The Philosophy of MATHEMATICS

An Introduction

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MATHEMATICS AS THE SCIENCE OF FORMAL SYSTEMS: EXPOSITION

We turn now to another line of thought with another historical root. As Leibniz sought the source of the self-evidence and the content of mathematics in logical relations between propositions and concepts, so Kant sought it in perception. And, just as Leibniz conceived the guiding principles of logicism, so Kant was led to anticipate the guiding principles of two modern movements in the philosophy of mathematics: formalism and intuitionism.

For Kant the role of logic in mathematics is precisely the role it has in any other field of knowledge. He holds that in mathematics, although the theorems follow from the axioms *according to* principles of logic, the axioms and theorems are not *themselves* principles of logic, or any application of such principles. He regards them, on the contrary, as descriptive, namely as describing the structure of two perceptual data, space and time. Their structure manifests itself as something which we find in perception, when we abstract its varying empirical content. Thus in perceiving two apples, the iteration which is perceived is a feature of the space and time in which the apples are located. The same structure manifests itself further in our deliberate geometrical constructions, both in making such constructions possible and in confining them within limits—permitting the construction, for example, of three-dimensional objects but not of four-dimensional.

Hilbert, who in his practical programme adapted Kant's guiding idea, expresses Kant's 'fundamental philosophical position', and his own, in the following words: '. . . something which is presupposed in the making of logical inferences and in the carrying out of logical operations, is already given in representation (*Vorstellung*): *i.e.* certain extra-logical concrete objects, which are intuitively present as immediate experience, and underlie all thought. If logical thinking is to be secure, these objects must be capable of being exhaustively surveyed, in their parts; and the exhibition, the distinction, the succession of

their parts, and their arrangement beside each other, must be given, with the objects themselves, as something that cannot be reduced to anything else or indeed be in any need of such reduction.'[1]

Hilbert shares this fundamental position with Brouwer and his school as well as with Kant. If mathematics is to be restricted—entirely and without qualification—to the description of concrete objects of a certain kind, and logical relations between such descriptions, then no inconsistencies can arise within it: precise descriptions of concrete objects are always mutually compatible. In particular, in this kind of mathematics, there are no antinomies to trouble us, generated by the notion of actual infinity; and for the simplest of reasons, namely that the concept of actual infinity does not describe any concrete object.

Yet—and here is the root of the disagreement between formalists such as Hilbert and intuitionists like Brouwer—Hilbert does not think his position requires him to abandon Cantor's transfinite mathematics. The task he sets himself is the accommodating of transfinite mathematics within a mathematics conceived, in Kantian fashion, as concerned with concrete objects. 'No one will ever be able to expel us', he says, 'from the paradise which Cantor has created for us.'

His way of reconciling concrete, finite mathematics with the abstract and transfinite theory of Cantor is something Hilbert again owes—at least fundamentally—to Kant.[2] It was not, indeed, in the philosophy of mathematics that Kant employed the principle on which Hilbert's reconciliation proceeds. Kant employed it in a part of philosophy which for him was much more important—the reconciliation of moral freedom and religious faith with natural necessity. Arguing in this context, Kant first pointed out that the notion of moral freedom (and some other notions, including that of actual infinity) were Ideas of Reason which were unrelated to perception, in the sense of being neither abstracted from it nor applicable to it. He then argued that any system containing notions applicable primarily to concrete objects (such as the mathematics and physics of his day) could indeed be amplified by Ideas, but only provided the amplified system could be shown to be consistent. Proving consistency, within a system embracing both the findings of theoretical science on the one hand and, on the other, the Ideas of morals and faith, was Kant's way as he himself put it 'of making room for faith'.

In quite similar fashion Hilbert distinguishes between the concrete

[1] Hilbert, *Die Grundlagen der Mathematik*, Sem. der Hamburger Universität, vol. 6, p. 65. Also Becker, p. 371.

