SYMBOLIC LOGIC

BY

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APPENDIX II

THE STRUCTURE OF THE SYSTEM OF STRICT IMPLICATION [1]

The System of Strict Implication, as presented in Chapter V of *A Survey of Symbolic Logic* (University of California Press, 1918), contained an error with respect to one postulate. This was pointed out by Dr. E. L. Post, and was corrected by me in the *Journal of Philosophy, Psychology, and Scientific Method* (XVII [1920], 300). The amended postulates (set A below) compare with those of Chapter VI of this book (set B below) as follows:

[1] This appendix is written by Mr. Lewis, but the points demonstrated are, most of them, due to other persons.

Groups II and III, below, were transmitted to Mr. Lewis by Dr. M. Wajsberg, of the University of Warsaw, in 1927. Dr. Wajsberg's letter also contained the first proof ever given that the System of Strict Implication is not reducible to Material Implication, as well as the outline of a system which is equivalent to that deducible from the postulates of Strict Implication with the addition of the postulate later suggested in Becker's paper and cited below as C11. It is to be hoped that this and other important work of Dr. Wajsberg will be published shortly.

Groups I, IV, and V are due to Dr. William T. Parry, who also discovered independently Groups II and III. Groups I, II, and III are contained in his doctoral dissertation, on file in the Harvard University Library. Most of the proofs in this appendix have been given or suggested by Dr. Parry.

It follows from Dr. Wajsberg's work that there is an unlimited number of groups, or systems, of different cardinality, which satisfy the postulates of Strict Implication. Mr. Paul Henle, of Harvard University, later discovered another proof of this same fact. Mr. Henle's proof, which can be more easily indicated in brief space, proceeds by demonstrating that any group which satisfies the Boole-Schröder Algebra will also satisfy the postulates of Strict Implication if $\Diamond p$ be determined as follows:

$$\Diamond p = 1 \text{ when and only when } p \neq 0;$$

$$\Diamond p = 0 \text{ when and only when } p = 0.$$

This establishes the fact that there are as many distinct groups satisfying the postulates as there are powers of 2, since it has been shown by Huntington that there is a group satisfying the postulates of the Boole-Schröder Algebra for every power of 2 ("Sets of Independent Postulates for the Algebra of Logic," *Trans. Amer. Math. Soc.*, V [1904], 309).

The proof of (14), p. 498, is due to Y. T. Shen (Shen Yuting).

A1. $p\,q \,.\prec.\, q\,p$
A2. $q\,p \,.\prec.\, p$
A3. $p \,.\prec.\, p\,p$
A4. $p(q\,r) \,.\prec.\, q(p\,r)$
A5. $p \prec \sim(\sim p)$
A6. $p \prec q \,.\, q \prec r :\prec.\, p \prec r$
A7. $\sim\Diamond p \prec \sim p$
A8. $p \prec q \,.\prec.\, \sim\Diamond q \prec \sim\Diamond p$

B1. $p\,q \,.\prec.\, q\,p$
B2. $p\,q \,.\prec.\, p$
B3. $p \,.\prec.\, p\,p$
B4. $(p\,q)r \,.\prec.\, p(q\,r)$
B5. $p \prec \sim(\sim p)$
B6. $p \prec q \,.\, q \prec r :\prec.\, p \prec r$
B7. $p \,.\, p \prec q :\prec.\, q$
B8. $\Diamond(p\,q) \prec \Diamond p$
B9. $(\exists\, p, q) : \sim(p \prec q) \,.\, \sim(p \prec \sim q)$

The primitive ideas and definitions are not identical in the two cases; but they form equivalent sets, in connection with the postulates.

Comparison of these two sets of postulates, as well as many other points concerning the structure of Strict Implication, will be facilitated by consideration of the following groups. Each of these is based upon the same matrix for the relation $p\,q$ and the function of negation $\sim p$. (This is a four-valued matrix which satisfies the postulates of the Boole-Schröder Algebra.) The groups differ by their different specification of the function $\Diamond p$. We give the fundamental matrix for $p\,q$ and $\sim p$ in the first case only. The matrix for $p \prec q$, resulting from this and the particular determination of $\Diamond p$, is given for each group:

GROUP I

$p\,q$	q: 1	2	3	4	$\sim p$	$\Diamond$	$\prec$	1	2	3	4
p: 1	1	2	3	4	4	1	1	2	4	4	4
2	2	2	4	4	3	1	2	2	2	4	4
3	3	4	3	4	2	1	3	2	4	2	4
4	4	4	4	4	1	3	4	2	2	2	2

GROUP II

$\Diamond$	$\prec$	1	2	3	4
1	1	1	4	3	4
2	2	1	1	3	3
1	3	1	4	1	4
4	4	1	1	1	1

GROUP III

$\Diamond$	$\prec$	1	2	3	4
1	1	1	4	4	4
1	2	1	1	4	4
1	3	1	4	1	4
4	4	1	1	1	1

Group IV

$\Diamond$	$\prec$	1	2	3	4
2	1	1	3	3	3
2	2	1	1	3	3
2	3	1	3	1	3
4	4	1	1	1	1

Group V

$\Diamond$	$\prec$	1	2	3	4
1	1	2	4	3	4
2	2	2	2	3	3
1	3	2	4	2	4
3	4	2	2	2	2

The 'designated values,' for all five groups, are 1 and 2; that is, the group is to be taken as satisfying any principle whose values, for all combinations of the values of its variables, are confined to 1 and 2. (In Groups II, III, and IV, 1 alone might be taken as the designated value: but in that case it must be remembered that, since

$$(\exists\, p, q) : \sim(p \prec q) \,.\, \sim(p \prec \sim q) :.=:. \sim[(p, q) : p \prec q \,.\vee.\, p \prec \sim q],$$

B9 would be satisfied *unless* $p \prec q \,.\vee.\, p \prec \sim q$ *always* has the value 1. It is simpler to take 1 and 2 both as designated values; in which case B9 is satisfied if and only if $\sim(p \prec q) \,.\, \sim(p \prec \sim q)$ has the value 1 or the value 2 for some combination of the values of p and q.)

All of these groups satisfy the operations of 'Adjunction,' 'Inference,' and the substitution of equivalents. If P and Q are functions having a designated value, then $P\,Q$ will have a designated value. If P has a designated value, and $P \prec Q$ has a designated value, then Q will have a designated value. And if $P = Q$—that is, if $P \prec Q \,.\, Q \prec P$ has a designated value—then P and Q will have the same value, and for any function f in the system, $f(P)$ and $f(Q)$ will have the same value.

The following facts may be established by reference to these groups:

(1) The system, as deduced from either set of postulates, is consistent. Group I, Group II, and Group III each satisfy all postulates of either set. For any one of these three groups, B9 is satisfied by the fact that $\sim(p \prec q) \,.\, \sim(p \prec \sim q)$ has a designated value when $p = 1$ and $q = 2$, and when $p = 1$ and $q = 3$.

