, Aristotle

[1] *An. pr.* ii. 8–10.

[2] Ibid. 8, 59^b3 ἀνάγκη γὰρ τοῦ συμπεράσματος ἀντιστραφέντος καὶ τῆς ἑτέρας μενούσης προτάσεως ἀναιρεῖσθαι τὴν λοιπήν· εἰ γὰρ ἔσται, καὶ τὸ συμπέρασμα ἔσται. Cf. *Top.* viii. 14, 163^a34 ἀνάγκη γάρ, εἰ τὸ συμπέρασμα μή ἐστι, μίαν τινα ἀναιρεῖσθαι τῶν προτάσεων, εἴπερ πασῶν τεθεισῶν ἀνάγκη ἦν τὸ συμπέρασμα εἶναι.

[3] *An. pr.* ii. 8, 59^b28 ἔστω γὰρ κατηγορικὸς ὁ συλλογισμός, καὶ ἀντιστρεφέσθω οὕτως (i.e. ἀντικειμένως). οὐκοῦν εἰ τὸ *A* οὐ παντὶ τῷ *Γ*, τῷ δὲ *B* παντί, τὸ *B* οὐ παντὶ τῷ *Γ*· καὶ εἰ τὸ μὲν *A* μὴ παντὶ τῷ *Γ*, τὸ δὲ *B* παντί, τὸ *A* οὐ παντὶ τῷ *B*.

[4] Ibid. i. 23, 41^a23 πάντες γὰρ οἱ διὰ τοῦ ἀδυνάτου περαίνοντες τὸ μὲν ψεῦδος συλλογίζονται, τὸ δ' ἐξ ἀρχῆς ἐξ ὑποθέσεως δεικνύουσιν, ὅταν ἀδύνατόν τι συμβαίνῃ τῆς ἀντιφάσεως τεθείσης, οἷον ὅτι ἀσύμμετρος ἡ διάμετρος διὰ τὸ γίνεσθαι τὰ περιττὰ ἴσα

concludes, are all other hypothetical arguments; for in every case the syllogism leads to a proposition that is different from the original thesis, and the original thesis is reached by an admission or some other hypothesis.[1] All this is, of course, not true; Aristotle does not understand the nature of hypothetical arguments. The proof of Baroco and Bocardo by the law of transposition is not reached by an admission or some other hypothesis, but performed by an evident logical law; besides, it is certainly a proof of one categorical syllogism on the ground of another, but it is not performed by a categorical syllogism.

At the end of Book I of the *Prior Analytics* Aristotle remarks that there are many hypothetical arguments that ought to be considered and described, and promises to do so in the sequel.[2] This promise he nowhere fulfils.[3] It was reserved for the Stoics to include the theory of hypothetical arguments in their system of propositional logic, in which the compound law of transposition found its proper place. On the occasion of an argument of Aenesidemus (which is irrelevant for our purpose) the Stoics analysed the following rule of inference which corresponds to the compound law of transposition: 'If the first and the second, then the third; but not the third, yet the first; therefore not the second.'[4] This rule is reduced to the second and third indemonstrable syllogisms of the Stoic logic. We already know the first indemonstrable syllogism, it is the *modus ponens*; the second is the *modus tollens*: 'If the first, then the second; but not the second; therefore not the first.' The third indemonstrable syllogism starts from a denied conjunction and reads: 'Not (the first and the second); but the first; therefore not the second.' According to Sextus Empiricus the analysis runs thus: By the second indemonstrable syllogism we get from the implication 'if the first and the second,

τοῖς ἀρτίοις συμμέτρου τεθείσης. τὸ μὲν οὖν ἴσα γίνεσθαι τὰ περιττὰ τοῖς ἀρτίοις συλλογίζεται, τὸ δ' ἀσύμμετρον εἶναι τὴν διάμετρον ἐξ ὑποθέσεως δείκνυσιν, ἐπεὶ ψεῦδος συμβαίνει διὰ τὴν ἀντίφασιν.

[1] *An. pr.* i. 23, $41^{a}37$ *ὡσαύτως δὲ καὶ οἱ ἄλλοι πάντες οἱ ἐξ ὑποθέσεως· ἐν ἅπασι γὰρ ὁ μὲν συλλογισμὸς γίνεται πρὸς τὸ μεταλαμβανόμενον, τὸ δ' ἐξ ἀρχῆς περαίνεται δι' ὁμολογίας ἤ τινος ἄλλης ὑποθέσεως.*

[2] Ibid. 44, $50^{a}39$ *πολλοὶ δὲ καὶ ἕτεροι περαίνονται ἐξ ὑποθέσεως, οὓς ἐπισκέψασθαι δεῖ καὶ διασημῆναι καθαρῶς. τίνες μὲν οὖν αἱ διαφοραὶ τούτων, καὶ ποσαχῶς γίνεται τὸ ἐξ ὑποθέσεως, ὕστερον ἐροῦμεν.*

[3] Alexander 389. 32, commenting on this passage says: *λέγει καὶ ἄλλους πολλοὺς ἐξ ὑποθέσεως περαίνεσθαι, περὶ ὧν ὑπερτίθεται μὲν ὡς ἐρῶν ἐπιμελέστερον, οὐ μὴν φέρεται αὐτοῦ σύγγραμμα περὶ αὐτῶν.*

[4] The Stoics denote proposition-variables by ordinal numbers.

then the third', and the negation of its consequent 'not the third', the negation of its antecedent 'not (the first and the second)'. From this proposition, which is virtually contained in the premisses, but not explicitly expressed in words, together with the premiss 'the first', there follows the conclusion 'not the second' by the third indemonstrable syllogism.[1] This is one of the neatest arguments we owe to the Stoics. We see that competent logicians reasoned 2,000 years ago in the same way as we are doing today.

§ 19. *The proofs by ecthesis*

The proofs by conversion and *per impossibile* are sufficient to reduce all imperfect syllogisms to perfect ones. But there is still a third kind of proof given by Aristotle, viz. the so-called proofs by exposition or *ἔκθεσις*. Although of little importance for the system, they have an interest in themselves, and it is worth while to study them carefully.

There are only three passages in the *Prior Analytics* where Aristotle gives a short characterization of this kind of proof. The first is connected with the proof of conversion of the *E*-premiss, the second is a proof of the mood Darapti, the third of the mood Bocardo. The word *ἐκθέσθαι* occurs only in the second passage, but there can be no doubt that the other two passages also are meant as proofs by ecthesis.[2]

Let us begin with the first passage, which runs thus: 'If *A*

[1] Sextus Empiricus (ed. Mutschmann), *Adv. math.* viii. 235–6 *συνέστηκε γὰρ ὁ τοιοῦτος λόγος* (scil. *ὁ παρὰ τῷ Αἰνησιδήμῳ ἐρωτηθείς*) *ἐκ δευτέρου ἀναποδείκτου καὶ τρίτου, καθὼς πάρεστι μαθεῖν ἐκ τῆς ἀναλύσεως, ἥτις σαφεστέρα μᾶλλον γενήσεται ἐπὶ τοῦ τρόπου ποιησαμένων ἡμῶν τὴν διδασκαλίαν, ἔχοντος οὕτως· 'εἰ τὸ πρῶτον καὶ τὸ δεύτερον, τὸ τρίτον· οὐχὶ δέ γε τὸ τρίτον, ἀλλὰ καὶ τὸ πρῶτον· οὐκ ἄρα τὸ δεύτερον.' ἐπεὶ γὰρ ἔχομεν συνημμένον ἐν ᾧ ἡγεῖται συμπεπλεγμένον ⟨τὸ⟩ 'τὸ πρῶτον καὶ τὸ δεύτερον', λήγει δὲ ⟨τὸ⟩ 'τὸ τρίτον', ἔχομεν δὲ καὶ τὸ ἀντικείμενον τοῦ λήγοντος τὸ 'οὐ τὸ τρίτον', συναχθήσεται ἡμῖν καὶ τὸ ἀντικείμενον τοῦ ἡγουμένου τὸ 'οὐκ ἄρα τὸ πρῶτον καὶ τὸ δεύτερον' δευτέρῳ ἀναποδείκτῳ. ἀλλὰ δὴ τοῦτο αὐτὸ κατὰ μὲν τὴν δύναμιν ἔγκειται τῷ λόγῳ, ἐπεὶ ἔχομεν τὰ συνακτικὰ αὐτοῦ λήμματα, κατὰ δὲ τὴν προφορὰν παρεῖται. ὅπερ τάξαντες μετὰ τοῦ λειπομένου λήμματος τοῦ 'τὸ πρῶτον',* ἕξομεν συναγόμενον τὸ συμπέρασμα τὸ 'οὐκ ἄρα τὸ δεύτερον' τρίτῳ ἀναποδείκτῳ.* [* *τοῦ πρώτου* codd., *τοῦ τρόπου* Kochalsky, *τοῦ 'τὸ πρῶτον'* scripsi. (*τρόπος* = mood expressed in variables, *συνημμένον* = implication, *ἡγούμενον* = antecedent, *λῆγον* = consequent, *συμπεπλεγμένον* = conjunction.)]

[2] There are two other passages dealing with ecthesis, *An. pr.* $30^{a}6$–14 and $30^{b}31$–40 (I owe this remark to Sir David Ross), but both are related to the scheme of modal syllogisms.

belongs to no *B*, neither will *B* belong to any *A*. For if it should belong to some, say *C*, it would not be true that *A* belongs to no *B*; for *C* is some of the *B*'s.'[1] The conversion of the *E*-premiss is here proved *per impossibile*, but this proof *per impossibile* is based on the conversion of the *I*-premiss which is proved by exposition. The proof by exposition requires the introduction of a new term, called the 'exposed term'; here it is *C*. Owing to the obscurity of the passage the very meaning of this *C* and of the logical structure of the proof can be reached only by conjecture. I shall try to explain the matter on the ground of modern formal logic.

We have to prove the law of conversion of the *I*-premiss: 'If *B* belongs to some *A*, then *A* belongs to some *B*.' Aristotle introduces for this purpose a new term, *C*; it follows from his words that *C* is included in *B* as well as in *A*, so that we get two premisses: '*B* belongs to all *C*' and '*A* belongs to all *C*'. From these premisses we can deduce syllogistically (by the mood Darapti) the conclusion '*A* belongs to some *B*'. This is the first interpretation given by Alexander.[2] But it may be objected that this interpretation presupposes the mood Darapti which is not yet proved. Alexander prefers, therefore, another interpretation which is not based on a syllogism: he maintains that the term *C* is a singular term given by perception, and the proof by exposition consists in a sort of perceptual evidence.[3] This explanation, however, which is accepted by Maier,[4] has no support in the text of the *Prior Analytics*: Aristotle does not say that *C* is an individual term. Moreover, a proof by perception is not a logical proof. If we

[1] *An. pr.* i. 2, 25a15 *εἰ οὖν μηδενὶ τῷ B τὸ A ὑπάρχει, οὐδὲ τῷ A οὐδενὶ ὑπάρξει τὸ B. εἰ γάρ τινι, οἷον τῷ Γ, οὐκ ἀληθὲς ἔσται τὸ μηδενὶ τῷ B τὸ A ὑπάρχειν· τὸ γὰρ Γ τῶν B τί ἐστιν.* [Corr. W. D. Ross.]