[2] See, *e.g.*, *op. cit.*, p. 71.

ideal notions of infinity

or real notions of finite mathematics and the ideal notions (Ideas) of transfinite mathematics. In order to justify the adjunction of ideal notions to the real, he too requires a proof that the system is consistent. Hilbert's task is thus to prove the consistency of a system comprising finite and transfinite mathematics. He adopts the Kantian theses (i) that mathematics includes descriptions of concrete objects and constructions and (ii) that the adjunction of ideal elements to a theory requires a proof of the consistency of the system thus amplified. In his hands these have been transformed into what is claimed to be a practical programme for founding mathematics upon what is perceived or perceivable. We have now to examine this.

1. *The programme*

To show that a system of propositions—*e.g.* the theorems of a mathematical theory—is internally consistent is to show that it does not contain two propositions one of which is the negation of the other or a proposition from which any other proposition would follow. (The second formulation also holds for systems in which negation is not available.) Only in the case of very simple systems is it possible to compile a list of all their propositions and to check the list for inconsistency. In general, a more complex investigation into the structure of the system as a whole will be necessary.

Such an investigation presupposes that the system is clearly demarcated and capable of being surveyed. The demarcation, as Frege saw, is secured to some extent by axiomatization: *i.e.* by listing the undefined concepts in the system, the presupposed assumptions in it, and lastly, the inference-rules (the rules for deducing theorems —from the assumptions and already deduced theorems). We have mentioned (in chapter II above) various axiomatizations of the logic of propositions, of classes, and of quantification. Similar axiomatizations have often been given for other systems, such, *e.g.*, as (unarithmetized) geometry and parts of theoretical physics. Axiomatization may be more or less strict, depending on the extent to which the rules of sentence-formation and of inferential procedure are more or less explicitly and precisely formulated.

For proving the consistency of a system two methods are available: the direct and the indirect. In some cases it can be shown by combinatorial means that inconsistent statements are not deducible in a given theory. In other cases the direct method proceeds by exhibiting a perceptual model of the theory. More precisely it consists (i) in identifying the objects of the theory with concrete objects, (ii) in identifying the postulates with exact descriptions of these objects and their mutual

relations; and (iii) in showing that an inference within the system will not lead to any other than exact descriptions. Since mathematics abounds in concepts of actual infinities which cannot be identified with perceptual objects, the use of the direct method is restricted to certain small parts of mathematics.[1]

A theory involving actual infinities can—at least *prima facie*—be tested for consistency only by the indirect method. One proceeds in this by establishing a one-one correspondence between (a) the postulates and theorems of the original theory and (b) all or some of the postulates and theorems of a second theory, which is assumed to be consistent. The consistency of this theory can in some cases be reduced to a third one. But none of these theories can have a concrete model.

Amongst indirect proofs of the consistency of any geometrical or physical theory the most common are based on arithmetization, *i.e.* on representing the objects of these theories by real numbers or systems of such. This is by no means surprising. For on the one hand the creative work of mathematicians, at least since Descartes, has been characterized by the demand that all mathematics should be capable of being embedded in arithmetic; and, on the other hand, the creative work of physicists, at least since Galileo, has been characterized by the demand that all physics should be mathematized. These are philosophical demands and convictions and they have led to extensions of mathematics so as to make it capable of accommodating all physical formalisms; and they have led to such extensions of arithmetic as to make it capable—by the use of one-one correspondences—of accommodating all mathematics, in particular all geometry and abstract algebra. It cannot indeed be said *a priori* that this arithmetization of science has no limits. But the reducibility to arithmetic of physical and mathematical theories which contain ideal notions, and which cannot be proved consistent by the direct method, raises the question of the consistency of arithmetic itself. Before Hilbert, no practical programme for proving the consistency of arithmetic had been suggested. (If mathematics should be found reducible to an obviously consistent logic, this problem would not, of course, arise.)