(2) The system, as deduced from either set, is not reducible to Material Implication. For any one of the five groups, $\sim(p \sim q) \,.\prec.\, p \prec q$ has the value 3 or 4 when $p = 1$ and $q = 2$. None of the 'paradoxes' of Material Implication, such as $p \,.\supset.\, q \supset p$ and $\sim p \,.\supset.\, p \supset q$, will hold for any of these groups if the sign of material implication, $\supset$, is replaced by $\prec$ throughout.

(3) The Consistency Postulate, B8, is independent of the set (B1–7 and B9) and of the set A1–7. Group V satisfies B1–7, and satisfies $\sim(p \prec q) \,.\, \sim(p \prec \sim q)$ for the values $p = 1$, $q = 2$. It also satisfies A1–7. But Group V fails to satisfy B8: B8 has the value 4 when $p = 2$ and $q = 3$, and when $p = 2$ and $q = 4$.

(4) Similarly, A8 is independent of the set A1–7, and of the set (B1–7 and B9). For Group V, A8 has the value 4 when $p = 1$ and $q = 3$, and when $p = 2$ and $q = 3$.

(5) Postulate B7 is independent of the set (B1–6 and B8, 9), and of the set (A1–6 and A8). Group IV satisfies B1–6, B8, and B9, and satisfies A1–6 and A8. But for this group, B7 has the value 3 when $p = 1$ and $q = 2$, and for various other combinations of the values of p and q.

(6) Similarly, A7 is independent of the set (A1–6 and A8) and of the set (B1–6 and B8, 9). For Group IV, A7 has the value 3 when $p = 1$ and when $p = 3$.

That the Existence Postulate, B9, is independent of the set B1–8, and of the set A1–8, is proved by the following two-element group, which satisfies B1–8 and A1–8:

$p\,q$	1	0	$\sim p$
1	1	0	0
0	0	0	1

$\Diamond$	$\prec$	1	0
1	1	1	0
0	0	1	1

(This is, of course, the usual matrix for Material Implication, with the function $\Diamond p$ specified as equivalent to p.) For this group, $\sim(p \prec q) \,.\, \sim(p \prec \sim q)$ has the value 0 for all combinations of the values of p and q.

Dr. Parry has been able to deduce B2 from the set (B1 and B3–9). However, the omission of B2 from the postulate set of Chapter VI would have been incompatible with the order of exposition there adopted, since the Consistency Postulate is required for the derivation of B2. Whether with this exception the members of set B are mutually independent has not been fully determined.

The question naturally arises whether the two sets A1–8 and B1–8 are equivalent. I have discovered no proof but believe that they are not. B1–8 are all deducible from A1–8: and A1–7 are all deducible from B1–8. The question is whether A8 is deducible from B1–8. If it is not, then the system as deduced from the postulate set of Chapter VI, B1–9, is somewhat 'stricter' than as deduced in the *Survey* from set A.

The logically important issue here concerns certain consequences which enter the system when A8 is introduced. Both Dr. Wajsberg and Dr. Parry have proved that the principle

$$p \prec q . \prec : q \prec r . \prec . p \prec r$$

is deducible from A1–8. I doubt whether this proposition should be regarded as a valid principle of deduction: it would never lead to any inference $p \prec r$ which would be questionable when $p \prec q$ and $q \prec r$ are given premises; but it gives the inference $q \prec r . \prec . p \prec r$ whenever $p \prec q$ is a premise. Except as an elliptical statement for "$p \prec q . q \prec r : \prec . p \prec r$ and $p \prec q$ is true," this inference seems dubious.

Now as has been proved under (3) above, the Consistency Postulate, B8, is not deducible from the set (B1–7 and B9). Likewise the principle mentioned in the preceding paragraph is independent of the set (B1–7 and B9): Group V satisfies this set, but for that group the principle in question has the value 4 when $p = 1$, $q = 3$, and $r = 1$, as well as for various other values of p, q, and r. But Group V also fails to satisfy B8, as was pointed out in (3) above. If it should hereafter be discovered that the dubious principle of the preceding paragraph is deducible from the set B1–9, then at least it is not contained in the system deducible from the set (B1–7 and B9); and I should then regard that system—to be referred to hereafter as S1—as the one which coincides in its properties with the strict principles of deductive inference. As the reader will have noted, Chapter VI was so developed that the theorems belonging to this system, S1, are readily distinguishable from those which require the Consistency Postulate, B8.

The system as deduced either from set A or from set B leaves undetermined certain properties of the modal functions, $\Diamond p$, $\sim\Diamond p$, $\Diamond\sim p$, and $\sim\Diamond\sim p$. In view of this fact, Professor Oskar Becker[2] has proposed the following for consideration as further postulates, any one or more of which might be added to either set:

C10.	$\sim\Diamond\sim p \prec \sim\Diamond\sim\sim\Diamond\sim p$	$\sim\Diamond\sim\sim\Diamond\sim p = \sim\Diamond\sim p$
C11.	$\Diamond p \prec \sim\Diamond\sim\Diamond p$	$\Diamond p = \sim\Diamond\sim\Diamond p$
C12.	$p \prec \sim\Diamond\sim\Diamond p$	

(Becker calls C12 the "Brouwersche Axiom.")

When A1–8, or B1–9, are assumed, the second form in which C10 is given can be derived from the first, since the converse implication, $\sim\Diamond\sim\sim\Diamond\sim p \prec \sim\Diamond\sim p$, is an immediate consequence of the general principle, $\sim\Diamond\sim p \prec p$ (18·42 in Chapter VI). The second form of C11 is similarly deducible from the first.

An alternative and notationally simpler form of C10 would be

C10·1 $\Diamond\Diamond p \prec \Diamond p$ $\qquad$ $\Diamond\Diamond p = \Diamond p$

(As before, the second form of the principle can be derived from the first; since the converse implication, $\Diamond p \prec \Diamond\Diamond p$, is an instance of the general principle $p \prec \Diamond p$, which is 18·4 in Chapter VI, deducible from A1–8, or B1–9.)

Substituting $\sim p$ for p, in C10·1, we have

$$\Diamond\Diamond\sim p \prec \Diamond\sim p \qquad \text{(a)}$$
$$\text{(a)} . = . \sim\Diamond\sim p \prec \sim\Diamond\Diamond\sim p . = . \sim\Diamond\sim p \prec \sim\Diamond\sim\sim\Diamond\sim p.$$

And substituting $\sim p$ for p in C10, we have

$$\sim\Diamond\sim(\sim p) \prec \sim\Diamond\sim\sim\Diamond\sim(\sim p) \qquad \text{(b)}$$
$$\text{(b)} . = . \sim\Diamond p \prec \sim\Diamond\sim\sim\Diamond p . = . \sim\Diamond p \prec \sim\Diamond\Diamond p . = . \Diamond\Diamond p \prec \Diamond p.$$

(The principles used in these proofs are 12·3 and 12·44 in Chapter VI.)