[2] Alexander 32. 12 *εἰ γὰρ τὸ B τινὶ τῷ A ὑπάρχει . . . ὑπαρχέτω τῷ Γ· ἔστω γὰρ τοῦτο τὶ τοῦ A, ᾧ ὑπάρχει τὸ B. ἔσται δὴ τὸ Γ ἐν ὅλῳ τῷ B καὶ τὶ αὐτοῦ, καὶ τὸ B κατὰ παντὸς τοῦ Γ· ταὐτὸν γὰρ τὸ ἐν ὅλῳ καὶ κατὰ παντός. ἀλλ' ἦν τὸ Γ τὶ τοῦ A· ἐν ὅλῳ ἄρα καὶ τῷ A τὸ Γ ἐστίν· εἰ δὲ ἐν ὅλῳ, κατὰ παντὸς αὐτοῦ ῥηθήσεται τὸ A. ἦν δὲ τὸ Γ τὶ τοῦ B· καὶ τὸ A ἄρα κατὰ τινὸς τοῦ B κατηγορηθήσεται.*

[3] Ibid. 32 *ἢ ἄμεινόν ἐστι καὶ οἰκειότατον τοῖς λεγομένοις τὸ δι' ἐκθέσεως καὶ αἰσθητικῶς λέγειν τὴν δεῖξιν γεγονέναι, ἀλλὰ μὴ τὸν εἰρημένον τρόπον μηδὲ συλλογιστικῶς. ὁ γὰρ διὰ τῆς ἐκθέσεως τρόπος δι' αἰσθήσεως γίνεται καὶ οὐ συλλογιστικῶς· τοιοῦτον γάρ τι λαμβάνεται τὸ Γ τὸ ἐκτιθέμενον, ὃ αἰσθητὸν ὂν μόριόν ἐστι τοῦ A· εἰ γὰρ κατὰ μορίου τοῦ A ὄντος τοῦ Γ αἰσθητοῦ τινος καὶ καθ' ἕκαστα λέγοιτο τὸ B, εἴη ἂν καὶ τοῦ B μόριον τὸ αὐτὸ Γ ὄν γε ἐν αὐτῷ· ὥστε τὸ Γ εἴη ἂν ἀμφοτέρων μόριον καὶ ἐν ἀμφοτέροις αὐτοῖς.*

[4] Op. cit., vol. ii *a*, p. 20: 'Die Argumentation bedient sich also nicht eines Syllogismus, sondern des Hinweises auf den Augenschein.'

want to prove logically that the premiss '*B* belongs to some *A*' may be converted, and the proof is to be performed by means of a third term *C*, we must find a thesis that connects the above premiss with a proposition containing *C*.

It would not, of course, be true to say simply that if *B* belongs to some *A*, then *B* belongs to all *C* and *A* belongs to all *C*; but a little modification of the consequent of this implication easily solves our problem. We must put before the consequent an existential quantifier, the words 'there exists', binding the variable *C*. For if *B* belongs to some *A*, there always exists a term *C* such that *B* belongs to all *C* and *A* belongs to all *C*. *C* may be the common part of *A* and *B* or a term included in this common part. If, for example, some Greeks are philosophers, there exists a common part of the terms 'Greek' and 'philosopher', viz. 'Greek philosopher', and it is evident that all Greek philosophers are Greeks, and all Greek philosophers are philosophers. We may state, therefore, the following thesis:

(1) If *B* belongs to some *A*, then there exists a *C* such that *B* belongs to all *C* and *A* belongs to all *C*.

This thesis is evident. But also the converse of (1) is evident. If there exists a common part of *A* and *B*, *B* must belong to some *A*. We get, therefore:

(2) If there exists a *C* such that *B* belongs to all *C* and *A* belongs to all *C*, then *B* belongs to some *A*.

It is probable that Aristotle intuitively felt the truth of these theses without being able to formulate them explicitly, and that he grasped their connexion with the conversion of the *I*-premiss without seeing all the deductive steps leading to this result. I shall give here the full formal proof of the conversion of the *I*-premiss, starting from theses (1) and (2), and applying to them some laws of the propositional logic and the rules of existential quantifiers.

The following thesis of the propositional logic was certainly known to Aristotle:

(3) If *p* and *q*, then *q* and *p*.

It is the commutative law of conjunction.[1] Applying this law to the premisses '*B* belongs to all *C*' and '*A* belongs to all *C*', we get:

(4) If *B* belongs to all *C* and *A* belongs to all *C*, then *A* belongs to all *C* and *B* belongs to all *C*.

[1] See *Principia Mathematica*, p. 116, thesis *3·22.

To this thesis I shall apply the rules of existential quantifiers. There are two such rules; both are stated with respect to a true implication. The first rule reads: It is permissible to put before a consequent of a true implication an existential quantifier, binding a free variable occurring in the consequent. It results from this rule that:

(5) If *B* belongs to all *C* and *A* belongs to all *C*, then there exists a *C* such that *A* belongs to all *C* and *B* belongs to all *C*.

The second rule reads: It is permissible to put before the antecedent of a true implication an existential quantifier, binding a free variable occurring in the antecedent, provided that this variable does not occur as a free variable in the consequent. In (5) *C* is already bound in the consequent; therefore according to this rule we may bind *C* in the antecedent, thus getting the formula:

(6) If there exists a *C* such that *B* belongs to all *C* and *A* belongs to all *C*, then there exists a *C* such that *A* belongs to all *C* and *B* belongs to all *C*.

The antecedent of this formula is identical with the consequent of thesis (1); it results, therefore, by the law of the hypothetical syllogism that:

(7) If *B* belongs to some *A*, then there exists a *C* such that *A* belongs to all *C* and *B* belongs to all *C*.

From (2) by interchanging *B* and *A* we get the thesis:

(8) If there exists a *C* such that *A* belongs to all *C* and *B* belongs to all *C*, then *A* belongs to some *B*,

and from (7) and (8) we may deduce by the hypothetical syllogism the law of conversion of the *I*-premiss:

(9) If *B* belongs to some *A*, then *A* belongs to some *B*.

We see from the above that the true reason of the convertibility of the *I*-premiss is the commutability of the conjunction. The perception of an individual term belonging to both *A* and *B* may intuitively convince us of the convertibility of this premiss, but is not sufficient for a logical proof. There is no need to assume *C* as a singular term given by perception.

The proof of the mood Darapti by exposition can now be easily understood. Aristotle reduces this mood to the first figure by conversion, and then he says: 'It is possible to demonstrate this also *per impossibile* and by exposition. For if both *P* and *R* belong to all *S*, should some of the *S*'s, e.g. *N*, be taken, both *P* and *R* will belong to this, and then *P* will belong to some *R*.'[1] Alexander's commentary on this passage deserves our attention. It begins with a critical remark. If *N* were a universal term included in *S*, we should get as premisses '*P* belongs to all *N*' and '*R* belongs to all *N*'. But this is just the same combination of premisses, *συζυγία*, as '*P* belongs to all *S*' and '*R* belongs to all *S*', and the problem remains the same as before. Therefore, Alexander continues, *N* cannot be a universal term; it is a singular term given by perception, a term evidently existing in *P* as well as in *R*, and the whole proof by ecthesis is a proof by perception.[2] We have already met this opinion above. In support of it Alexander adduces three arguments: First, if his explanation were rejected, we should have no proof at all; secondly, Aristotle does not say that *P* and *R* belong to all *N*, but simply to *N*; thirdly, he does not convert the propositions with *N*.[3] None of these arguments is convincing: in our example there is no need of conversion; Aristotle often omits the mark of universality where it should be used,[4] and as to the first argument, we know already that there exists another and a better explanation.

The mood Darapti:

(10) If *P* belongs to all *S* and *R* belongs to all *S*, then *P* belongs to some *R*,

[1] *An. pr.* i. 5, 28ᵃ22 *ἔστι δὲ καὶ διὰ τοῦ ἀδυνάτου καὶ τῷ ἐκθέσθαι ποιεῖν τὴν ἀπόδειξιν· εἰ γὰρ ἄμφω* (scil. *Π καὶ Ρ*) *παντὶ τῷ Σ ὑπάρχει, ἂν ληφθῇ τι τῶν Σ, οἷον τὸ Ν, τούτῳ καὶ τὸ Π καὶ τὸ Ρ ὑπάρξει, ὥστε τινὶ τῷ Ρ τὸ Π ὑπάρξει.*

[2] Alexander 99. 28 *τί γὰρ διαφέρει τῷ Σ ὑπάρχειν λαβεῖν παντὶ τό τε Π καὶ τὸ Ρ καὶ μέρει τινὶ τοῦ Σ τῷ Ν; τὸ γὰρ αὐτὸ καὶ ἐπὶ τοῦ Ν ληφθέντος μένει· ἡ γὰρ αὐτὴ συζυγία ἐστίν, ἄν τε κατὰ τοῦ Ν παντὸς ἐκείνων ἑκάτερον, ἄν τε κατὰ τοῦ Σ κατηγορῆται. ἢ οὐ τοιαύτη ἡ δεῖξις, ᾗ χρῆται· ὁ γὰρ δι' ἐκθέσεως τρόπος δι' αἰσθήσεως γίνεται. οὐ γὰρ ἵνα τοιοῦτόν τι τοῦ Σ λάβωμεν, καθ' οὗ ῥηθήσεται παντὸς καὶ τὸ Π καὶ τὸ Ρ, λέγει . . . ἀλλ' ἵνα τι τῶν ὑπ' αἴσθησιν πιπτόντων, ὃ φανερόν ἐστιν ὂν καὶ ἐν τῷ Π καὶ ἐν τῷ Ρ.*

[3] Ibid. 100. 7 *ὅτι γὰρ αἰσθητὴ ἡ διὰ τῆς ἐκθέσεως δεῖξις, σημεῖον πρῶτον μὲν τὸ εἰ μὴ οὕτως λαμβάνοιτο, μηδεμίαν γίνεσθαι δεῖξιν· ἔπειτα δὲ καὶ τὸ αὐτὸν μηκέτι χρήσασθαι ἐπὶ τοῦ Ν, ὃ ἦν τι τοῦ Σ, τῷ παντὶ αὐτῷ ὑπάρχειν τό τε Π καὶ τὸ Ρ, ἀλλ' ἁπλῶς θεῖναι τὸ ὑπάρχειν· ἀλλὰ καὶ τὸ μηδετέραν ἀντιστρέψαι.*

[4] See, for instance, p. 2, n.

results from a substitution of thesis (2)—take *P* for *B*, and *R* for *A*:

(11) If there exists a *C* such that *P* belongs to all *C* and *R* belongs to all *C*, then *P* belongs to some *R*,

and from the thesis:

(12) If *P* belongs to all *S* and *R* belongs to all *S*, then there exists a *C* such that *P* belongs to all *C* and *R* belongs to all *C*.

Thesis (12) we may prove by applying to the identity:

(13) If *P* belongs to all *C* and *R* belongs to all *C*, then *P* belongs to all *C* and *R* belongs to all *C*,

the second rule of existential quantifiers, getting thus:

(14) If *P* belongs to all *C* and *R* belongs to all *C*, then there exists a *C* such that *P* belongs to all *C* and *R* belongs to all *C*,

and substituting in (14) the letter *S* for the free variable *C*, i.e. performing the substitution in the antecedent only, as it is not permissible to substitute anything for a bound variable.

From (12) and (11) the mood Darapti results by the hypothetical syllogism. We see again that the exposed term *C* is a universal term like *A* or *B*. It is of no consequence, of course, to denote this term by *N* rather than by *C*.

Of greater importance seems to be the third passage, containing the proof by exposition of the mood Bocardo. This passage reads: 'If *R* belongs to all *S*, but *P* does not belong to some *S*, it is necessary that *P* should not belong to some *R*. For if *P* belongs to all *R*, and *R* belongs to all *S*, then *P* will belong to all *S*; but we assumed that it did not. Proof is possible also without reduction *ad impossibile*, if some of the *S*'s be taken to which *P* does not belong.'[1] I shall analyse this proof in the same way as the other proofs by exposition.