And Hilbert's basic idea, here, is as ingenious as it is simple. The mathematician deals with concrete objects or systems of such. He can therefore rely on 'finite methods'; in other words he can rest content with the employment of concepts which can be instantiated in perception, with statements in which these concepts are correctly applied, and with inferences from statements of this type to other such statements.

[1] See, *e.g.*, Hilbert-Bernays, *op. cit.*, p. 12.

Finite methods do not lead to inconsistencies, especially in mathematics where the concrete objects can be effectively demarcated.

Classical arithmetic does, of course, deal with such abstract and ideal objects as actual infinities. But even when on this account non-finite methods have to be used *within* arithmetic it may nevertheless be possible to regard or reconstruct arithmetic *itself* as a concrete object which can be dealt with by finite methods. It would be natural to expect this concrete object to possess properties capable of throwing light on classical arithmetic as usually conceived. It may in particular be expected to have a property the possession of which would guarantee the consistency of the classical arithmetic.

Before attempting a more detailed exposition of these points one can hardly do better than formulate the programme for proving the consistency of the classical arithmetic in Hilbert's own words: 'Consider the essence and method of the ordinary finite theory of numbers: This can certainly be developed through number-construction by means of concrete, intuitive (*inhaltlicher, anschaulicher*) considerations. But the science of mathematics is in no way exhausted by number-equations and is not entirely reducible to such. Yet one can assert that it is an apparatus which in its application to whole numbers must always yield correct numerical equations. But then there arises the demand to inquire into the structure of the apparatus to an extent sufficient for the truth of the assertion to be recognized. And here we have at our disposal, as an aid, that same concrete (*konkret inhaltliche*) manner of contemplation, and finite attitude of thinking, which had been applied in the development of the theory of numbers itself for the derivation of numerical equations. This scientific demand can indeed be fulfilled, *i.e.* it is possible to achieve in a purely intuitive and finite manner—just as is the case with the truths of the theory of numbers—those insights which guarantee the reliability of the mathematical apparatus.'[1]

The consistency of the classical arithmetic—including, we may say, the main parts of Cantor's theory—is to be proved and the programme would appear to be (i) to define with all possible clarity what is meant in mathematics by finite methods as opposed to non-finite, (ii) to reconstruct as much as possible of classical arithmetic as a precisely demarcated concrete object which is given to, or realizable in, perception and (iii) to show that this object has a property which clearly guarantees the consistency of classical arithmetic.

The formalist not only needs the assurance that his formalism formalizes a consistent theory, but also that it completely formalizes

[1] *Op. cit.*, p. 71; Becker, p. 372.

what it is meant to formalize. A formalism is complete, if every formula which—in accordance with its intended interpretation—is provable within the formalism, embodies a true proposition, and if, conversely, every true proposition is embodied in a provable formula. (This is the original meaning of the term 'completeness' which has also other, though related, meanings in the literature some of which have no reference to an original, non-formalized, theory.) For some such formalisms there are available mechanical methods—decision procedures—by which one can decide for any formula whether it is provable or not and whether consequently the embodied proposition is true or false. The ideal would be a consistent, complete and mechanically decidable formalism for all mathematics.

2. *Finite methods and infinite totalities*

Incompatibility is a relationship between propositions or concepts. Perceivable objects and processes cannot be incompatible with each other. Again, propositions cannot be incompatible with each other if they *precisely* describe such objects and processes; for a description implying incompatibility between entities that cannot be incompatible could not be precise. Yet the trouble is that there is no general test for deciding whether a description is or is not precise. Attempts such as Russell's sense-data theory to mark out in general objects which can be precisely described—or such attempts as are made by theories like Neurath's theory of 'protocol sentences' to mark out propositions which are precisely descriptive—are by no means universally accepted as successful. In mathematics it seems to be otherwise. Here it seems comparatively easy to demarcate a narrow field of perceptual objects and processes which will be capable of precise description, or at least of a description free from contradictions. In the elementary theory of numbers we deal with such objects and processes. The methods of dealing with them, the so-called finite (or 'finitary') methods, are explained in the above mentioned papers by Hilbert and in the classic *Die Grundlagen der Mathematik* by Hilbert and Bernays.[1] Consistently with these texts the point of view might be put as follows.