For reasons which will appear, we add, to this list of further postulates to be considered, the following:

C13. $\Diamond\Diamond p$

That is, "For every proposition p, the statement 'p is self-consistent' is a self-consistent statement."

[2] See his paper "Zur Logik der Modalitäten," *Jahrbuch für Philosophie und Phänomenologische Forschung*, XI (1930), 497–548.

Concerning these proposed additional postulates, the following facts may be established by reference to Groups I, II, and III, above, all of which satisfy the set A1–8 and the set B1–9:

(7) C10, C11, and C12 are all consistent with A1–8 and with B1–9 and with each other. Group III satisfies C10, C11, and C12.

(8) C10, C11, and C12 are each independent of the set A1–8 and of the set B1–9. For Group I, C10, C11, and C12 all fail to hold when $p = 3$.

(9) Neither C11 nor C12 is deducible from the set (A1–8 and C10) or from the set (B1–9 and C10). Group II satisfies C10; but C11 fails, for this group, when $p = 2$ or $p = 4$; and C12 fails when $p = 2$.

(10) C13 is consistent with the set A1–8 and with the set B1–9. Group I satisfies C13.

(11) C13 is independent of the set A1–8 and of the set B1–9, and of (A1–8 and C10, C11, and C12) or (B1–9 and C10, C11, and C12). Group III satisfies all these sets; but for this group, C13 fails when $p = 4$.

When A1–8, or B1–9, are assumed, the relations of C10, C11, and C12 to each other are as follows:

(12) C10 is deducible from C11. By C11 and the principle $\sim(\sim p) = p$,

$$\begin{aligned}\sim\Diamond\sim p &= \sim[\Diamond(\sim p)] = \sim[\sim\Diamond \sim\Diamond(\sim p)] = \Diamond[\sim\Diamond(\sim p)]\\ &= \sim\Diamond \sim\Diamond(\sim\Diamond \sim p) = \sim\Diamond[\sim\Diamond \sim\Diamond(\sim p)]\\ &= \sim\Diamond[\Diamond(\sim p)] = \sim\Diamond\{\sim[\sim(\Diamond\sim p)]\} = \sim\Diamond\sim \sim\Diamond\sim p.\end{aligned}$$

(13) C12 also is deducible from C11. By 18·4 in Chapter VI, $p \prec \Diamond p$; and this, together with C11, implies C12, by A6 or by B6.

(14) From C10 and C12 together, C11 is deducible. Substituting $\Diamond p$ for p in C12, we have

$$\Diamond p \prec \sim\Diamond \sim\Diamond \Diamond p. \qquad \text{(a)}$$

And by C10·1, $\sim\Diamond \sim\Diamond \Diamond p = \sim\Diamond \sim\Diamond p$. Hence (a) is equivalent to C11.

From (12), (13), and (14), it follows that as additional postulates to the set A1–8, or the set B1–9, C11 is exactly equivalent to C10 and C12 together. But as was proved in (9), the addition of C10 alone, gives a system in which neither C11 nor C12 is deducible.

Special interest attaches to C10. The set A1–8, or the set B1–9, *without* C10, gives the theorem

$$\sim\Diamond\sim \sim\Diamond\sim p \,.\prec :\, \sim\Diamond\sim p = p.$$

This is deducible from 19·84 in Chapter VI. It follows from this that if there should be some proposition p such that $\sim\Diamond\sim \sim\Diamond\sim p$ is true, then the equivalences

$$p = \sim\Diamond\sim p \qquad \text{and} \qquad \sim\Diamond\sim p = \sim\Diamond\sim \sim\Diamond\sim p$$

would hold for that particular proposition. And since, by 19·84 itself, all necessary propositions are equivalent, it follows that if there is *any* proposition p which is necessarily-necessary—such that $\sim\Diamond\sim \sim\Diamond\sim p$ is true—then *every* proposition which is necessary is also necessarily-necessary; and the principle stated by C10 holds universally. But as was proved in (8), this principle, $\sim\Diamond\sim p = \sim\Diamond\sim \sim\Diamond\sim p$, is not deducible from A1–8 or from B1–9. Hence the two possibilities, with respect to necessary propositions, which the system, as deduced from A1–8 or from B1–9, leaves open are: (a) that there exist propositions which are necessarily-necessary, and that for every proposition p, $\sim\Diamond\sim p = \sim\Diamond\sim \sim\Diamond\sim p$; and (b) that there exist propositions which are necessary—as 20·21 in Chapter VI requires—but *no* propositions which are *necessarily*-necessary. This last is exactly what is required by C13, $\Diamond \Diamond p$. Substituting here $\sim p$ for p, we have, as an immediate consequence of C13, $\Diamond \Diamond\sim p$. This is equivalent to the theorem "For every proposition p, 'p is necessarily-necessary' is false": $\Diamond \Diamond\sim p = \Diamond\sim \sim\Diamond\sim p = \sim(\sim\Diamond\sim \sim\Diamond\sim p)$ [by the principle $\sim(\sim p) = p$]. Thus C10 expresses alternative (a) above; and C13 expresses alternative (b). Hence as additional postulates, C10 and C13 are contrary assumptions.

(As deduced from A1–8, the system leaves open the further alternative that there should be no necessary propositions, or that the class of necessary propositions should merely coincide with the class of true propositions; but in that case the system becomes a redundant form of Material Implication.)

From the preceding discussion it becomes evident that there is a group of systems of the general type of Strict Implication and each distinguishable from Material Implication. We shall arrange these in the order of increasing comprehensiveness and decreasing 'strictness' of the implication relation:

S1, deduced from the set B1–7, contains all the theorems of Sections 1–4 in Chapter VI. It contains also all theorems of Section 5, in the form of T-principles, but not with omission of the T. This system does not contain A8 or the principle

$$p \prec q \,.\, \prec : q \prec r \,.\, \prec \,.\, p \prec r.$$

However, it does contain, in the form of a T-principle, any theorem which could be derived by using A8 as a principle of inference: because it contains

$$p \prec q \,.\, \sim\Diamond q : \prec \,.\, \sim\Diamond p;$$

and hence if (by substitutions) $p \prec q$ becomes an asserted principle, we shall have

$$T \,.\, \sim\Diamond q : \prec \,.\, \sim\Diamond p.$$

When the Existence Postulate, B9, is added, this system S1 contains those existence theorems which are indicated in Section 6 of Chapter VI as not requiring the Consistency Postulate, B8.

S2, deduced from the set B1–8, contains all the theorems of Sections 1–5 in Chapter VI, any T-principle being replaceable by the corresponding theorem without the T. When the Existence Postulate, B9, is added, it contains all the existence theorems of Section 6. Whether S2 contains A8 and the principle

$$p \prec q \,.\, \prec : q \prec r \,.\, \prec \,.\, p \prec r$$

remains undetermined. If that should be the case, then it will be equivalent to S3.