Let us denote the part of *S* to which *P* does not belong by *C*; we get two propositions: '*S* belongs to all *C*' and '*P* belongs to no *C*'. From the first of these propositions and the premiss '*R*

[1] *An. pr.* i. 6, 28b17 *εἰ γὰρ τὸ P παντὶ τῷ Σ, τὸ δὲ Π τινὶ μὴ ὑπάρχει, ἀνάγκη τὸ Π τινὶ τῷ P μὴ ὑπάρχειν. εἰ γὰρ παντί, καὶ τὸ P παντὶ τῷ Σ, καὶ τὸ Π παντὶ τῷ Σ ὑπάρξει· ἀλλ' οὐχ ὑπῆρχεν. δείκνυται δὲ καὶ ἄνευ τῆς ἀπαγωγῆς, ἐὰν ληφθῇ τι τῶν Σ ᾧ τὸ Π μὴ ὑπάρχει.*

belongs to all *S*' we get by the mood Barbara the consequence '*R* belongs to all *C*', which yields together with the second proposition '*P* belongs to no *C*' the required conclusion '*P* does not belong to some *R*' by the mood Felapton. The problem is how we can get the propositions with *C* from the original premisses '*R* belongs to all *S*' and '*P* does not belong to some *S*'. The first of these premisses is useless for our purpose as it does not contain *P*; from the second premiss we cannot get our propositions in the ordinary way, since it is particular, and our propositions are universal. But if we introduce the existential quantifier we can get them, for the following thesis is true:

(15) If *P* does not belong to some *S*, then there exists a *C* such that *S* belongs to all *C* and *P* belongs to no *C*.

The truth of this thesis will be obvious if we realize that the required condition for *C* is always fulfilled by that part of *S* to which *P* does not belong.

Starting from thesis (15) we can prove the mood Bocardo on the basis of the moods Barbara and Felapton by means of some laws of propositional logic and the second rule of existential quantifiers. As the proof is rather long, I shall give here only a sketch.

We take as premisses, besides (15), the mood Barbara with transposed premisses:

(16) If *S* belongs to all *C* and *R* belongs to all *S*, then *R* belongs to all *C*,

and the mood Felapton, also with transposed premisses:

(17) If *R* belongs to all *C* and *P* belongs to no *C*, then *P* does not belong to some *R*.

To these premisses we may apply a complicated thesis of propositional logic which, curiously enough, was known to the Peripatetics and is ascribed by Alexander to Aristotle himself. It is called the 'synthetic theorem', *συνθετικὸν θεώρημα*, and runs thus: 'If α and β imply γ, and γ together with δ implies ϵ, then α and β together with δ imply ϵ.'[1] Take for α, β, and γ the first

[1] Alexander 274. 19 *δι' ὧν δὲ λέγει νῦν, ὑπογράφει ἡμῖν φανερώτερον τὸ λεγόμενον 'συνθετικὸν θεώρημα', οὗ αὐτός ἐστιν εὑρετής. ἔστι δὲ ἡ περιοχὴ αὐτοῦ τοιαύτη· 'ὅταν ἔκ τινων συνάγηταί τι, τὸ δὲ συναγόμενον μετὰ τινὸς ἢ τινῶν συνάγῃ τι, καὶ τὰ συνακτικὰ αὐτοῦ μεθ' οὗ ἢ μεθ' ὧν συνάγεται ἐκεῖνο, καὶ αὐτὰ τὸ αὐτὸ συνάξει.'* The following example is given ibid. 26 *ἐπεὶ γὰρ τὸ 'πᾶν δίκαιον ἀγαθὸν' συναγόμενον ὑπὸ τῶν 'πᾶν δίκαιον καλόν, πᾶν καλὸν ἀγαθὸν' συνάγει μετὰ τοῦ 'πᾶν ἀγαθὸν συμφέρον'*

premiss, the second premiss, and the conclusion respectively of Barbara, for δ and ϵ the second premiss and the conclusion respectively of Felapton; we get the formula:

(18) If S belongs to all C and R belongs to all S and P belongs to no C, then P does not belong to some R.

This formula may be transformed by another law of propositional logic into the following:

(19) If S belongs to all C and P belongs to no C, then if R belongs to all S, P does not belong to some R.

To this formula may be applied the second rule of existential quantifiers. For C is a free variable occurring in the antecedent of (19), but not in the consequent. According to this rule we get the thesis:

(20) If there exists a C such that S belongs to all C and P belongs to no C, then if R belongs to all S, P does not belong to some R.

From premiss (15) and thesis (20) there results by the hypothetical syllogism the consequence:

(21) If P does not belong to some S, then if R belongs to all S, P does not belong to some R,

and this is the implicational form of the mood Bocardo.

It is, of course, highly improbable that Aristotle saw all the steps of this deduction; but it is important to know that his intuitions with regard to the proof by ecthesis were right. Alexander's commentary on this proof of the mood Bocardo is worthy of quotation. 'It is possible', he says, 'to prove this mood without assuming some S given by perception and singular, but taking such an S, to none of which P would belong. For P will belong to none of this S, and R to all, and this combination of premisses yields as conclusion that P does not belong to some R.'[1] Here at last Alexander concedes that the exposed term may be universal.

The proofs by exposition have no importance for Aristotle's syllogistic as a system. All theorems proved by ecthesis can be proved by conversion or *per impossibile*. But they are highly important in themselves, as they contain a new logical element the meaning of which was not entirely clear for Aristotle. This was perhaps the reason why he dropped this kind of proof in his final chapter (7) of Book I of the *Prior Analytics*, where he sums up his systematic investigation of syllogistic.[1] Nobody after him understood these proofs. It was reserved for modern formal logic to explain them by the idea of the existential quantifier.

τὸ ' πᾶν δίκαιον συμφέρον', καὶ τὰ ' πᾶν δίκαιον καλόν, πᾶν καλὸν ἀγαθὸν' ὄντα συνακτικὰ τοῦ ' πᾶν δίκαιον ἀγαθὸν' μετὰ τοῦ ' πᾶν ἀγαθὸν συμφέρον' συνάξει τὸ ' πᾶν δίκαιον συμφέρον'.

[1] Alexander 104. 3 *δύναται δ' ἐπὶ τῆς συζυγίας ταύτης δεικνύναι, καὶ εἰ μὴ αἰσθητόν τι τοῦ Σ λαμβάνοιτο καὶ καθ' ἕκαστα, ἀλλὰ τοιοῦτον, οὗ κατὰ μηδενὸς κατηγορηθήσεται τὸ Π. ἔσται γὰρ τὸ μὲν Π κατ' οὐδενὸς αὐτοῦ, τὸ δὲ P κατὰ παντός· ἡ δ' οὕτως ἔχουσα συζυγία συλλογιστικῶς δέδεικται συνάγουσα τὸ τινὶ τῷ P τὸ Π μὴ ὑπάρχειν.*

§ 20. *The rejected forms*

Aristotle in his systematic investigation of syllogistic forms not only proves the true ones but also shows that all the others are false, and must be rejected. Let us see by means of an example how Aristotle proceeds to reject false syllogistic forms. The following two premisses are given: A belongs to all B and B belongs to no C. It is the first figure: A is the first or the major term, B is the middle, and C is the last or the minor term. Aristotle writes:

'If the first term belongs to all the middle, but the middle to none of the last, there will be no syllogism of the extremes; for nothing necessary follows from the terms being so related; for it is possible that the first should belong to all as well as to none of the last, so that neither a particular nor a universal conclusion is necessary. But if there is no necessary consequence by means of these premisses, there cannot be a syllogism. Terms of belonging to all: animal, man, horse; to none: animal, man, stone.'[2]

In contrast to the shortness and obscurity of the proofs by ecthesis, the above passage is rather full and clear. Nevertheless I am afraid it has not been properly understood by the commentators. According to Alexander, Aristotle shows in this passage that from the same combination of premisses there can be

[1] Cf. the comment of Alexander, who maintains to the end his idea of the perceptual character of proofs by ecthesis, 112. 33: *ὅτι δὲ ἡ δι' ἐκθέσεως δεῖξις ἦν αἰσθητικὴ καὶ οὐ συλλογιστική, δῆλον καὶ ἐκ τοῦ νῦν αὐτὸν μηκέτι μνημονεύειν αὐτῆς ὡς διὰ συλλογισμοῦ τινος γινομένης.*

[2] *An. pr.* i. 4, $26^{a}2$ *εἰ δὲ τὸ μὲν πρῶτον παντὶ τῷ μέσῳ ἀκολουθεῖ, τὸ δὲ μέσον μηδενὶ τῷ ἐσχάτῳ ὑπάρχει, οὐκ ἔσται συλλογισμὸς τῶν ἄκρων· οὐδὲν γὰρ ἀναγκαῖον συμβαίνει τῷ ταῦτα εἶναι· καὶ γὰρ παντὶ καὶ μηδενὶ ἐνδέχεται τὸ πρῶτον τῷ ἐσχάτῳ ὑπάρχειν, ὥστε οὔτε τὸ κατὰ μέρος οὔτε τὸ καθόλου γίνεται ἀναγκαῖον· μηδενὸς δὲ ὄντος ἀναγκαίου διὰ τούτων οὐκ ἔσται συλλογισμός. ὅροι τοῦ παντὶ ὑπάρχειν ζῷον, ἄνθρωπος, ἵππος· τοῦ μηδενὶ ζῷον, ἄνθρωπος, λίθος.*

derived (δυνάμενον συνάγεσθαι) for some concrete terms a universal affirmative conclusion, and for some other concrete terms a universal negative conclusion. This is, Alexander asserts, the most obvious sign that such a combination of premisses has no syllogistic force, since opposite and contradictory propositions which destroy each other are proved by it (δείκνυται).[1] What Alexander says is certainly misleading, for nothing can be formally derived from an asyllogistic combination of premisses, and nothing can be proved by it. Besides, propositions with different concrete subjects and predicates are neither opposite to each other nor contradictory. Maier again puts the terms pointed out by Aristotle into a syllogistical form:

all men are animals	all men are animals
no horse is a man	no stone is a man
all horses are animals	no stone is an animal

(the premisses are underlined by him, as in a syllogism), and says that there results (*ergibt sich*) from logically equivalent premisses a universal affirmative proposition as well as a universal negative.[2] We shall see below that the terms given by Aristotle are not intended to be put into the form of a syllogism, and that nothing results formally from the premisses of the would-be syllogisms quoted by Maier. In view of these misunderstandings a logical analysis of the matter seems to be necessary.

If we want to prove that the following syllogistic form:

(1) If *A* belongs to all *B* and *B* belongs to no *C*, then *A* does not belong to some *C*,

is not a syllogism, and consequently not a true logical theorem, we must show that there exist such values of the variables *A*, *B*, and *C* as verify the premisses without verifying the conclusion. For an implication containing variables is true only when all the

[1] Alexander 55. 22 καὶ γὰρ καθόλου καταφατικὸν ἐπί τινος ὕλης δείξει δυνάμενον συνάγεσθαι καὶ πάλιν ἐπ' ἄλλης καθόλου ἀποφατικόν, ὃ ἐναργέστατον σημεῖον τοῦ μηδεμίαν ἔχειν τὴν συζυγίαν ταύτην ἰσχὺν συλλογιστικήν, εἴ γε τά τε ἐναντία καὶ τὰ ἀντικείμενα ἐν αὐτῇ δείκνυται, ὄντα ἀλλήλων ἀναιρετικά.

[2] Op. cit., vol. ii. *a*, p. 76: 'Es handelt sich also um folgende Kombinationen:

aller Mensch ist Lebewesen	aller Mensch ist Lebewesen
kein Pferd ist Mensch	kein Stein ist Mensch
alles Pferd ist Lebewesen	kein Stein ist Lebewesen

So wird an Beispielen gezeigt, dass bei der in Frage stehenden Prämissenzusammenstellung von logisch völlig gleichen Vordersätzen aus sowohl ein allgemein bejahender, als ein allgemein verneinender Satz sich ergeben könne.'

values of variables that verify the antecedent verity the consequent also. The easiest way of showing this is to find concrete terms verifying the premisses '*A* belongs to all *B*' and '*B* belongs to no *C*', but not verifying the conclusion '*A* does not belong to some *C*'. Aristotle found such terms: take 'animal' for *A*, 'man' for *B*, 'horse' for *C*. The premisses 'Animal belongs to all man' or 'All men are animals', and 'Man belongs to no horse' or 'No horses are men', are verified; but the conclusion 'Animal does not belong to some horse' or 'Some horses are not animals' is false. Formula (1), therefore, is not a syllogism. For the same reason neither will the following form:

(2) If *A* belongs to all *B* and *B* belongs to no *C*, then *A* belongs to no *C*,

be a syllogism, because the premisses are verified for the same concrete terms as before, but the conclusion 'Animal belongs to no horse' or 'No horses are animals' is false. It follows from the falsity of (1) and (2) that no negative conclusion can be drawn from the given premisses.