The subject matter of the elementary theory of numbers consists of the signs '1', '11', '111', etc., *plus* the process of producing these signs by starting with '1' and putting always another stroke beyond the last stroke of the previous sign. The initial figure '1' and the production-rule together provide the objects of the theory; these objects can be abbreviated by use of the ordinary notation, the numeral '111', *e.g.*, being written as '3'. The small letters *a*, *b*, *c*, etc.

[1] See also Kleene's *Introduction to Metamathematics*, Amsterdam, 1952.

are employed to designate unspecified figures. For operations performed on the figures one uses further signs: brackets, the sign '≡' (to indicate that two figures have the same structure) and the sign '<' (to indicate that one figure is in an obvious and perceivable way contained in another). Thus $11 < 111$, *i.e.* if beginning with '1' we build up '11' and '111' by parallel steps the former will be finished before the latter.

Within this elementary theory of numbers, one can perform and describe concrete addition, subtraction, multiplication and division. The associative, commutative and distributive laws, and the principle of induction are nothing else than obvious features of these operations. Thus '$11+111=111+11$' is an instance of '$a+b=b+a$', an equation which asserts in a general way that the production of figures by iterating the stroke does not depend on order.

Again the principle of induction, the most characteristic of all the principles of arithmetic, is, in the words of Hilbert and Bernays[1] not an 'independent principle' but 'a consequence which we take from the concrete construction (*Aufbau*) of the figures'. Indeed if (a) '1' has a certain property and (b) if, provided the property is possessed by any stroke-expression, it is also possessed by the succeeding stroke-expression (the expression formed by putting a further '1' after the original) then this property will be seen to be possessed by any stroke-expression that can be produced. Having defined the concrete fundamental operations by means of the concrete principle of induction, one can define the notion of prime numbers, and construct for any given prime number a bigger prime number. The process of recursive definition can also be defined and performed concretely. For example the factorial function $\rho\ (n)=1.2.3\ldots n$ is recursively defined by (a) $\rho(1)=1$ and (b) $\rho(n+1)=\rho(n).(n+1)$. This definition prescribes in an obvious way how, beginning with $\rho(1)$, and using nothing but concrete addition and multiplication, we can build up $\rho(n)$ for any perceptually given figure n.

Elementary arithmetic is the paradigm of mathematical theory. It is an apparatus which produces formulae, and which can be entirely developed by finite methods. This statement, however, the meaning of which has just been illustrated from the development of elementary arithmetic, is still needlessly imprecise, and requires an actual and explicit characterization of what is to be meant by 'finite methods'.

First, every mathematical concept or characteristic must be such that its possession or non-possession by any object can be decided by inspection of either the actually constructed object or the constructive

[1] *Op. cit.*, p. 23.

process which would produce the object. The second of these alternatives introduces a certain latitude in determining finite characteristics and the finite methods consisting in their employment. Thus one is reasonably content with a process of construction which is 'in principle' performable. Indeed it is at this point, namely when the choice arises between making the formalist programme less strict or sacrificing it, that some relaxation of the finite point of view may be expected.

Secondly, a truly universal proposition—a proposition about all stroke-expressions for example—is not finite: no totality of an unlimited number of objects can be made available for inspection, either in fact or 'in principle'. It is, however, permissible to interpret any such statement as being about each constructed object. Thus, that all numbers divisible by four are divisible by two means that if one constructs an object divisible by four, this object will have the property of being divisible by two. Clearly this assertion does not imply that the class of all numbers divisible by four is actually and completely available.