S3, deduced from the set A1–8, as in the *Survey*, contains all the theorems of S2 and contains such consequences of A8 as

$$p \prec q \,.\, \prec : q \prec r \,.\, \prec \,.\, p \prec r.$$

If B9 is added, the consequences include all existence theorems of S2.

For each of the preceding systems, S1, S2, and S3, any one of the additional postulates, C10, C11, C12, and C13, is consistent

with but independent of the system (but C10 and C13 are mutually incompatible).

S4, deduced from the set (B1–7 and C10), contains all theorems of S3, and in addition the consequences of C10. A8 and B8 are deducible theorems. S4 is incompatible with C13. C11 and C12 are consistent with but independent of S4. If B9 be added, the consequences include all existence theorems of S2.

S5, deduced from the set (B1–7 and C11), or from the set (B1–7, C10, and C12), contains all theorems of S4 and in addition the consequences of C12. If B9 be added, all existence theorems of S2 are included. A8 and B8 are deducible theorems. S5 is incompatible with C13.

Dr. Wajsberg has developed a system mathematically equivalent to S5, and has discovered many important properties of it, notably that it is the limiting member of a certain family of systems. Mr. Henle has proved that S5 is mathematically equivalent to the Boole-Schröder Algebra (*not* the Two-valued Algebra), if that algebra be interpreted for propositions, and the function $\Diamond p$ be determined by:

$$\Diamond p = 1 \text{ when and only when } p \neq 0;$$

$$\Diamond p = 0 \text{ when and only when } p = 0.$$

In my opinion, the principal logical significance of the system S5 consists in the fact that it divides all propositions into two mutually exclusive classes: the intensional or modal, and the extensional or contingent. According to the principles of this system, all intensional or modal propositions are either necessarily true or necessarily false. As a consequence, for any modal proposition—call it p_m—

$$\Diamond(p_m) = (p_m) = \sim\Diamond\sim(p_m),$$

$$\text{and } \Diamond\sim(p_m) = \sim(p_m) = \sim\Diamond(p_m).$$

For extensional or contingent propositions, however, possibility, truth, and necessity remain distinct.

Prevailing good use in logical inference—the practice in mathematical deductions, for example—is not sufficiently precise and self-conscious to determine clearly which of these five systems expresses the acceptable principles of deduction. (The meaning of 'acceptable' here has been discussed in Chapter VIII.) The issues concern principally the nature of the relation of 'implies' which is to be relied upon for inference, and certain subtle questions about the meaning of logical 'necessity,' 'possibility' or 'self-consistency,' etc.—for example, whether C10 is true or false. (Professor Becker has discussed at length a number of such questions, in his paper above referred to.) Those interested in the merely mathematical properties of such systems of symbolic logic tend to prefer the more comprehensive and less 'strict' systems, such as S5 and Material Implication. The interests of logical study would probably be best served by an exactly opposite tendency.

APPENDIX III

FINAL NOTE ON SYSTEM S_2

January 5, 1959

This Appendix is intended to supplement Chapter VI which presents the calculus of Strict Implication, S2, and Appendix II concerning the series of related systems S1—S5.

An adequate summary of the literature pertinent to these two topics which has appeared since the first publication of this book in 1932 would not be possible within reasonable limits of space here. But inasmuch as what is included in this present edition will stand as the permanent record of Strict Implication there are four items brief account of which should be set down.

J. C. C. McKinsey has shown that the postulate 11•5 in Chapter VI (B5 in Appendix II) is redundant, being deducible from the remainder of the set.[1]

E. V. Huntington contributed additional theorems which are important for understanding the logical import of the Consistency Postulate, 19•01, and for the comparison of Strict Implication with Boolean Algebra, to be mentioned later.[2]

Two basic theorems in Section 5 of Chapter VI—supposedly theorems requiring the Consistency Postulate—can be proved without that assumption and are included in S1.

W. T. Parry has supplied the proof that the systems S2 and S3 are distinct, thus completing proof that all five of the systems S1—S5 are distinct from one another.[3]

The first three of these topics can be covered summarily by proving additional theorems, so numbered that the place where they can be interpolated in the development as given in Chapter VI will be indicated. A few other theorems, omitted in the original edition but helpful for one reason or another, will be included here.

The first group of theorems gives the derivation of 11•5; 12•29 below. The proof as given by McKinsey is here simplified, taking advantage of the fact that, in Chapter VI, no use of 11•5 is made in proof of any theorem prior to 12•3.

[1] J. C. C. McKinsey, "A Reduction in the Number of Postulates for C. I. Lewis' System of Strict Implication," *Bull. Amer. Math. Soc.*, Vol. 40 (1934), pp. 425–427.

[2] E. V. Huntington, "Postulates for Assertion, Conjunction, Negation, and Equality," *Proc. Amer. Acad. of Arts and Sciences*, Vol. 72 (1937), No. 1, pp. 1–44.

[3] W. T. Parry, "The Postulates for 'Strict Implication'," *Mind*, Vol. XLIII, N. S. (1934), No. 169, pp. 78–80.

I am also indebted to Professor Parry for adding to my list of the errata needing correction and other assistance.

12·27 $\sim\{\sim[\sim(\sim p)]\} ⥽ p$
[12·25: $\sim(\sim p)/p$] $\sim\{\sim[\sim(\sim p)]\} ⥽ \sim(\sim p)$ (1)
[11·6] (1) . 12·25: $⥽$ Q.E.D.

12·28 $\sim p = \sim[\sim(\sim p)]$
[12·2: $\sim[\sim(\sim p)]/p; p/q$] 12·27 . = : $\sim p ⥽ \sim[\sim(\sim p)]$ (1)
[12·25 : $\sim p/p$] $\sim[\sim(\sim p)] ⥽ \sim p$ (2)
[11·03] (1) . (2) : = Q.E.D.

12·29 $p ⥽ \sim(\sim p)$
[12·1] $\sim\Diamond(p \sim p) . ⥽ . \sim\Diamond(p \sim p)$ (1)
[12·28] (1) $\therefore = : . \sim\Diamond(p \sim p) . ⥽ . \sim\Diamond\{p \sim[\sim(\sim p)]\}$ (2)
[11·02] (2) . = : 12·1 . $⥽$ Q.E.D.

12·3 $p = \sim(\sim p)$
[11·03] 12·25 . 12·29 : = Q.E.D.

The next group are simple and obvious theorems, omitted in Chapter VI but helpful for the comparison of Strict Implication with Boolean Algebra.

12·91 $p \sim p . ⥽ . q$
[11·2 : $\sim q/q$] $p \sim q . ⥽ . p$ (1)
[12·3, 12·6] (1) = Q.E.D.

12·92 $p \sim p . = . q \sim q$
[12·91 : $q \sim q/q$] $p \sim p . ⥽ . q \sim q$ (1)
[12·91 : $q/p; p \sim p/q$] $q \sim q . ⥽ . p \sim p$ (2)
[11·03] (1) . (2) : = Q.E.D.