Nor can an affirmative conclusion be drawn from them. Take the next syllogistical form:

(3) If *A* belongs to all *B* and *B* belongs to no *C*, then *A* belongs to some *C*.

There exist values for *A*, *B*, and *C*, i.e. concrete terms, that verify the premisses without verifying the conclusion. Aristotle again gives such terms: take 'animal' for *A*, 'man' for *B*, 'stone' for *C*. The premisses are verified, for it is true that 'All men are animals' and 'No stone is a man', but the conclusion 'Some stone is an animal' is obviously false. Formula (3), therefore, is not a syllogism. Neither can the last form:

(4) If *A* belongs to all *B* and *B* belongs to no *C*, then *A* belongs to all *C*,

be a syllogism, since for the given terms the premisses are verified as before, but the conclusion 'All stones are animals' is not verified. It results from the above that no conclusion whatever can be derived from the combination of premisses '*A* belongs to all *B*' and '*B* belongs to no *C*', where *A* is the predicate and *B* is the subject of the conclusion. This combination of premisses is useless for syllogistic.

The main point of this process of rejection is to find a true universal affirmative proposition (like 'All horses are animals') and a true universal negative proposition (like 'No stone is an animal'), both compatible with the premisses. It is not sufficient to find, for instance, for some terms a true universal affirmative statement, and for some other terms a true particular negative statement. This opinion was put forward by Alexander's teacher Herminus and some older Peripatetics, and was rightly refuted by Alexander.[1] This is again a proof that Aristotle's ideas of rejection have not been properly understood.

The syllogistic forms (1)–(4) are rejected by Aristotle on the basis of some concrete terms that verify the premisses without verifying the conclusion. Aristotle, however, knows yet another kind of proof for rejection. Investigating the syllogistic forms of the second figure, Aristotle states generally that in this figure neither two affirmative nor two negative premisses yield a necessary conclusion, and then continues thus:

'Let *M* belong to no *N*, and not to some *X*. It is possible then for *N* to belong either to all *X* or to no *X*. Terms of belonging to none: black, snow, animal. Terms of belonging to all cannot be found, if *M* belongs to some *X*, and does not belong to some *X*. For if *N* belonged to all *X*, and *M* to no *N*, then *M* would belong to no *X*; but it is assumed that it belongs to some *X*. In this way, then, it is not possible to take terms, and the proof must start from the indefinite nature of the particular premiss. For since it is true that *M* does not belong to some *X*, even if it belongs to no *X*, and since if it belongs to no *X* a syllogism is not possible, clearly it will not be possible either.'[2]

Aristotle here begins the proof of rejection by giving concrete terms, as in the first example. But then he breaks off his proof, as he cannot find concrete terms that would verify the premisses

[1] Cf. Alexander 89. 34–90. 27. The words of Herminus are quoted 89. 34: Ἑρμῖνος δὲ λέγει 'ἐφ' ἧς γὰρ συζυγίας τὴν ἀντίφασιν ἔνεστι συναγομένην δεῖξαι, εὔλογον ταύτην μηδὲν ἔλαττον ἀσυλλόγιστον λέγειν τῆς ἐν ᾗ τὰ ἐναντία συνάγεται· ἀσυνύπαρκτα γὰρ καὶ ταῦτα ὁμοίως ἐκείνοις.'

[2] *An. pr.* i. 5, 27^b12–23 ἔστωσαν γὰρ . . . στερητικαί, οἷον τὸ *M* τῷ μὲν *N* μηδενὶ τῷ δὲ Ξ τινὶ μὴ ὑπαρχέτω· ἐνδέχεται δὴ καὶ παντὶ καὶ μηδενὶ τῷ Ξ τὸ *N* ὑπάρχειν. ὅροι τοῦ μὲν μὴ ὑπάρχειν· μέλαν, χιών, ζῷον· τοῦ δὲ παντὶ ὑπάρχειν οὐκ ἔστι λαβεῖν, εἰ τὸ *M* τῷ Ξ τινὶ μὲν ὑπάρχει, τινὶ δὲ μή. εἰ γὰρ παντὶ τῷ Ξ τὸ *N*, τὸ δὲ *M* μηδενὶ τῷ *N*, τὸ *M* οὐδενὶ τῷ Ξ ὑπάρξει· ἀλλ' ὑπέκειτο τινὶ ὑπάρχειν. οὕτω μὲν οὖν οὐκ ἐγχωρεῖ λαβεῖν ὅρους, ἐκ δὲ τοῦ ἀδιορίστου δεικτέον· ἐπεὶ γὰρ ἀληθεύεται τὸ τινὶ μὴ ὑπάρχειν τὸ *M* τῷ Ξ καὶ εἰ μηδενὶ ὑπάρχει, μηδενὶ δὲ ὑπάρχοντος οὐκ ἦν συλλογισμός, φανερὸν ὅτι οὐδὲ νῦν ἔσται.

'*M* belongs to no *N*' and '*M* does not belong to some *X*', without verifying the proposition '*N* does not belong to some *X*', provided *M*, which does not belong to some *X*, belongs at the same time to some (other) *X*. The reason is that from the premisses '*M* belongs to no *N*' and '*M* belongs to some *X*' the proposition '*N* does not belong to some *X*' follows by the mood Festino. But it is not necessary that *M* should belong to some *X*, when it does not belong to some (other) *X*; *M* might belong to no *X*. Concrete terms verifying the premisses '*M* belongs to no *N*' and '*M* belongs to no *X*', and not verifying the proposition '*N* does not belong to some *X*', can easily be chosen, and in fact Aristotle found them, rejecting the syllogistic form of the second figure with universal negative premisses; the required terms are: *M*—'line', *N*—'animal', *X*—'man'.[1] The same terms may be used to disprove the syllogistic form:

(5) If *M* belongs to no *N* and *M* does not belong to some *X*, then *N* does not belong to some *X*.

For the premiss 'No animal is a line' is true, and the second premiss 'Some man is not a line' is also true, as it is true that 'No man is a line', but the conclusion 'Some man is not an animal' is false. Aristotle, however, does not finish his proof in this way,[2] because he sees another possibility: if the form with universal negative premisses:

(6) If *M* belongs to no *N* and *M* belongs to no *X*, then *N* does not belong to some *X*,

is rejected, (5) must be rejected too. For if (5) stands, (6), having a stronger premiss than (5), must also stand.

Modern formal logic, as far as I know, does not use 'rejection' as an operation opposed to Frege's 'assertion'. The rules of rejection are not yet known. On the ground of the above proof of Aristotle we may state the following rule:

(*c*) If the implication 'If α, then β' is asserted, but its consequent β is rejected, then its antecedent α must be rejected too.

[1] Ibid. 27^a20 οὐδ' (scil. ἔσται συλλογισμός) ὅταν μήτε τοῦ *N* μήτε τοῦ Ξ μηδενὸς κατηγορῆται τὸ *M*. ὅροι τοῦ ὑπάρχειν γραμμή, ζῷον, ἄνθρωπος, τοῦ μὴ ὑπάρχειν γραμμή, ζῷον, λίθος.

[2] Alexander completed this proof, 88. 12: τοῦ παντὶ τὸ *N* τῷ Ξ ὑπάρχειν ὅροι· γραμμὴ τὸ *M*, ζῷον τὸ *N*, ἄνθρωπος τὸ Ξ· ἡ μὲν γὰρ γραμμὴ οὐδενὶ ζῴῳ καὶ τινὶ οὐχ ὑπάρχει ἀνθρώπῳ ἐπεὶ καὶ μηδενί, ζῷον δὲ παντὶ ἀνθρώπῳ.

This rule can be applied not only to reject (5) if (6) is rejected, but also to reject (2) if (1) is rejected. For from an *E*-premiss an *O*-premiss follows, and if (2) is true, then (1) must be true. But if (1) is rejected, so must (2) be rejected.

The rule (*c*) for rejection corresponds to the rule of detachment for assertion. We may accept another rule for rejection corresponding to the rule of substitution for assertion. It can be formulated thus:

(*d*) If α is a substitution for β, and α is rejected, then β must be rejected too.

Example: suppose that '*A* does not belong to some *A*' is rejected; then '*A* does not belong to some *B*' must be rejected too, since, if the second expression were asserted, we should obtain from it by substitution the first expression, which is rejected.

The first of these rules was anticipated by Aristotle, the second was unknown to him. Both enable us to reject some forms, provided that some other forms have already been rejected. Aristotle rejects some forms by means of concrete terms, as 'man', 'animal', 'stone'. This procedure is correct, but it introduces into logic terms and propositions not germane to it. 'Man' and 'animal' are not logical terms, and the proposition 'All men are animals' is not a logical thesis. Logic cannot depend on concrete terms and statements. If we want to avoid this difficulty, we must reject some forms axiomatically. I have found that if we reject the two following forms of the second figure axiomatically:

(7) If *A* belongs to all *B* and *A* belongs to all *C*, then *B* belongs to some *C*, and

(8) If *A* belongs to no *B* and *A* belongs to no *C*, then *B* belongs to some *C*,

all the other forms may be rejected by the rules (*c*) and (*d*).

§ 21. *Some unsolved problems*

The Aristotelian system of non-modal syllogisms is a theory of four constants which may be denoted by 'All — is', 'No — is', 'Some — is', and 'Some — is not'. These constants are functors of two arguments which are represented by variables having as values only concrete universal terms. Singular, empty, and also negative terms are excluded as values. The constants together

with their arguments form four kinds of proposition called premisses, viz. 'All *A* is *B*', 'No *A* is *B*', 'Some *A* is *B*', and 'Some *A* is not *B*'. The system may be called 'formal logic', as concrete terms, like 'man' or 'animal', belong not to it but only to its applications. The system is not a theory of the forms of thought, nor is it dependent on psychology; it is similar to a mathematical theory of the relation 'greater than', as was rightly observed by the Stoics.

The four kinds of premiss form theses of the system by means of two functors 'if — then' and 'and'. These functors belong to propositional logic, which is an auxiliary theory of the system. In some proofs we meet a third propositional functor, viz. the propositional negation 'It is not true that', denoted shortly by 'not'. The four Aristotelian constants 'All — is', 'No — is', 'Some — is' and 'Some — is not', together with the three propositional constants 'if — then', 'and', and 'not', are the sole elements of the syllogistic.

All theses of the system are propositions regarded as true for all values of the variables that occur in them. No Aristotelian syllogism is formulated as a rule of inference with the word 'therefore', as is done in the traditional logic. The traditional logic is a system different from the Aristotelian syllogistic, and should not be mixed up with the genuine logic of Aristotle. Aristotle divided syllogisms into three figures, but he knew and accepted all the syllogistic moods of the fourth figure. The division of syllogisms into figures is of no logical importance and has only a practical aim: we want to be sure that no valid syllogistical mood is omitted.

The system is axiomatized. As axioms Aristotle takes the two first moods of the first figure, Barbara and Celarent. To these two axioms we have to add two laws of conversion, as these cannot be proved syllogistically. If we wish to have the law of identity, 'All *A* is *A*,' in the system we have to assume it axiomatically. The simplest basis we can get is to take the constants 'All — is' and 'Some — is' as primitive terms, to define the two other constants by means of those terms with the help of propositional negation, and to assume as axioms four theses, viz. the two laws of identity and the moods Barbara and Datisi, or Barbara and Dimaris. It is not possible to build up the system on one axiom only. To look for the principle of the Aristotelian syllogistic is a vain attempt, if 'principle' means the same as 'axiom'. The so-called *dictum de omni et nullo* cannot be the principle of syllogistic in this sense, and was never stated to be such by Aristotle himself.