Thirdly, a truly existential proposition—to the effect, *e.g.*, that there exists a stroke-expression with a certain property—is equally not finite: we cannot go through *all* stroke-expression (of a certain kind) to find one which has the property in question. But we may regard an existential proposition as an incomplete statement to be supplemented by an indication either of a concrete object which possesses the property or of the constructive process yielding such an object. In the words of Hermann Weyl,[1] an existential proposition is 'merely a document indicating the presence of a treasure without disclosing its location'. Propositions which involve both universal and existential assertions—*e.g.* to the effect that *there exists* an object which stands to *every* object in a certain relation—can again only be suffered as *façons de parler* promising the exhibition of perceivable or constructible relationships.

Fourthly, the law of excluded middle is not universally valid. In finitist mathematics one permits neither the statement that *all* stroke-expressions possess a property P nor the statement that there *exists* a stroke-expression which does not possess P—unless these statements are backed by an actual construction. One consequently cannot admit as universally valid the unqualified disjunction of these two statements, that is to say the law of excluded middle.

Even in elementary arithmetic there is occasion for using, in a restricted way, transfinite methods, in particular the principle of

[1] *Philosophy of Mathematics and Natural Science*, Princeton, 1949, p. 51.

excluded middle. But whereas transfinite methods here are easily replaceable by finite ones quite sufficient for their perceivable or constructible subject matter, the situation is different, as we have seen already at various stages of the argument, in analysis. This fundamental difference between elementary arithmetic and analysis in its classical form is due—as has frequently been pointed out—to the fact that the central notion of analysis, that of a real number, is defined in terms of actual infinite totalities. (See Appendix A.)

We have seen that every real number between 0 and 1 (we can disregard the real numbers outside this interval without loss of generality) can be represented by a decimal fraction of the form $0{\cdot}a_1a_2a_3\ldots$ where the dots indicate that the number of decimal places is α, *i.e.* denumerably infinite. If the numbers to the right of the decimal point do not terminate, *i.e.* if they are not from a certain place onwards all zeros, and if their sequence shows no periodicity, then the infinite decimal fraction represents an irrational number. Every place of the decimal fraction can be occupied by one of the numbers 0 to 9. The totality of these possibilities, which represents the totality of all real numbers in any interval is, we have seen, greater than the totality of all integers and greater than the totality of all rational numbers. Its cardinal number c is greater than α, the cardinal number of any denumerable set.

In order to appreciate the nature of this statement about real numbers it will be well to consider the representation of real numbers by binary fractions of the form $0{\cdot}b_1b_2b_3\ldots$. Here, just as the first place to the right of the decimal point indicates tenths, the second hundredths, the third thousandths and so on, so the first place to the right of the binary point indicates halves, the second quarters, the third eighths, etc. Again, just as every place of a decimal fraction can be occupied by any number from 0 to 9 inclusive, so every place of a binary fraction—every b—is occupied by either 0 or 1. Moreover just as all real numbers can be represented by all decimal fractions, so all real numbers can be represented by all binary fractions—the choice of the decimal, the binary or any other system being a purely external matter.

Assume now that all natural numbers are given in their natural order and in their totality thus: 1, 2, 3, 4, 5, 6, ... Now form a finite or infinite subclass from the totality, indicating *the choice* of a number for the subclass by writing 1 in its place, and indicating the *rejection* of a number by writing in its place 0. If we choose 2, 4, 5, ... and reject 1, 3, 6, we shall thus write 010110.... It is clear that every infinite sequence of zeros and ones determines one and only one subclass of the class of natural numbers in their natural order. But we have just seen that every infinite sequence of zeros and ones determines one and only one real number between 0 and 1 (in the binary representation). There is thus a one-one correspondence between the class of all subclasses of natural numbers and the class of all real numbers between 0 and 1 and, as can be easily shown, the class of all real numbers in any interval. In speaking of a real number the classical analyst is committed to the assumption that it is 'possible' to pick out a subclass from the *actual* totality of all natural numbers. In speaking of all real numbers he is not only committed to assuming the actual totality of all natural numbers but also the *greater actual* infinite totality of all subclasses of this class (see p. 63). The assumption of such totalities implied in speaking of a real number, or even of all real numbers, transcends the finite point of view and the employment of finite methods.