12·93 $\sim\Diamond p . = . p ⥽ \sim p$
[12·7, 12·3] $\sim\Diamond p . = . \sim\Diamond[p \sim(\sim p)]$ (1)
[11·02] (1) = Q.E.D.

12·94 $\sim\Diamond\sim p . = . \sim p ⥽ p$
[12·93 : $\sim p/p$] $\sim\Diamond\sim p . = : \sim p . ⥽ . \sim(\sim p)$ (1)
[12·3] (1) = Q.E.D.

12·95 $\Diamond p . = . \sim(p ⥽ \sim p)$
[12·11] $\sim(\sim\Diamond p) . = . \sim(\sim\Diamond p)$ (1)
[12·3, 12·93] (1) = Q.E.D.

12·96 $\Diamond\sim p . = . \sim(\sim p ⥽ p)$
[12·95, 12·3]

13·6 $p \vee \sim p . = . q \vee \sim q$
[12·11, 12·92, 11·01, 12·3]

13·7 $q . ⥽ . p \vee \sim p$
[12·91 : $\sim p/p; \sim q/q$] $\sim p \sim(\sim p) . ⥽ . \sim q$ (1)
[12·42, 11·01] (1) $⥽$ Q.E.D.

Theorems 19·57 and 19·58 of Section 5 in Chapter VI do not require the Consistency Postulate, 19·01. Proof of them from postulates and theorems of S1 exclusively follows. Two consequences are added.

16·36 $p . q \sim q : = . q \sim q$ (See 19·57)
[11·2] $q \sim q . p : ⥽ : q \sim q$ (1)
[12·91] $q \sim q . ⥽ . p$ (2)
[16·33] (2) $. = : . q \sim q . ⥽ : q \sim q . p$ (3)
[11·03] (1) . (3) $: = : q \sim q . p : = . q \sim q$ (4)
[12·15] (4) = Q.E.D.

16·37 $p . = : p . \vee . q \sim q$ (See 19·58)
[12·91] $q \sim q . ⥽ . p$ (1)
[16·34, 13·11] (1) $. = : . p . \vee . q \sim q : ⥽ . p$ (2)
[13·2] $p . ⥽ : p . \vee . q \sim q$ (3)
[11·03] (2) . (3) : = Q.E.D.

16·38 $p . = : p . q \vee \sim q$
[12·3] $p = \sim(\sim p)$
[16·37] $= : \sim(\sim p . \vee . q \sim q)$
[11·01, 12·3] $= : \sim(\sim p) . \sim(q \sim q)$
[12·3, 11·01] $= : p . \sim q \vee q$
[13·11] $= : p . q \vee \sim q$

16·39 $p \vee \sim p . = : p \vee \sim p . \vee . q$
[12·11] $\sim(p \sim p) . = . \sim(p \sim p)$ (1)
[16·36] (1) $. = : \sim(p \sim p) . = . \sim(p \sim p . \sim q)$ (2)
[11·01] (2) $. = : . \sim p \vee p . = : . \sim p \vee p . \vee . q$ (3)
[13·11] (3) = Q.E.D.

The group of theorems which follows are all of them important in connection with the topic of the 'paradoxes of Strict Implication', to be discussed in conclusion. Proofs of 16·395, 19·86 and 19·87 are due to Huntington, and proofs of the others are to be found in his paper cited above. The first, 16·395, is a theorem in S1; the remainder of the group are theorems in S2.

16·395 $p q . ⥽ . p : = . \sim\Diamond(r \sim r)$
[12·11] $\sim\Diamond(r \sim r) . = . \sim\Diamond(r \sim r)$
[12·92] $= . \sim\Diamond(p \sim p)$
[16·36, 12·15] $= . \sim\Diamond(p \sim p . q)$
[12·5] $= . \sim\Diamond(p q . \sim p)$
[11·02] $= : p q . ⥽ . p$

19·85 $\sim\Diamond(p \sim q) . ⥽ . \sim\Diamond(r \sim r)$
[19·01 : $p \sim q/p$] $\Diamond(p \sim q . q) . ⥽ . \Diamond(p \sim q)$ (1)
[12·5, 16·36] (1) $. = : \Diamond(q \sim q) . ⥽ . \Diamond(p \sim q)$ (2)
[12·92] (2) $. = : \Diamond(r \sim r) . ⥽ . \Diamond(p \sim q)$ (3)
[12·44] (3) = Q.E.D.

19·86 $\sim\Diamond p \,.\, \strictif \,.\, r \strictif r$

[19·85 : $\sim p/q$] $\sim\Diamond[p \sim(\sim p)] \,.\, \strictif \,.\, \sim\Diamond(r \sim r)$ (1)

[12·3, 12·7] (1) $. = : \sim\Diamond p \,.\, \strictif \,.\, \sim\Diamond(r \sim r)$ (2)

[11·02] (2) = Q.E.D.

19·87 $p \strictif q \,.\, \strictif : p \,.\, = \,.\, p q$

[16·33] 19·85 $. = : . \sim\Diamond(p \sim q) \,.\, \strictif : \sim(p \sim q) \,.\, \sim\Diamond(r \sim r)$

[11·02] $= : . p \strictif q \,.\, \strictif : p \strictif q \,.\, \sim\Diamond(r \sim r)$

[16·395] $= : : p \strictif q \,.\, \strictif : . p \strictif q : p q \,.\, \strictif \,.\, p$

[16·33] $= : : p \strictif q \,.\, \strictif : . \; p \,.\, \strictif \,.\, p q : p q \,.\, \strictif \,.\, p$

[11·03] = Q.E.D.

19·88 $p \strictif q \,.\, = : p \,.\, = \,.\, p q$

[11·03, 12·1] $p \,.\, = \,.\, p q : \strictif : p \,.\, \strictif \,.\, p q : p q \,.\, \strictif \,.\, p$ (1)

[11·2] $p \,.\, \strictif \,.\, p q : p q \,.\, \strictif \,.\, p : . \strictif : p \,.\, \strictif \,.\, p q$ (2)

[11·6] (1) . (2) $: \strictif : . p \,.\, = \,.\, p q : \strictif : p \,.\, \strictif \,.\, p q$ (3)

[16·33] (3) $. = : . p \,.\, = \; p q : \strictif \,.\, p \strictif q$ (4)

[11·03] (4) . 19·87 : = Q.E.D.

19·89 $\sim\Diamond p \,.\, = : p \,.\, = \,.\, q \sim q$

[19·88] $p \strictif \sim p \,.\, = : p \,.\, = \,.\, p \sim p$ (1)

[12·92] (1) $. = : . p \strictif \sim p \,.\, = : p \,.\, = \,.\, q \sim q$ (2)

[18·12] (2) = Q.E.D.