Aristotle reduces the so-called imperfect syllogisms to the perfect, i.e. to the axioms. Reduction here means proof or deduction of a theorem from the axioms. He uses three kinds of proof: by conversion, by *reductio ad impossibile*, and by ecthesis. Logical analysis shows that in all the proofs of the first two kinds there are involved theses of the most elementary part of propositional logic, the theory of deduction. Aristotle uses them intuitively, but soon after him the Stoics, who were the inventors of the first system of propositional logic, stated some of them explicitly—the compound law of transposition and the so-called 'synthetic theorem', which is ascribed to Aristotle but does not exist in his extant logical works. A new logical element seems to be implied by the proofs by ecthesis: they can be explained with the help of existential quantifiers. The systematic introduction of quantifiers into the syllogistic would completely change this system: the primitive term 'Some — is' could be defined by the term 'All — is', and many new theses would arise not known to Aristotle. As Aristotle himself has dropped the proofs by ecthesis in his final summary of the syllogistic, there is no need to introduce them into his system.

Another new logical element is contained in Aristotle's investigation of the inconclusive syllogistic forms: it is rejection. Aristotle rejects invalid forms by exemplification through concrete terms. This procedure is logically correct, but it introduces into the system terms and propositions not germane to it. There are, however, cases where he applies a more logical procedure, reducing one invalid form to another already rejected. On the basis of this remark a rule of rejection could be stated corresponding to the rule of detachment by assertion; this can be regarded as the commencement of a new field of logical inquiries and of new problems that have to be solved.

Aristotle does not systematically investigate the so-called polysyllogisms, i.e. syllogisms with more than three terms and two premisses. As we have seen, Galen studied compound syllogisms consisting of four terms and three premisses. It is an old error to ascribe to Galen the authorship of the fourth figure: Galen divided the compound syllogisms of four terms into four figures, but not the simple ones known to us by their medieval names. His investigations were entirely forgotten. But compound syllogisms also belong to the syllogistic and have to be taken into account, and here is another problem that has to be studied systematically. An essential contribution to this problem is the set of formulae given by C. A. Meredith, and mentioned above at the end of section 14.

There still remains one problem not seen by Aristotle, but of the utmost importance for his whole system: it is the problem of decision. The number of significant expressions of the syllogistic is infinite; most of them are certainly false, but some of them may be true, like valid polysyllogisms of *n* terms where *n* is any integer whatever. Can we be sure that our axioms together with our rules of inference are sufficient to prove all the true expressions of the syllogistic? And similarly, can we be sure that our rules of rejection, formulated at the end of section 20, are sufficient to reject all the false expressions, provided that a finite number of them is rejected axiomatically? I raised these problems in 1938 in my Seminar on Mathematical Logic at the University of Warsaw. One of my former pupils, now Professor of Logic and Methodology at the University of Wrocław, J. Słupecki, found the solution to both problems. His answer to the first question was positive, to the second negative. According to Słupecki it is not possible to reject all the false expressions of the syllogistic by means of the rules (*c*) and (*d*) quoted in section 20, provided a finite number of them is rejected axiomatically. However many false expressions we may reject axiomatically, there always exist other false expressions that cannot be rejected otherwise than axiomatically. But it is impossible to establish an infinite set of axioms. A new rule of rejection must be added to the system to complete the insufficient characterization of the Aristotelian logic given by the four axioms. This rule was found by Słupecki.

Słupecki's rule of rejection peculiar to Aristotle's syllogistic can be formulated in the following way: Let α and β denote negative premisses of the Aristotelian logic, i.e. premisses of the type 'No *A* is *B*' or 'Some *A* is not *B*', and let γ denote either a simple premiss (of any kind) or an implication the consequent of which is a simple premiss and the antecedent a conjunction of such premisses: if the expressions 'If α, then γ' and 'If β, then γ'

are rejected, then the expression 'If α and β, then γ' must be rejected too.[1] This rule, together with the rules of rejection (*c*) and (*d*) and the axiomatically rejected expression 'If all *C* is *B* and all *A* is *B*, then some *A* is *C*', enables us to reject any false expression of the system. Besides, we suppose as given the four asserted axioms of the syllogistic, the definitions of the *E*- and the *O*-premiss, the rules of inference for asserted expressions, and the theory of deduction as an auxiliary system. In this way the problem of decision finds its solution: for any given significant expression of the system we can decide whether it is true and may be asserted or whether it is false and must be rejected.

By the solution of this problem the main investigations on Aristotle's syllogistic are brought to an end. There remains only one problem, or rather one mysterious point waiting for an explanation: in order to reject all the false expressions of the system it is necessary and sufficient to reject axiomatically only one false expression, viz. the syllogistic form of the second figure with universal affirmative premisses and a particular affirmative conclusion. There exists no other expression suitable for this purpose. The explanation of this curious logical fact may perhaps lead to new discoveries in the field of logic.

[1] J. Słupecki, 'Z badań nad sylogistyką Arystotelesa' (Investigation on Aristotle's Syllogistic), *Travaux de la Société des Sciences et des Lettres de Wrocław*, Sér. B, No. 9, Wrocław (1948). See chapter v, devoted to the problem of decision.

CHAPTER IV

ARISTOTLE'S SYSTEM IN SYMBOLIC FORM

§ 22. *Explanation of the symbolism*

This chapter does not belong to the history of logic. Its purpose is to set out the system of non-modal syllogisms according to the requirements of modern formal logic, but in close connexion with the ideas set forth by Aristotle himself.

Modern formal logic is strictly formalistic. In order to get an exactly formalized theory it is more convenient to employ a symbolism invented for this purpose than to make use of ordinary language which has its own grammatical laws. I have therefore to start from the explanation of such a symbolism. As the Aristotelian syllogistic involves the most elementary part of the propositional logic called theory of deduction, I shall explain the symbolic notation of both these theories.

In both theories there occur variables and constants. Variables are denoted by small Latin letters, constants by Latin capitals. By the initial letters of the alphabet *a*, *b*, *c*, *d*, ..., I denote term-variables of the Aristotelian logic. These term-variables have as values universal terms, as 'man' or 'animal'. For the constants of this logic I employ the capital letters *A*, *E*, *I*, and *O*, used already in this sense by the medieval logicians. By means of these two kinds of letters I form the four functions of the Aristotelian logic, writing the constants before the variables:

Aab	means	All *a* is *b*	or *b* belongs to all *a*,
Eab	,,	No *a* is *b*	,, *b* belongs to no *a*,
Iab	,,	Some *a* is *b*	,, *b* belongs to some *a*,
Oab	,,	Some *a* is not *b*	,, *b* does not belong to some *a*.

The constants *A*, *E*, *I*, and *O* are called functors, *a* and *b* their arguments. All Aristotelian syllogisms are composed of these four types of function connected with each other by means of the words 'if' and 'and'. These words also denote functors, but of a different kind from the Aristotelian constants: their arguments are not term-expressions, i.e. concrete terms or term-variables, but propositional expressions, i.e. propositions like

'All men are animals', propositional functions like '*Aab*', or propositional variables. I denote propositional variables by *p*, *q*, *r*, *s*, ..., the functor 'if' by *C*, the functor 'and' by *K*. The expression *Cpq* means 'if *p*, then *q*' ('then' may be omitted) and is called 'implication' with *p* as the antecedent and *q* as the consequent. *C* does not belong to the antecedent, it only combines the antecedent with the consequent. The expression *Kpq* means '*p* and *q*' and is called 'conjunction'. We shall meet in some proofs a third functor of propositional logic, propositional negation. This is a functor of one argument and is denoted by *N*. It is difficult to render the function *Np* either in English or in any other modern language, as there exists no single word for the propositional negation.[1] We have to say by circumlocution 'it-is-not-true-that *p*' or 'it-is-not-the-case-that *p*'. For the sake of brevity I shall use the expression 'not-*p*'.

The principle of my notation is to write the functors before the arguments. In this way I can avoid brackets. This symbolism without brackets, which I invented and have employed in my logical papers since 1929,[2] can be applied to mathematics as well as to logic. The associative law of addition runs in the ordinary notation thus:

$$(a+b)+c = a+(b+c),$$

and cannot be stated without brackets. If you write, however, the functor + before its arguments, you get:

$$(a+b)+c = ++abc \quad \text{and} \quad a+(b+c) = +a+bc.$$

The law of association can be now written without brackets:

$$++abc = +a+bc.$$

Now I shall explain some expressions written down in this symbolic notation. The symbolic expression of a syllogism is easy to understand. Take, for instance, the mood Barbara:

If all *b* is *c* and all *a* is *b*, then all *a* is *c*.

It reads in symbols:

CKAbcAabAac.

[1] The Stoics used for propositional negation the single word οὐχί.

[2] See, for instance, Łukasiewicz and Tarski, 'Untersuchungen über den Aussagenkalkül', *Comptes Rendus des séances de la Société des Sciences et des Lettres de Varsovie*, xxiii (1930), Cl. III, pp. 31–2.

The conjunction of the premisses *Abc* and *Aab*, viz. *KAbcAab*, is the antecedent of the formula, the conclusion *Aac* is its consequent.

Some expressions of the theory of deduction are more complicated. Take the symbolic expression of the hypothetical syllogism:

If (if *p*, then *q*), then [if (if *q*, then *r*), then (if *p*, then *r*)].

It reads:

CCpqCCqrCpr.

In order to understand the construction of this formula you must remember that *C* is a functor of two propositional arguments which follow immediately after *C*, forming together with *C* a new compound propositional expression. Of this kind are the expressions *Cpq*, *Cqr*, and *Cpr* contained in the formula. Draw brackets around each of them; you will get the expression:

C(*Cpq*)*C*(*Cqr*)(*Cpr*).

Now you can easily see that (*Cpq*) is the antecedent of the whole formula, and the rest, i.e. *C*(*Cqr*)(*Cpr*), is the consequent, having (*Cqr*) as its antecedent and (*Cpr*) as its consequent.

In the same way we may analyse all the other expressions, for instance the following, which contains *N* and *K* besides *C*:

CCKpqrCKNrqNp.

Remember that *K*, like *C*, is a functor of two arguments, and that *N* is a functor of one argument. By using different kinds of brackets we get the expression:

C[*C*(*Kpq*)*r*]{*C*[*K*(*Nr*)*q*](*Np*)}.

[*C*(*Kpq*)*r*] is here the antecedent of the whole formula while {*C*[*K*(*Nr*)*q*](*Np*)} is its consequent, having the conjunction [*K*(*Nr*)*q*] as its antecedent and the negation (*Np*) as its consequent.

§ 23. *Theory of deduction*

The most fundamental logical system on which all the other logical systems are built up is the theory of deduction. As every logician is bound to know this system, I shall here describe it in brief.

The theory of deduction can be axiomatized in several different ways, according to which functors are chosen as primitive terms. The simplest way is to follow Frege, who takes as primitive terms the functors of implication and negation, in our symbolism *C* and *N*. There exist many sets of axioms of the *C–N*-system; the simplest of them and the one almost universally accepted was discovered by myself before 1929.[1] It consists of three axioms:

T1. *CCpqCCqrCpr*
T2. *CCNppp*
T3. *CpCNpq*.

The first axiom is the law of the hypothetical syllogism already explained in the foregoing section. The second axiom, which reads in words 'If (if not-*p*, then *p*), then *p*', was applied by Euclid to the proof of a mathematical theorem.[2] I call it the law of Clavius, as Clavius (a learned Jesuit living in the second half of the sixteenth century, one of the constructors of the Gregorian calendar) first drew attention to this law in his commentary on Euclid. The third axiom, in words 'If *p*, then if not-*p*, then *q*', occurs for the first time, as far as I know, in a commentary on Aristotle ascribed to Duns Scotus; I call it the law of Duns Scotus.[3] This law contains the venom usually imputed to contradiction: if two contradictory sentences, like α and $N\alpha$, were true together, we could derive from them by means of this law the arbitrary proposition *q*, i.e. any proposition whatever.

There belong to the system two rules of inference: the rule of substitution and the rule of detachment.