Classical analysis transcends the finite point of view not only by assuming actual infinite totalities, but by using the law of excluded middle without qualification. If not all members of a class have a certain property P then at least one member has the property *not-P* and *vice-versa*—indifferently whether the class in question be finite, denumerably infinite or greater than these. Another non-constructive principle of classical analysis and the theory of sets was made explicit by Zermelo. This is the so-called principle or axiom of choice (*Auswahlprinzip*). Hilbert and Bernays formulate it as follows:[1] 'If to every object x of a genus $\mathfrak{G}_1$ there exists at least one object y of genus $\mathfrak{G}_2$, which stands to x in the relation $B(x, y)$, then there exists a function ϕ, which correlates with every object x of genus $\mathfrak{G}_1$, a unique object $\phi(x)$ of genus $\mathfrak{G}_2$ such that this object stands in the relation $B(x, \phi(x))$ to x.'

Another way of expressing the axiom of choice is to say that given a class of classes, each of which has at least one member, there always exists a selector-function which selects one member from each of these classes. (One might 'picture' the selector-function as a man with as many hands as there are non-empty classes—picking out one element from each of them.) It is obviously possible to exhibit a selector-function for a class consisting of a finite number of finite classes. When it comes, however, to picking out one member from each of an infinite number of finite classes, still more from an infinite number of infinite classes, the exhibition of the selector-function, as a feature of perceivable or constructible objects or processes, is clearly out of the question. That the axiom of choice is implicitly assumed in a great deal of analysis and set-theory only became clear to mathematicians

[1] *Op. cit.*, p. 41.

after Zermelo discovered it to have been a tacit assumption in the proof that every class can be well-ordered, and that in consequence the cardinal numbers of any two (finite or infinite) classes are comparable (see p. 64).[1]

Thus, on Hilbert's showing, classical mathematics has as its hard core a perceivable, or at least in principle perceptually constructible, subject-matter, to which fictitious, imperceivable and perceptually non-constructible objects, in particular various infinite totalities, are adjoined. To this adjunction of 'fictitious' subject-matter there correspond (i) ideal concepts which are characteristic of it—*e.g.* Cantor's actual infinities, and transfinite cardinal and ordinal numbers—(ii) ideal statements describing either it or operations upon it—*e.g.* the unqualified law of excluded middle, or the axiom of choice—(iii) ideal inferences leading either from statements of finite mathematics to ideal statements or from ideal statements to other ideal statements.

This adjunction of ideal concepts, statements and inferences to a theory is, of course, not at all new in mathematics. Thus in projective geometry it has proved of great use to introduce an ideal point at infinity on every straight line and to define it as the point at which all lines parallel to the given line intersect; and to introduce, in every plane, an ideal line containing all the points at infinity of all the lines in the plane. There can, of course, be no question of 'the ideal point common to two parallel lines' denoting any perceptually-given or constructible entity; the reasons for demanding points of intersection of parallel lines require any set of parallel lines to have *one* point of intersection, *not two* points of intersection, one, as it were, at each end of the parallel lines.[2] By adjoining ideal points, lines and planes to the 'real' ones, one creates concepts which, although logically related to the concepts to which they have been adjoined, are even less characteristic of perception than the former. Even if 'real point' and 'real line' can *cum grano salis* be said to describe perceptual objects, no amount of salt will make it plausible to say that 'ideal point' and 'ideal line' are perceptual characteristics.