19·891 $\sim\Diamond\sim p \,.\, = : p \,.\, = \,.\, q \vee \sim q$

[19·89 : $\sim p/p$] $\sim\Diamond\sim p \,.\, = : \sim p \,.\, = \,.\, q \sim q$

[11·03] $= : \sim p \,.\, \strictif \,.\, q \sim q : q \sim q \,.\, \strictif \,.\, \sim p$

[12·45, 12·3] $= : \sim(q \sim q) \strictif p : p \strictif \sim(q \sim q)$

[11·01, 12·3] $= : q \vee \sim q \,.\, \strictif p : p \,.\, \strictif \,.\, q \vee \sim q$

[11·03] $= : p \,.\, = \,.\, q \vee \sim q$

We turn to the fourth item mentioned in our introductory remarks: Parry's proof that the principle

$$p \strictif q \,.\, \strictif : q \strictif r \,.\, \strictif \,.\, p \strictif r,$$

deducible from the postulates of S3 (A1—8 in Appendix II) is not deducible from the postulates of S2. On this point, we could not do better than to quote his succinct demonstration in the paper cited above:

"In answer to this question as to the two sets of postulates, it will here be shown that A8, and the consequence of A1—8 referred to by Mr. Lewis, are *independent* of B1—9, *i. e.*, that they cannot be deduced from set B by the admitted operations. This fact can be demonstrated by the matrix method as follows:—

"(1) The eight numbers, 0 to 7, are taken as 'elements'; the numbers 6 and 7 are said to be the 'designated values'.

"(2) The 'primitive ideas' of the System of Strict Implication are given matrix (or extensional) definitions in terms of the elements, as follows:

$\sim p = 7 - p$

p	$\Diamond p$
0	1
1	5
2	7
3	7
4	7
5	7
6	7
7	7

$p q$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	0	1	0	1	0	1
2	0	0	2	2	0	0	2	2
3	0	1	2	3	0	1	2	3
4	0	0	0	0	4	4	4	4
5	0	1	0	1	4	5	4	5
6	0	0	2	2	4	4	6	6
7	0	1	2	3	4	5	6	7

"From these tables, with the definitions 11·02 and 11·03, the tables for $p \strictif q$ and $p = q$ are readily constructed. $(\exists p) f(p)$ has a designated value if and only if $f(p)$ has a designated value for some substitution of an element for p.

"(3) All the postulates of set B are found to be 'satisfied'; *i. e.*, if we make any substitution of elements for the variables of a postulate, and calculate according to the matrix definitions, the result is a designated value.

"(4) If certain principles are satisfied, any principle derivable from them by means of the operations used for proof is also satisfied.

"(5) Finally, A8 and certain consequences of set A are *not* satisfied, hence are not derivable from set B by the admitted operations. If we substitute 1 for p, 0 for q, in A8, the result is 0. (Even if we weaken the main relation of A8 to material implication, the principle is not satisfied.) That the principle mentioned above,

$$p \strictif q \,.\, \strictif : q \strictif r \,.\, \strictif \,.\, p \strictif r$$

is not satisfied, is shown by the substitution 1,0,0 for p,q,r respectively."

Let us add, however, that although the above-mentioned principle is not provable in S2, we can derive in S2, from any premise of the form $P \strictif Q$, a corresponding conclusion $Q \strictif R \,.\, \strictif \,.\, P \strictif R$, as Parry has shown by the following proof-schema:

Given $P \strictif Q$; To prove $Q \strictif R \,.\, \strictif \,.\, P \strictif R$.

[Hypothesis] $P \strictif Q$ (1)

[19·87] (1) $. \strictif : P \,.\, = \,.\, P Q$ (2)

[19·52] $Q \strictif R \,.\, \strictif : P Q \,.\, \strictif \,.\, R$ (3)

[(2)] (3) $. = : Q \strictif R \,.\, \strictif \,.\, P \strictif R$

This completes the proof, otherwise established in Appendix II, that all five of the systems S1—S5 are distinct, and that each later system in the series requires some postulate which is independent of its predecessor system in the series.

In a paper published in 1946, Dr. Ruth C. Barcan Marcus has shown that the system S2 can be extended to first-order propositional functions.[1] Though it happens that I hold certain logical convictions in the light of which I should prefer to approach the logic of propositional functions in a different way, I appreciate this demonstration that there is a calculus of functions which bears to the calculus of Strict Implication a relation similar to that which holds between the calculus of functions in *Principia Mathematica* (*9—*11) and the calculus of propositions (*1—*5) in that work.

For anyone who should be interested in Strict Implication as logic, and not merely in the mathematical and metamathematical structure of it as a system, there are two further topics which I think it is of some importance to consider. First, it may be of historical interest, and it certainly will be of logical interest, to compare both Strict Implication and Material Implication with the Boolean Algebra in the classic form given it by Schröder. Second, it should be the prime desideratum of deductive logic to identify correctly and develop the properties of that logical nexus which holds between a premise p and a consequence q when and only when q is validly deducible from p. And in that connection, attention to the 'paradoxes', both of Strict Implication and of Material Implication, will be called for. The first of these two topics can be notably illuminating for any consideration of the second.

Schröder developed the two systems referred to in the early chapters of this book: first, the general Boolean Algebra interpreted by him as the logic of classes; second, the Two-Valued Algebra which he interprets as the logic of propositions. However, it will be more convenient for our purposes to suppose that we have both these systems before us in the form of uninterpreted mathematical systems, defined after the manner made familiar by Huntington; *i. e.*, as "a class K of elements $a, b, c, \ldots$, and an operation o such that;"—the postulates being appended. Also, for simplicity, let us use the notations of the propositional calculus throughout— $\sim p$, $p.q$, $p \vee q$, etc.—whether it is the interpretation for classes or that for propositions which is under consideration at the moment. For brevity, we may refer to the general calculus interpreted for classes as K1, the Two-Valued Algebra as K2.

K2 is derived from the same postulates as K1, with the *addition* of a postulate expressed by Schröder as "$p = (p = 1)$,"

[1] Ruth C. Barcan, "A Functional Calculus of the first order, based on Strict Implication;" *Journal for Symbolic Logic*, Vol. II, No. 1, pp. 1–16.

which is equivalent in force to the assumption, "For any element p, either $p = 0$ or $p = 1$." The basic mathematical properties of K1 and K2 both, are determined by the fact that the zero element of the system is the modulus of the operation of (logical) multiplication; the 'and'-relation of elements. For any element p,

$$p \sim p = 0. \qquad \text{Hence } p \sim p = q \sim q = r \sim r, \text{ etc.}$$

And the element 1, which is the inverse of 0, is the modulus of the operation of (logical) addition; the 'or'-relation of elements. For any element p,

$$p \vee \sim p = 1. \qquad \text{Hence } p \vee \sim p = q \vee \sim q = r \vee \sim r, \text{ etc.}$$

Let us here write that relation which is read, for propositions, as 'p implies q', and for classes as 'The class p is contained in the class q', as $p \rightarrow q$. Both in K1 and in K2, this relation is so defined that it holds when and only when $p \sim q = 0$ and $\sim p \vee q = 1$. Thus, even in the uninterpreted system, $p \rightarrow q$ is a *statement*, equivalent to the equations just mentioned, which holds when and only when, for any element r,

$$p \sim q = 0 = r \sim r \quad \text{and} \quad \sim p \vee q = 1 = r \vee \sim r.$$

Let us remember also that, both in K1 and in K2, $0 \rightarrow p$ and $p \rightarrow 1$ for any element p; *e. g.*, the null class is contained in every class, and every class is contained in the universal class, 'everything.'