The rule of substitution allows us to deduce new theses from a thesis asserted in the system by writing instead of a variable a significant expression, everywhere the same for the same variable. Significant expressions are defined inductively in the following way: (*a*) any propositional variable is a significant expression; (*b*) $N\alpha$ is a significant expression provided α is a

[1] First published in Polish: 'O znaczeniu i potrzebach logiki matematycznej' (On the Importance and Requirements of Mathematical Logic), *Nauka Polska*, vol. x, Warsaw (1929), pp. 610–12. Cf. also the German contribution quoted in p. 78, n. 2: Satz 6, p. 35.

[2] See above, section 16.

[3] Cf. my paper quoted in p. 48, n.

significant expression; (*c*) $C\alpha\beta$ is a significant expression provided α and β are significant expressions.

The rule of detachment is the *modus ponens* of the Stoics referred to above: if a proposition of the type $C\alpha\beta$ is asserted and its antecedent α is asserted too, it is permissible to assert its consequent β, and detach it from the implication as a new thesis.

By means of these two rules we can deduce from our set of axioms all the true theses of the *C–N*-system. If we want to have in the system other functors besides *C* and *N*, e.g. *K*, we must introduce them by definitions. This can be done in two different ways, as I shall show on the example of *K*. The conjunction '*p* and *q*' means the same as 'it-is-not-true-that (if *p*, then not-*q*)'. This connexion between *Kpq* and *NCpNq* may be expressed by the formula:

$$Kpq = NCpNq,$$

where the sign = corresponds to the words 'means the same as'. This kind of definition requires a special rule of inference allowing us to replace the *definiens* by the *definiendum* and vice versa. Or we may express the connexion between *Kpq* and *NCpNq* by an equivalence, and as equivalence is not a primitive term of our system, by two implications converse to each other:

$$CKpqNCpNq \quad \text{and} \quad CNCpNqKpq.$$

In this case a special definition-rule is not needed. I shall use definitions of the first kind.

Let us now see by an example how new theses can be derived from the axioms by the help of rules of inference. I shall deduce from T1–T3 the law of identity *Cpp*. The deduction requires two applications of the rule of substitution and two applications of the rule of detachment; it runs thus:

T1. *q*/*CNpq* × *C*T3–T4
T4. *CCCNpqrCpr*
T4. *q*/*p*, *r*/*p* × *C*T2–T5
T5. *Cpp*.

The first line is called the derivational line. It consists of two parts separated from each other by the sign ×. The first part, T1. *q*/*CNpq*, means that in T1 *CNpq* has to be substituted for *q*. The thesis produced by this substitution is omitted in order to save space. It would be of the following form:

(I) *CCpCNpqCCCNpqrCpr*.

The second part, *C*T3–T4, shows how this omitted thesis is constructed, making it obvious that the rule of detachment may be applied to it. Thesis (I) begins with *C*, and then there follow axiom T3 as antecedent and thesis T4 as consequent. We can therefore detach T4 as a new thesis. The derivational line before T5 has a similar explanation. The stroke (/) is the sign of substitution and the short rule (–) the sign of detachment. Almost all subsequent deductions are performed in the same manner.

One must be very expert in performing such proofs if one wants to deduce from the axioms T1–T3 the law of commutation *CCpCqrCqCpr* or even the law of simplification *CpCqp*. I shall therefore explain an easy method of verifying expressions of our system without deducing them from the axioms. This method, invented by the American logician Charles S. Peirce about 1885, is based on the so-called principle of bivalence, which states that every proposition is either true or false, i.e. that it has one and only one of two possible truth-values: truth and falsity. This principle must not be mixed up with the law of the excluded middle, according to which of two contradictory propositions one must be true. It was stated as the basis of logic by the Stoics, in particular by Chrysippus.[1]

All functions of the theory of deduction are truth-functions, i.e. their truth and falsity depend only upon the truth and falsity of their arguments. Let us denote a constant false proposition by *o*, and a constant true proposition by *1*. We may define negation in the following way:

$$No = 1 \quad \text{and} \quad N1 = o.$$

This means: the negation of a false proposition means the same as a true proposition (or, shortly, is true) and the negation of a true proposition is false. For implication we have the following four definitions:

$$Coo = 1, \quad Co1 = 1, \quad C1o = o, \quad C11 = 1.$$

[1] Cicero, *Acad. pr.* ii. 95 'Fundamentum dialecticae est, quidquid enuntietur (id autem appellant ἀξίωμα) aut verum esse aut falsum'; *De fato* 21 'Itaque contendit omnes nervos Chrysippus ut persuadeat omne ἀξίωμα aut verum esse aut falsum.' In the Stoic terminology ἀξίωμα means 'proposition', not 'axiom'.

This means: an implication is false only when its antecedent is true and its consequent false; in all the other cases it is true. This is the oldest definition of implication, stated by Philon of Megara and adopted by the Stoics.[1] For conjunction we have the four evident equalities:

$$Koo = o, \quad Ko1 = o, \quad K1o = o, \quad K11 = 1.$$

A conjunction is true only when both its arguments are true; in all the other cases it is false.

Now if we want to verify a significant expression of the theory of deduction containing all or some of the functors *C*, *N*, and *K* we have to substitute for the variables occurring in the expression the symbols *o* and *1* in all possible permutations, and reduce the formulae thus obtained on the basis of the equalities given above. If after the reduction all the formulae give *1* as the final result, the expression is true or a thesis; if any one of them gives *o* as the final result, the expression is false. Let us take as an example of the first kind the law of transposition *CCpqCNqNp*; we get:

For $p/o,\ q/o$: $CCooCNoNo = C1C11 = C11 = 1$,
„ $p/o,\ q/1$: $CCo1CN1No = C1Co1 = C11 = 1$,
„ $p/1,\ q/o$: $CC1oCNoN1 = CoC1o = Coo = 1$,
„ $p/1,\ q/1$: $CC11CN1N1 = C1Coo = C11 = 1$.

As for all substitutions the final result is *1*, the law of transposition is a thesis of our system. Let us now take as an example of the second kind the expression *CKpNqq*. It suffices to try only one substitution:

$$p/1,\ q/o: CK1Noo = CK11o = C1o = o.$$

This substitution gives *o* as the final result, and therefore the expression *CKpNqq* is false. In the same way we may check all the theses of the theory of deduction employed as auxiliary premisses in Aristotle's syllogistic.

§ 24. *Quantifiers*

Aristotle had no clear idea of quantifiers and did not use them in his works; consequently we cannot introduce them into his syllogistic. But, as we have already seen, there are two points in his system which we can understand better if we explain them

[1] Sextus Empiricus, *Adv. math.* viii. 113 ὁ μὲν Φίλων ἔλεγεν ἀληθὲς γίνεσθαι τὸ συνημμένον, ὅταν μὴ ἄρχηται ἀπ' ἀληθοῦς καὶ λήγῃ ἐπὶ ψεῦδος, ὥστε τριχῶς μὲν γίνεσθαι κατ' αὐτὸν ἀληθὲς συνημμένον, καθ' ἕνα δὲ τρόπον ψεῦδος.

by employing quantifiers. Universal quantifiers are connected with the so-called 'syllogistic necessity', existential or particular quantifiers with the proofs by ecthesis. I shall now translate into symbols the proofs with existential quantifiers set down in section 19, and then the argument dependent on universal quantifiers mentioned in section 5.

I denote quantifiers by Greek capitals, the universal quantifier by *Π*, and the particular or existential quantifier by *Σ*. *Π* may be read 'for all', and *Σ* 'for some' or 'there exists'; e.g. *ΣcKAcbAca* means in words: 'There exists a *c* such that all *c* is *b* and all *c* is *a*', or more briefly: 'For some *c*, all *c* is *b* and all *c* is *a*.' Every quantified expression, for instance *ΣcKAcbAca*, consists of three parts: part one, in our example *Σ*, is always a quantifier; part two, here *c*, is always a variable bound by the preceding quantifier; part three, here *KAcbAca*, is always a propositional expression containing the variable just bound by the quantifier as a free variable. It is by putting *Σc* before *KAcbAca* that the free variable *c* in this last formula becomes bound. We may put it briefly: *Σ* (part one) binds *c* (part two) in *KAcbAca* (part three).

The rules of existential quantifiers have already been set out in section 19. In derivational lines I denote by *Σ1* the rule allowing us to put *Σ* before the antecedent, and by *Σ2* the rule allowing us to put it before the consequent of a true implication. The following deductions will be easily understood, as they are translations of the deductions given in words in section 19, the corresponding theses bearing the same running number and having corresponding small letters as variables instead of capitals.

Proof of conversion of the I-premiss

Theses assumed as true without proof:

(1) *CIabΣcKAcbAca*
(2) *CΣcKAcbAcaIab*

Theses (1) and (2) can be used as a definition of the *I*-premiss.

(3) *CKpqKqp* (commutative law of conjunction)

(3) *p/Acb, q/Aca* × (4)

(4) *CKAcbAcaKAcaAcb*

(4) *Σ2c* × (5)

(5) *CKAcbAcaΣcKAcaAcb*

(5) *Σ1c* × (6)

(6) *CΣcKAcbAcaΣcKAcaAcb*

T1. *CCpqCCqrCpr* (law of the hypothetical syllogism)

T1. *p/Iab, q/ΣcKAcbAca, r/ΣcKAcaAcb* × *C*(1)–*C*(6)–(7)

(7) *CIabΣcKAcaAcb*

(2) *b/a, a/b* × (8)

(8) *CΣcKAcaAcbIba*

T1. *p/Iab, q/ΣcKAcaAcb, r/Iba* × *C*(7)–*C*(8)–(9)

(9) *CIabIba*

The derivational lines show that (4) and (8) result from other theses by substitution only, and (7) and (9) by substitution and two detachments. Upon this pattern the reader himself may try to construct the proof of the mood Darapti, which is easy.

Proof of the mood Bocardo

(The variables *P*, *R*, and *S* used in section 19 must be re-lettered, as the corresponding small letters *p*, *r*, and *s* are reserved to denote propositional variables: write *d* for *P*, *a* for *R*, and *b* for *S*.)

Thesis assumed without proof:

(15) *CObdΣcKAcbEcd*

Two syllogisms taken as premisses:

(16) *CKAcbAbaAca* (Barbara)
(17) *CKAcaEcdOad* (Felapton)

T6. *CCKpqrCCKrstCKKpqst*

This is the 'synthetic theorem' ascribed to Aristotle.

T6. *p/Acb, q/Aba, r/Aca, s/Ecd, t/Oad* × *C*(16)–*C*(17)–(18)

(18) *CKKAcbAbaEcdOad*

T7. *CCKKpqrsCKprCqs* (auxiliary thesis)

T7. *p/Acb, q/Aba, r/Ecd, s/Oad* × *C*(18)–(19)

(19) *CKAcbEcdCAbaOad*

(19) *Σ1c* × (20)

(20) *CΣcKAcbEcdCAbaOad*

T1. *CCpqCCqrCpr*

T1. *p/Obd, q/ΣcKAcbEcd, r/CAbaOad* × *C*(15)–*C*(20)–(21)

(21) *CObdCAbaOad*

This is the implicational form of the mood Bocardo. If we wish to have the usual conjunctional form of this mood, we must apply to (21) the so-called law of importation:

T8. *CCpCqrCKpqr.*

We get:

T8. *p/Obd, q/Aba, r/Oad* × *C*(21)–(22)

(22) *CKObdAbaOad* (Bocardo).

By the so-called law of exportation,

T9. *CCKpqrCpCqr,*

which is the converse of the law of importation, we can get the implicational form of the mood Bocardo back from its conjunctional form.

The rules of universal quantifiers are similar to the rules of particular quantifiers set out in section 19. The universal quantifier can be put before the antecedent of a true implication unconditionally, binding a free variable occurring in the antecedent, and before the consequent of a true implication only under the condition that the variable which is to be bound in the consequent does not occur in the antecedent as a free variable. I denote the first of these rules by *Π1*, the second by *Π2*.