The introduction of ideal elements into projective geometry, into the algebraic theory of numbers and mathematical theories in general, has, according to Hilbert, been one of the glories of creative mathematical thinking. The emergence of antinomies as a result of this adjoining of infinite totalities to elementary arithmetic requires according to him not their abandonment but some proof that an extended arithmetic—the combination into one system of finite and transfinite objects and methods—is free from contradiction. How this is to be achieved is, he argues, suggested by considering elementary arithmetic.

[1] As to the use of the axiom in topology, in the theory of Lebesgue measure, etc., see J. B. Rosser, *Logic for Mathematicians*, New York, 1953, pp. 510 ff.

[2] For an explanation of the reasons for the introduction of ideal points, lines and planes and for further details see, *e.g.*, Courant and Robbins, *What is Mathematics?*, Oxford, 1941, and later editions, especially chapter IV.

His crucial point here is that elementary arithmetic can be conceived of in two different ways; on the one hand, quite naturally, as being a *theory about* the regulated activity of constructing stroke-expressions, and, on the other hand, somewhat artificially, as being a *formalism*, *i.e.* as itself a regulated activity of constructing perceptual objects—this time, of course, not stroke-expressions but formulae. The arithmetical theory consists of statements, the arithmetical formalism of symbol-manipulations and their results. The formalism can, just like the regulated activity of constructing stroke-expressions, become the subject-matter of another theory, usually called a 'metatheory'. We are thus led to distinguish between two kinds of constructing activities—stroke-construction and formula-construction; and between two kinds of theory—the original theory about stroke-construction and the new 'metatheory' about formula-construction.

The connection between arithmetical theory, arithmetical formalism and metatheory about the arithmetical formalism is obviously quite intimate. In its broad outlines it is founded on the fact that the *same* physical objects, *e.g.* ⟨1+1=2⟩ or ⟨1+1=3⟩ (the objects between the French quotes), function in distinct though corresponding ways, in the arithmetical theory and in the arithmetical formalism. The formalism may be built up in such a manner that it becomes possible to distinguish among its rules two kinds in particular: (a) rules for the production of such formulae as correspond (like our two examples) to statements of the theory and which we shall call statement-formulae; (b) rules for the production of such as (like the first example, but unlike the second) correspond to true statements or theorems of the theory and which we shall call theorem-formulae.

In asserting that a certain physical object is, in the context of the formalism, a statement-formula or a theorem-formula, we are speaking *about* formula-construction and are making a statement of metatheory. This statement is *finite*, in that it asserts of a perceptual object, or of the process by which it is produced, a purely perceptual or (literally!) formal characteristic. The *formal* characteristic of a statement-formula's being a theorem-formula corresponds to the *logical* characteristic of a statement's being a theorem.

To this correspondence between the formal characteristics of the formalism and the logical characteristics of the theory, others can be added. Perhaps the most important of these is the correspondence between the formal consistency of the formalism and the logical consistency of the theory. To assert that the *theory* is logically consistent is to assert that not every statement of the theory is also a theorem of the theory. (This definition, as has been indicated before, has the advantage of avoiding the use of the notion of negation.) To assert that the *formalism* is formally consistent is to assert that not every statement-formula of the formalism is also a theorem-formula. In view of the correspondence (mediated by their embodiment in the same physical objects) between statement-formulae and theorem-formulae on the one hand, and statements and theorems on the other, we are entitled to say that to demonstrate formal consistency is at the same time to demonstrate logical consistency.

We now turn to non-elementary arithmetic. The subject-matter of this arithmetical *theory* is, of course, no longer finite. But it may be possible to construct an arithmetical *formalism*—with statement-formulae and theorem-formulae corresponding as before to statements and theorems of the theory; and this formalism could then be the subject-matter of a metatheory. Since the subject-matter, namely formula-construction, would be finite, the metatheory would be just as finite as elementary arithmetic, from which it would differ only by being about a different kind of perceptual construction. If a formalism corresponding, in the required manner, to the theory of non-elementary arithmetic can be constructed, then we can again, by demonstrating *formal* consistency of the formalism, *eo ipso* establish *logical* consistency of the theory. Indeed we can do this by strictly *finite* methods, since our subject matter—the regulated activity of formula-construction—is perceptual, or at least in principle perceptually constructible. Our next task, therefore, must be to consider the formula-constructing activities, or formalisms—both formalisms considered by themselves and formalisms which are at the same time formalizations of theories.