For the propositional interpretation, Schröder interpreted $p = 1$—equivalent to $p = r \vee \sim r$—as 'p is true'; and $p = 0$—equivalent to $p = r \sim r$—as 'p is false.' On this interpretation, the added postulate of K2 is called for: every proposition is either true or false. But all the postulates of K1 are also consistent with the *contradictory* of this added postulate: "For some element p, $p \neq 0$ and $p \neq 1$." (It is false that all classes are either empty or universal).

The properties of the relation $p \supset q$ in the truth-value calculus of Material Implication are correlative throughout with those of $p \rightarrow q$ in K2, since $p \supset q$ is, by reason of its definition, equivalent to $\sim(p \sim q)$—in K2, $p \sim q = 0$. In Boolean Algebra generally (K1 and K2 both) $r = 0$ is equivalent to $\sim r = 1$; and $1 \cdot 1 = 1$, $1 \cdot 0 = 0$, $0 \cdot 1 = 0$, and $0 \cdot 0 = 0$. And all the paradoxes of the truth-value interpretation of $p \rightarrow q$, obliged by the added postulate of K2, are implicit in the fact that when the values of p and q are restricted to 0 and 1, there are only four alternatives:

$p = 1, q = 1\colon p \supset q, \sim p \supset q, \sim p \supset \sim q; \sim p \supset p, \sim q \supset q$
$p = 0, q = 0\colon p \supset q, p \supset \sim q, \sim p \supset \sim q; p \supset \sim p, q \supset \sim q$
$p = 0, q = 1\colon p \supset q, \sim p \supset q, p \supset \sim q; p \supset \sim p, \sim q \supset q$
$p = 1, q = 0\colon p \supset \sim q, \sim p \supset q, \sim p \supset \sim q; \sim p \supset q, q \supset \sim q$

Also, for any one of these alternatives, either $p \supset q$ or $q \supset p$, either $\sim p \supset q$ or $q \supset \sim p$, and so on. A relation so nearly ubiquitous could not coincide with that which holds when and only when q is deducible from p.

However, there is another interpretation of Boolean Algebra which can be imposed on K1 (not K2) for which it becomes a calculus of propositions. All that is necessary in order to assure the possibility of this second interpretation of K1 is to observe that, retaining the same meaning of $\sim p$ as the contradictory of p, of $p.q$ as 'p and q', and of $p \vee q$ as 'either p or q', a different meaning can be imposed on the element 0 ($= r \sim r$, for any element r) and the element 1 ($= r \vee \sim r$, for any element r). For any proposition r, 'r and $\sim r$' is not only a *false* proposition but also a formal contradiction, necessarily false, self-inconsistent, analytically certifiable as false. And 'r or $\sim r$' is not only a *true* statement but a formal tautology, necessarily true, analytic. And taking $p = 1$ to signify 'p is analytic', $p = 0$ to signify 'p is self-inconsistent, contradictory', it is evident at once that we have another possible reading for all the postulates and theorems of K1. For this second interpretation, $p = 1$ becomes the function $\sim\Diamond\sim p$ of Strict Implication; $p = 0$ becomes $\sim\Diamond p$; $p \neq 0$ becomes $\Diamond p$; and $p \neq 1$ becomes $\Diamond\sim p$. And $p \rightarrowtail q$, equivalent to $p \sim q = 0$, becomes the relation of strict implication, $p \prec q$. And for *this* interpretation of K1, the added postulate of K2, "For every element p, either $p = 0$ or $p = 1$," will be false, and its contradictory, "For some element p, $p \neq 0$ and $p \neq 1$," will be true. There are contingent propositions, neither analytic and logically necessary nor contradictory and logically impossible. It is interesting to conjecture what might have happened, in the further development of exact logic, if Schröder or Charles S. Peirce had taken thought upon this second possible interpretation of Boolean Algebra as a propositional calculus.

As this will suggest, every postulate and theorem of K2 can, with a suitable dictionary, be translated into a postulate or theorem of Material Implication; and, with a slightly different dictionary, every postulate and theorem of K1 can be translated into a postulate or theorem of Strict Implication. However, this relation of 'mathematical equivalence' is clouded, in both cases, by the fact that our dictionary would have to provide *symbolic* equivalents for some of the English (or German) in K1 or K2. The *presumed* logic and logical relations of any uninterpreted mathematical system are usually not written in symbols but in the vernacular. Incidentally it is this kind of fact which accounts, in part, for the possible distinctness of the systems S1—S5, turning upon such questions as the inclusion or non-inclusion of $\Diamond\Diamond p = \Diamond p$. (See Appendix II). In a Boolean Algebra, $(p \neq 0) \neq 0$ would have no meaning; and if it were given one by declaring it equivalent to $p \neq 0$, the significance of that assumption would be obscure.

As we have said, this comparison with K1 and K2 can throw light upon the paradoxes, both of Material Implication and of Strict Implication. Strict Implication escapes the paradoxes of the truth-value logic of propositions because, by the different interpretation of 0 ($= p \sim p$) and 1 ($= p \vee \sim p$) and by repudiating restriction of propositions to these two values, it also escapes restriction to the four alternatives tabulated above for the relation of material implication. It leaves the 'middle ground' of contingent propositions between the analytic and the self-contradictory.

As a consequence of these facts, the paradoxes of Strict Implication are confined to the implications which are assertable when one or both of the propositions related is contradictory (equivalent, for some r, to $r \sim r$) or when one or both is analytic (equivalent, for some r, to $r \vee \sim r$). They do not extend to implications assertable as holding between premises and conclusions both of which are contingent propositions, neither analytic and logically necessary nor contradictory and logically impossible. Thus they do not affect what implies what amongst empirically substantiated and factually informational premises and their empirically significant consequences. In contrast to this, the paradoxes of Material Implication affect the implications assertable between any chosen pair of propositions without restriction, because *all* propositions are either *truth-value* equivalent to $r \sim r$ or *truth-value* equivalent to $r \vee \sim r$.

However—and here is a main point of the comparison with the Boolean systems K1 and K2—every theorem of K1 (common to K1 and K2) which is paradoxical on the interpretation of $p = 1$ as 'p is true' and $p = 0$ as 'p is false', remains paradoxical when $p = 1$ is interpreted as 'p is analytic' and $p = 0$ as 'p is contradictory'.