Two derived rules result from the above primitive rules of universal quantifiers: first, it is permissible (by rule *Π2* and the law of simplification) to put universal quantifiers in front of a true expression binding free variables occurring in it; secondly, it is permissible (by rule *Π1* and the propositional law of identity) to drop universal quantifiers standing in front of a true expression. How these rules may be derived I shall explain by the example of the law of conversion of the *I*-premiss.

From the law of conversion

(9) *CIabIba*

there follows the quantified expression

(26) *ΠaΠbCIabIba,*

and from the quantified expression (26) there follows again the unquantified law of conversion (9).

First: from (9) follows (26).

T10. *CpCqp* (law of simplification)

T10. *p/CIabIba* × *C*(9)–(23)

(23) *CqCIabIba*

To this thesis we apply rule *Π2* binding *b*, and then *a*, as neither *b* nor *a* occurs in the antecedent:

(23) *Π2b* × (24)

(24) *CqΠbCIabIba*

(24) *Π2a* × (25)

(25) *CqΠaΠbCIabIba*

(25) *q/CpCqp* × *C*T10–(26)

(26) *ΠaΠbCIabIba*

Secondly: from (26) follows (9).

T5. *Cpp* (law of identity)

T5. *p/CIabIba* × (27)

(27) *CCIabIbaCIabIba*

To this thesis we apply rule *Π1* binding *b*, and then *a*:

(27) *Π1b* × (28)

(28) *CΠbCIabIbaCIabIba*

(28) *Π1a* × (29)

(29) *CΠaΠbCIabIbaCIabIba*

(29) × *C*(26)–(9)

(9) *CIabIba*

Aristotle asserts: 'If some *a* is *b*, it is necessary that some *b* should be *a*.' The expression 'it is necessary that' can have, in my opinion, only this meaning: it is impossible to find such values of the variables *a* and *b* as would verify the antecedent without verifying the consequent. That means, in other words: 'For all *a*, and for all *b*, if some *a* is *b*, then some *b* is *a*.' This is our quantified thesis (26). It has been proved that this thesis is equivalent to the unquantified law of conversion 'If some *a* is *b*, then some *b* is *a*', which does not contain the sign of necessity. Since the syllogistic necessity is equivalent to a universal quantifier it may be omitted, as a universal quantifier may be omitted at the head of a true formula.

§ 25. *Fundamentals of the syllogistic*

Every axiomatized deductive system is based on three fundamental elements: primitive terms, axioms, and rules of inference. I start from the fundamentals for asserted expressions, the fundamental elements for the rejected ones being given later.

As primitive terms I take the constants *A* and *I*, defining by them the two other constants, *E* and *O*:

Df 1. *Eab* = *NIab*
Df 2. *Oab* = *NAab*.

In order to abbreviate the proofs I shall employ instead of the above definitions the two following rules of inference:

Rule RE: *NI* may be everywhere replaced by *E* and conversely.

Rule RO: *NA* may be everywhere replaced by *O* and conversely.

The four theses of the system axiomatically asserted are the two laws of identity and the moods Barbara and Datisi:

1. *Aaa*
2. *Iaa*
3. *CKAbcAabAac* (Barbara)
4. *CKAbcIbaIac* (Datisi).

Besides the rules RE and RO I accept the two following rules of inference for the asserted expressions:

(*a*) Rule of substitution: If α is an asserted expression of the system, then any expression produced from α by a valid substitution is also an asserted expression. The only valid substitution is to put for term-variables *a*, *b*, *c* other term-variables, e.g. *b* for *a*.

(*b*) Rule of detachment: If $C\alpha\beta$ and α are asserted expressions of the system, then β is an asserted expression.

As an auxiliary theory I assume the *C–N*-system of the theory of deduction with *K* as a defined functor. For propositional variables propositional expressions of the syllogistic may be substituted, like *Aab*, *Iac*, *KEbcAab*, etc. In all subsequent proofs (and also for rejected expressions) I shall employ only the following fourteen theses denoted by roman numerals:

I.	*CpCqp*	(law of simplification)
II.	*CCqrCCpqCpr*	(law of hypothetical syllogism, 2nd form)
III.	*CCpCqrCqCpr*	(law of commutation)
IV.	*CpCNpq*	(law of Duns Scotus)
V.	*CCNppp*	(law of Clavius)
VI.	*CCpqCNqNp*	(law of transposition)
VII.	*CCKpqrCpCqr*	(law of exportation)
VIII.	*CpCCKpqrCqr*	
IX.	*CCspCCKpqrCKsqr*	
X.	*CCKpqrCCsqCKpsr*	
XI.	*CCrsCCKpqrCKqps*	
XII.	*CCKpqrCKpNrNq*	
XIII.	*CCKpqrCKNrqNp*	
XIV.	*CCKpNqNrCKprq*	

Thesis VIII is a form of the law of exportation, theses IX–XI are compound laws of hypothetical syllogism, and XII–XIV are compound laws of transposition. All of these can be easily verified by the *0–1* method explained in section 23. Theses IV and V give together with II and III the whole *C–N*-system, but IV and V are needed only in proofs for rejected expressions.

The system of axioms 1–4 is consistent, i.e. non-contradictory. The easiest proof of non-contradiction is effected by regarding term-variables as proposition-variables, and by defining the functions *A* and *I* as always true, i.e. by putting *Aab* = *Iab* = *KCaaCbb*. The axioms 1–4 are then true as theses of the theory of deduction, and as it is known that the theory of deduction is non-contradictory, the syllogistic is non-contradictory too.

All the axioms of our system are independent of each other. The proofs of this may be given by interpretation in the field of the theory of deduction. In the subsequent interpretations the term-variables are treated as propositional variables.

Independence of axiom 1: Take *K* for *A*, and *C* for *I*. Axiom 1 is not verified, for *Aaa* = *Kaa*, and *Kaa* gives *0* for *a*/*0*. The other axioms are verified, as can be seen by the *0–1* method.

Independence of axiom 2: Take *C* for *A*, and *K* for *I*. Axiom 2 is not verified, for *Iaa* = *Kaa*. The other axioms are verified.

Independence of axiom 4: Take *C* for *A* and *I*. Axiom 4 is not verified, for *CKAbcIbaIac* = *CKCbcCbaCac* gives *0* for *b*/*0*, *a*/*1*, *c*/*0*. The rest are verified.

Independence of axiom 3: it is impossible to prove the independence of this axiom on the ground of a theory of deduction with only two truth-values, *o* and *1*. We must introduce a third truth-value, let us say *2*, which may be regarded as another symbol for truth, i.e. for *1*. To the equivalences given for *C*, *N*, and *K* in section 23, we have to add the following formulae:

$$Co2 = C12 = C21 = C22 = 1, \quad C20 = 0, \quad N2 = 0,$$
$$Ko2 = K20 = 0, \quad K12 = K21 = K22 = 1.$$

It can easily be shown that under these conditions all the theses of the *C–N*-system are verified. Let us now define *Iab* as a function always true, i.e. $Iab = 1$ for all values of *a* and *b*, and *Aab* as a function with the values

$Aaa = 1$, $Ao1 = A12 = 1$, and $Ao2 = o$ (the rest is irrelevant).

Axioms 1, 2, and 4 are verified, but from 3 we get by the substitutions $b/1$, $c/2$, a/o: $CKA12Ao1Ao2 = CK1Io = C1o = o$.

It is also possible to give proofs of independence by interpretation in the field of natural numbers. If we want, for instance, to prove that axiom 3 is independent of the remaining axioms, we can define *Aab* as $a+1 \neq b$, and *Iab* as $a+b = b+a$. *Iab* is always true, and therefore axioms 2 and 4 are verified. Axiom 1 is also verified, for $a+1$ is always different from *a*. But axiom 3, i.e. 'If $b+1 \neq c$ and $a+1 \neq b$, then $a+1 \neq c$', is not verified: Take 3 for *a*, 2 for *b*, and 4 for *c*: the premisses will be true and the conclusion false.

It results from the above proofs of independence that there exists no single axiom or 'principle' of the syllogistic. The four axioms 1–4 may be mechanically conjoined by the word 'and' into one proposition, but they remain distinct in this inorganic conjunction without representing one single idea.

§ 26. *Deduction of syllogistic theses*

From axioms 1–4 we can derive all the theses of the Aristotelian logic by means of our rules of inference and by the help of the theory of deduction. I hope that the subsequent proofs will be quite intelligible after the explanations given in the foregoing sections. In all syllogistical moods the major term is denoted by *a*, the middle term by *b*, and the minor term by *c*.

The major premiss is stated first, so that it is easy to compare the formulae with the traditional names of the moods.[1]

A. The Laws of Conversion

VII. p/Abc, q/Iba, $r/Iac \times C4–5$

5. *CAbcCIbaIac*

5. b/a, c/a, $a/b \times C1–6$

6. *CIabIba* (law of conversion of the *I*-premiss)

III. p/Abc, q/Iba, $r/Iac \times C5–7$

7. *CIbaCAbcIac*

7. b/a, $c/b \times C2–8$

8. *CAabIab* (law of subordination for affirmative premisses)

II. q/Iab, $r/Iba \times C6–9$

9. *CCpIabCpIba*

9. $p/Aab \times C8–10$

10. *CAabIba* (law of conversion of the *A*-premiss)

6. a/b, $b/a \times 11$

11. *CIbaIab*

VI. p/Iba, $q/Iab \times C11–12$

12. *CNIabNIba*

12. RE × 13

13. *CEabEba* (law of conversion of the *E*-premiss)

VI. p/Aab, $q/Iab \times C8–14$

14. *CNIabNAab*

14. RE, RO × 15

15. *CEabOab* (law of subordination for negative premisses)

B. The Affirmative Moods

X. p/Abc, q/Iba, $r/Iac \times C4–16$

16. *CCsIbaCKAbcsIac*

16. $s/Iab \times C6–17$

17. *CKAbcIabIac* (Darii)

[1] In my Polish text-book, *Elements of Mathematical Logic*, published in 1929 (see p. 46, n. 3), I showed for the first time how the known theses of the syllogistic may be formally deduced from axioms 1–4 (pp. 180–90). The method expounded in the above text-book is accepted with some modifications by I. M. Bocheński, O.P., in his contribution: *On the Categorical Syllogism*, Dominican Studies, vol. i, Oxford (1948).

16. *s/Aab*×C10–18
18. *CKAbcAabIac* (Barbari)

8. *a/b, b/a*×19
19. *CAbaIba*

16. *s/Aba*×C19–20
20. *CKAbcAbaIac* (Darapti)

XI. *r/Iba, s/Iab*×C11–21
21. *CCKpqIbaCKqpIab*

4. *c/a, a/c*×22
22. *CKAbaIbcIca*

21. *p/Aba, q/Ibc, b/c*×C22–23
23. *CKIbcAbaIac* (Disamis)

17. *c/a, a/c*×24
24. *CKAbaIcbIca*

21. *p/Aba, q/Icb, b/c*×C24–25
25. *CKIcbAbaIac* (Dimaris)

18. *c/a, a/c*×26
26. *CKAbaAcbIca*

21. *p/Aba, q/Acb, b/c*×C26–27
27. *CKAcbAbaIac* (Bramantip)

C. The Negative Moods

XIII. *p/Ibc, q/Aba, r/Iac*×C23–28
28. *CKNIacAbaNIbc*

28. RE×29
29. *CKEacAbaEbc*

29. *a/b, b/a*×30
30. *CKEbcAabEac* (Celarent)

IX. *s/Eab, p/Eba*×C13–31
31. *CCKEbaqrCKEabqr*

31. *a/c, q/Aab, r/Eac*×C30–32
32. *CKEcbAabEac* (Cesare)

XI. *r/Eab, s/Eba*×C13–33
33. *CCKpqEabCKqpEba*

32. *c/a, a/c*×34
34. *CKEabAcbEca*

33. *p/Eab, q/Acb, a/c, b/a*×C34–35
35. *CKAcbEabEac* (Camestres)

30. *c/a, a/c*×36
36. *CKEbaAcbEca*

33. *p/Eba, q/Acb, a/c, b/a*×C36–37
37. *CKAcbEbaEac* (Camenes)