In Material Implication, the key paradoxes, implicating all the others, are: A false proposition implies any proposition; A true proposition is implied by any; Any two false propositions are equivalent; Any two true propositions are equivalent. Correspondingly, the key paradoxes of Strict Implication are: A contradictory (self-inconsistent) proposition implies any proposition; An analytic proposition is implied by any; Any two contradictory propositions are equivalent; Any two analytic propositions are equivalent. Other paradoxical theorems of Strict Implication include 16•95, 19•85 and 19•86 above.

Strict Implication, defining $p \prec q$ as a statement which holds when and only when the conjoint statement $p \sim q$, which asserts the premise and denies the conclusion, is self-inconsistent, is put forward with the intent to satisfy the requirement that $p \prec q$ hold when and only when q is a consequence validly deducible from the premise p. On acount of the paradoxes, there are many who doubt that it does so satisfy this requirement. And some amongst them have put forward alternative developments of a calculus of propositions designed to eliminate the paradoxes. It is especially regrettable to omit consideration of such proposals. But these have been too numerous and various to allow adequate report and discussion of them here. I shall, however, venture the conviction—recognizing my hazard in so doing—that without sacrifice or plain omission of some intuitively acceptable, analytically certifiable, and time-honored principle of inference, or the introduction of some assumption which fails of accord with such perduring logical principles, the paradoxes of Strict Implication are inescapable. They are unavoidable consequences of indispensable rules of inference.

The manner in which the paradoxes are involved in our logical commonplaces is, of course, a complex matter. But it will do no harm to remind ourselves of a trivial example or two. If the moon is a planet, and is not made of green cheese, then the moon is a planet. But if two premises together imply a conclusion and that conclusion is false while one of the premises is true, then the other premise must be false. So if the moon is a planet and also is not a planet, then the moon is made of green cheese. We need not dally with a suppositious amendment: "Whatever in the premises is non-essential to the conclusion is no part of what implies that conclusion." That dictum would condemn the syllogism, since in every syllogism the premises contain information not contained in the conclusion. So to alter the meaning of 'implies' that all implications would be reducible to the form 'p implies p', would destroy the usefulness of logic.

That analytic conclusions are implied by any premise, is inescapable once we recognize as valid the procedure by which, our premise being taken, alternatives to be conjoined can be exhaustively specified. In a simple case, use of the principle in question may appear jejune; but in more complex instances we hardly could proceed without it. If today is Monday, then either today is Monday and it is hot or today is Monday and it is not hot. And that implies that either it is hot or it is not hot. This has the appearance of clumsy sleight of hand; and we may say, "In what sense of 'If . . . then' and 'implies'?" It can be answered; "In the sense of 'presupposed'." A presupposition of X is a necessary condition of X. When p implies q and q implies r, the truth of p is a *sufficient* condition of the truth of q, but it is r the truth of which is a *necessary* condition of the truth of q. The analytic principle called the Law of the Excluded Middle is presupposed when we begin to logicize. When any premise is taken, all statements of the form $p \vee \sim p$ are already presumed, whether we think of them or not. Otherwise we could not move forward by an inference: p is true, hence either p and q or p and not-q. But p and not-q is not possible; p implies q. So q.[1]

[1] It would be sufficient here if we substitute "But p and not-q is not the case." But we need to *infer* q only when we do not know, without reference to some premise, whether q is true or not. We infer that it is the case by knowing the *incompatibility* of its falsity with the truth of the premise.

It remains to suggest why these paradoxes of Strict Implication are paradoxical. Let us observe that they concern two questions: What is to be taken as consequence of an assumption which, being self-contradictory, could not possibly be the case; and what is to be taken as sufficient premise for that which, being analytic and self-certifying, could not possibly fail to be the case? That to *infer* in such cases is affected with a sense of paradox, reflects the futility of drawing any inference when the premise is not only known false but is not even rational to suppose; and the gratuitous character of inferring what could be known true without reference to any premise. 'Deducible' and 'inferable' have a normative connotation: they do not concern what we are 'able' to infer in our foolish moments, but what, having taken commitment to our premises, it is rationally warranted to conclude and rationally forbidden to deny. And it becomes paradoxical to say; "From a premise of the form 'p and not-p' any and every conclusion is to be inferred." Such a statement invites the rejoinder; "From such a premise, no conclusion at all should be drawn, because no such premise should ever be asserted; the supposition of it is irrational." Somewhat similarly it has the air of paradox to say; "The Law of the Excluded Middle, and any conclusion reducible to the form 'either p or not-p,' is to be inferred from any and every premise." This might invite the rejoinder; "Any such conclusion is not to be inferred at all, being self-certifying to any clear and rational mind." But in this case, the rejoinder would have less force. Some rational minds—including those of human logicians and mathematicians—sometimes have a need to deduce something which they recognize will, if deducible, be analytic. They do not yet know whether what they have in mind is analytic or not, or even whether it is true. To deduce it will be to prove it. What manner of procedure is open to them in such cases? They *might* proceed by seeing whether they can reduce this statement which they wish to establish as a theorem to the form 'p or not-p' for some complex expression p. Ordinarily, they do not attempt that manner of proof. Instead, they seek to deduce what they have in mind from the postulates, definitions, and other theorems already assumed or proved. But, without knowing whether the hoped-for theorem is analytic or not, what steps of inference are open to them? Plainly, they must confine themselves to deductive steps which are universally justifiable modes of inference, valid whether the conclusion sought is analytic or is contingent on given premises. Perhaps this is so obvious as to be banal. It comes to the same thing to say: "That what is analytic is deducible from any and every premise, warrants the presumption that a particular statement, p, is deducible from any and every premise, only on the *additional* premise that p is analytic." The sense in which what is *known* to be reducible to the form 'r or not-r' is thereby known to be deducible from any and every premise, has already been illustrated. The point is that such a paradigm as $p . \prec . q \vee \sim q$, though it is a true and analytic statement about what is deducible from what, is incapable of any use as a rule for deducing any conclusion which is not already known with certainty. That is the paradox.

Thus none of the paradoxical paradigms, either those which indicate what is deducible from the logically impossible, or those which formulate sufficient conditions of the logically necessary, are capable of any use for the characteristic purpose of inferring—for establishing the truth or increasing the credibility of something by reference to premises which imply it. This holds, for paradigms of the former sort, for the double reason that the contradictory premise is not assertable, and that being deducible from such a premise adds no scintilla of credibility to the consequence of it. And it holds for paradigms of the latter type because the paradigms cannot be applied unless the conclusion is known with analytic certainty in advance.

In the light of these considerations affecting the paradoxes, we may observe the possibility of drawing a somewhat fine distinction between 'deducibility' and 'inferability'. (I do not propose this distinction.) We might say that no inference is to be drawn from anything the assertion of which is rationally contraindicated. And for that the acceptance of which is rationally dictated—the analytic—no premise is to be taken as a condition of assertion: the sense in which it is to be 'inferred' is extralogical. If we should be minded so to limit the sense of 'infer', we could then say that the paradoxes of Strict Implication are unexceptionable paradigms of deduction, but are not relevant to logically valid inference.

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