II. *q/Eab, r/Oab*×C15–38
38. *CCpEabCpOab*

38. *p/KEbcAab, b/c*×C30–39
39. *CKEbcAabOac* (Celaront)

38. *p/KEcbAab, b/c*×C32–40
40. *CKEcbAabOac* (Cesaro)

38. *p/KAcbEab, b/c*×C35–41
41. *CKAcbEabOac* (Camestrop)

38. *p/KAcbEba, b/c*×C37–42
42. *CKAcbEbaOac* (Camenop)

XIII. *p/Abc, q/Iba, r/Iac*×C4–43
43. *CKNIacIbaNAbc*

43. RE, RO×44
44. *CKEacIbaObc*

44. *a/b, b/a*×45
45. *CKEbcIabOac* (Ferio)

31. *a/c, q/Iab, r/Oac*×C45–46
46. *CKEcbIabOac* (Festino)

X. *p/Ebc, q/Iab, r/Oac*×C45–47
47. *CCsIabCKEbcsOac*

47. *s/Iba*×C11–48
48. *CKEbcIbaOac* (Ferison)

31. *a/c, q/Iba, r/Oac*×C48–49
49. *CKEcbIbaOac* (Fresison)

10. *a/b, b/a*×50
50. *CAbaIab*

47. *s/Aba*×C50–51
51. *CKEbcAbaOac* (Felapton)

31. *a/c, q/Aba, r/Oac*×C51–52
52. *CKEcbAbaOac* (Fesapo)

As a result of all these deductions one remarkable fact deserves our attention: it was possible to deduce twenty syllogistic moods without employing axiom 3, the mood Barbara. Even Barbari could be proved without Barbara. Axiom 3 is the most important thesis of the syllogistic, for it is the only syllogism that yields a universal affirmative conclusion, but in the system of simple syllogisms it has an inferior rank, being necessary to prove only two syllogistic moods, Baroco and Bocardo. Here are these two proofs:

XII. *p/Abc, q/Aab, r/Aac* × *C*3–53

53. *CKAbcNAacNAab*

53. RO × 54

54. *CKAbcOacOab*

54. *b/c, c/b* × 55

55. *CKAcbOabOac* (Baroco)

XIII. *p/Abc, q/Aab, r/Aac* × *C*3–56

56. *CKNAacAabNAbc*

56. RO × 57

57. *CKOacAabObc*

57. *a/b, b/a* × 58

58. *CKObcAbaOac* (Bocardo)

§ 27. *Axioms and rules for rejected expressions*

Of two intellectual acts, to assert a proposition and to reject it,[1] only the first has been taken into account in modern formal logic. Gottlob Frege introduced into logic the idea of assertion, and the sign of assertion (⊢), accepted afterwards by the authors of *Principia Mathematica*. The idea of rejection, however, so far as I know, has been neglected up to the present day.

We assert true propositions and reject false ones. Only true propositions can be asserted, for it would be an error to assert a proposition that was not true. An analogous property cannot be asserted of rejection: it is not only false propositions that have to be rejected. It is true, of course, that every proposition is either true or false, but there exist propositional expressions that are neither true nor false. Of this kind are the so-called propositional functions, i.e. expressions containing free variables

[1] I owe this distinction to Franz Brentano, who describes the acts of believing as *anerkennen* and *verwerfen*.

and becoming true for some of their values, and false for others. Take, for instance, *p*, the propositional variable: it is neither true nor false, because for *p*/1 it becomes true, and for *p*/0 it becomes false. Now, of two contradictory propositions, α and *N*α, one must be true and the other false, one therefore must be asserted and the other rejected. But neither of the two contradictory propositional functions, *p* and *Np*, can be asserted, because neither of them is true: they both have to be rejected.

The syllogistic forms rejected by Aristotle are not propositions but propositional functions. Let us take an example: Aristotle says that no syllogism arises in the first figure, when the first term belongs to all the middle, but to none of the last. The syllogistic form therefore:

(*i*) *CKAbcEabIac*

is not asserted by him as a valid syllogism, but rejected. Aristotle himself gives concrete terms disproving the above form: take for *b* 'man', for *c* 'animal', and for *a* 'stone'. But there are other values for which the formula (*i*) can be verified: by identifying the variables *a* and *c* we get a true implication *CKAbaEabIaa*, for its antecedent is false and its consequent true. The negation of the formula (*i*):

(*j*) *NCKAbcEabIac*

must therefore be rejected too, because for *c/a* it is false.

By introducing quantifiers into the system we could dispense with rejection. Instead of rejecting the form (*i*) we could assert the thesis:

(*k*) *ΣaΣbΣcNCKAbcEabIac*.

This means: there exist terms *a*, *b*, and *c* that verify the negation of (*i*). The form (*i*), therefore, is not true for all *a*, *b*, and *c*, and cannot be a valid syllogism. In the same way instead of rejecting the expression (*j*) we might assert the thesis:

(*l*) *ΣaΣbΣcCKAbcEabIac*.

But Aristotle knows nothing of quantifiers; instead of adding to his system new theses with quantifiers he uses rejection. As rejection seems to be a simpler idea than quantification, let us follow in Aristotle's steps.

Aristotle rejects most invalid syllogistic forms by exemplification through concrete terms. This is the only point where we cannot follow him, because we cannot introduce into logic such concrete terms as 'man' or 'animal'. Some forms must be rejected axiomatically. I have found[1] that if we reject axiomatically the two following forms of the second figure:

CKAcbAabIac
CKEcbEabIac,

all the other invalid syllogistic forms may be rejected by means of two rules of rejection:

(*c*) Rule of rejection by detachment: if the implication 'If α, then β' is asserted, but the consequent β is rejected, then the antecedent α must be rejected too.
(*d*) Rule of rejection by substitution: if β is a substitution of α, and β is rejected, then α must be rejected too.

Both rules are perfectly evident.

The number of syllogistic forms is $4 \times 4^3 = 256$; 24 forms are valid syllogisms, 2 forms are rejected axiomatically. It would be tedious to prove that the remaining 230 invalid forms may be rejected by means of our axioms and rules. I shall only show, by the example of the forms of the first figure with premisses *Abc* and *Eab*, how our rules of rejection work on the basis of the first axiom of rejection.

Rejected expressions I denote by an asterisk put before their serial number. Thus we have:

*59. *CKAcbAabIac* (Axiom)
*59*a*. *CKEcbEabIac*
I. *p/Iac*, *q/KAcbAab* × 60
60. *CIacCKAcbAabIac*
60 × *C**61–*59
*61. *Iac*.

Here for the first time is applied the rule of rejection by detachment. The asserted implication 60 has a rejected consequent, *59; therefore its antecedent, *61, must be rejected too. In this same way I get the rejected expressions *64, *67, *71, *74, and *77.

[1] See section 20.

V. *p/Iac* × 62
62. *CCNIacIacIac*
62. RE × 63
63. *CCEacIacIac*
63 × *C**64–*61
*64. *CEacIac*
1. *a/c* × 65
65. *Acc*
VIII. *p/Acc*, *q/Eac*, *r/Iac* × *C*65–66
66. *CCKAccEacIacCEacIac*
66 × *C**67–*64
*67. *CKAccEacIac*
*67 × *68. *b/c*
*68. *CKAbcEabIac*

Here the rule of rejection by substitution is applied. Expression *68 must be rejected, because by the substitution of *b* for *c* in *68 we get the rejected expression *67. The same rule is used to get* 75.

II. *q/Aab*, *r/Iab* × *C*8–69
69. *CCpAabCpIab*
69. *p/KAbcEab*, *b/c* × 70
70. *CCKAbcEabAacCKAbcEabIac*
70 × *C**71–*68
*71. *CKAbcEabAac*
XIV. *p/Acb*, *q/Iac*, *r/Aab* × 72
72. *CCKAcbNIacNAabCKAcbAabIac*
72. RE, RO × 73
73. *CCKAcbEacOabCKAcbAabIac*
73 × *C**74–*59
*74. *CKAcbEacOab*
*74 × *75. *b/c*, *c/b*
*75. *CKAbcEabOac*
38. *p/KAbcEab*, *b/c* × 76
76. *CCKAbcEabEacCKAbcEabOac*
76 × *C**77–*75
*77. *CKAbcEabEac*

The rejected expressions *68, *71, *75, and *77 are the four possible forms of the first figure having as premisses *Abc* and *Eab*. From these premisses no valid conclusion can be drawn in the first figure. We can prove in the same way on the basis of the two axiomatically rejected forms that all the other invalid syllogistic forms in all the four figures must be rejected too.

§ 28. *Insufficiency of our axioms and rules*

Although it is possible to prove all the known theses of the Aristotelian logic by means of our axioms and rules of assertion, and to disprove all the invalid syllogistic forms by means of our axioms and rules of rejection, the result is far from being satisfactory. The reason is that besides the syllogistic forms there exist many other significant expressions in the Aristotelian logic, indeed an infinity of them, so that we cannot be sure whether from our system of axioms and rules all the true expressions of the syllogistic can be deduced or not, and whether all the false expressions can be rejected or not. In fact, it is easy to find false expressions that cannot be rejected by means of our axioms and rules of rejection. Such, for instance, is the expression:

(F1) *CIabCNAabAba.*

It means: 'If some *a* is *b*, then if it is not true that all *a* is *b*, all *b* is *a*.' This expression is not true in the Aristotelian logic, and cannot be proved by the axioms of assertion, but it is consistent with them and added to the axioms does not entail any invalid syllogistic form. It is worth while to consider the system of the syllogistic as thus extended.

From the laws of the Aristotelian logic:

8. *CAabIab* and
50. *CAbaIab*

and the law of the theory of deduction:

(*m*) *CCprCCqrCCNpqr*

we can derive the following new thesis 78:

(*m*) *p*/*Aab*, *q*/*Aba*, *r*/*Iab* × *C*8–*C*50–78
78. *CCNAabAbaIab.*

This thesis is a converse implication with regard to (F1), and together with (F1) gives an equivalence. On the ground of this equivalence we may define the functor *I* by the functor *A*:

(F2) *Iab* = *CNAabAba.*

This definition reads: ' "Some *a* is *b*" means the same as "If it is not true that all *a* is *b*, then all *b* is *a*".' As the expression 'If not-*p*, then *q*' is equivalent to the alternation 'Either *p* or *q*', we can also say: ' "Some *a* is *b*" means the same as "Either all *a* is *b* or all *b* is *a*".' It is now easy to find an interpretation of this extended system in the so-called Eulerian circles. The terms *a*, *b*, *c* are represented by circles, as in the usual interpretation, but on the condition that no two circles shall intersect each other. Axioms 1–4 are verified, and the forms *59 *CKAcbAabIac* and *59*a* *CKEcbEabIac* are rejected, because it is possible to draw two circles lying outside each other and included in a third circle, which refutes the form *CKAcbAabIac*, and to draw three circles each excluding the two others, which refutes the form *CKEcbEabIac*. Consequently all the laws of the Aristotelian logic are verified, and all the invalid syllogistic forms are rejected. The system, however, is different from the Aristotelian syllogistic, because the formula (F1) is false, as we can see from the following example: it is true that 'Some even numbers are divisible by 3', but it is true neither that 'All even numbers are divisible by 3' nor that 'All numbers divisible by 3 are even'.

It results from this consideration that our system of axioms and rules is not categorical, i.e. not all interpretations of our system verify and falsify the same formulae or are isomorphic. The interpretation just expounded verifies the formula (F1) which is not verified by the Aristotelian logic. The system of our axioms and rules, therefore, is not sufficient to give a full and exact description of the Aristotelian syllogistic.

In order to remove this difficulty we could reject the expression (F1) axiomatically. But it is doubtful whether this remedy would be effective; there may be other formulae of the same kind as (F1), perhaps even an infinite number of such formulae. The problem is to find a system of axioms and rules for the Aristotelian syllogistic on which we could decide whether any given significant expression of this system has to be asserted or rejected. To this most important problem of decision the next chapter is devoted.