3. *Formal systems and formalizations*

Once a formal system has been constructed a new 'entity' has been brought into the world—a system of rules for the production of formulae. These formulae are perceptual objects which can be distinguished and classified by means of perceptual characteristics which are possessed either by the formulae themselves or by the process of their production, in particular by the sequence of formulae which

successively lead from an initial formula to the formula under consideration. In a formal argument we must ignore any correspondence between the formal properties of the formal system and the logical properties of any pre-existing theory, even though to establish such a correspondence was the guiding motive in constructing the formal system.

According to Hilbert the content of mathematics is still propositions; in the case of elementary arithmetic they are propositions about stroke-expressions and their production, in the case of the amplified (classical) arithmetic they include in addition propositions 'about' ideal objects. The formal systems which he constructs are merely means by which, in virtue of the correspondence between formal and logical properties, he studies the pre-existing mathematical theories. His formalisms are formalizations.

Yet since no insight derived from the pre-existing theory is permitted to enter the arguments concerning the formal system; since, that is to say, from the point of view of these arguments, no theory needs to exist of which the formal theory is a formalization, the possibility is opened to us of regarding the formal theory not merely as an instrument for investigating a pre-existing system of propositions, but as the subject-matter of mathematics itself. There are good grounds for this. On the one hand, there is no reason why the subject-matter of metamathematics should not be extended to any kind of formal manipulation of marks. On the other hand a phenomenalist philosopher, or one of a similar philosophical persuasion, might well—for philosophical reasons of a general kind—deny the existence of ideal propositions and thus declare, *e.g.*, the amplified arithmetic with its ideal objects and propositions to be meaningless or simply false. If so, he would, with H. B. Curry[1] propose to define mathematics as 'the science of formal systems'. In other words, whereas to Hilbert mathematics, or rather metamathematics, is the Leibnizian 'thread of Ariadne' leading him through the labyrinth of mathematical propositions and theories, the *strict formalist* regards mathematics as having this thread—and nothing more—for its subject-matter.

The change from Hilbert's formalist point of view to the strict formalism of Curry leaves the former's mathematical results untouched. It represents, however, a transition to a different philosophical point of view. Mathematics has now no truck *with anything but* formal systems, in particular not with ideal, non-perceptual entities. Hilbert's position is analogous to that of a moderate phenomenalist who would admit physical-object concepts as auxiliary

[1] *Outlines of a Formalist Philosophy of Mathematics*, Amsterdam, 1951.

—if fictitious—concepts, in terms of which sense-data would be ordered or purely phenomenalistic statements made—even if physical-object concepts could not be 'reduced' to sense-data, or to purely phenomenalist concepts. Strict formalism on the other hand is analogous to a phenomenalism which would admit only sense-data and purely phenomenalist statements.

Strict formalism as a philosophy of mathematics is nearer than Hilbert's view to Kant's doctrine in the *Transendental Aesthetic*. According to Kant a statement in pure mathematics has constructions for its subject-matter—constructions in space and time, which by the very nature of these intuitions are restricted. According to strict formalism the subject-matter of mathematics is constructions, the possibility of which is restricted by the limits under which perception is possible; and our statements about these constructions are *demonstrationes ad oculos*, read off, as it were, from perception. They are true synthetic statements. However, their self-evidence is neither that of logical tautologies, nor, as Kant held, that attaching to supposedly *a priori* particulars. It is the self-evidence of very simple phenomenalist or sense-data statements. Statements about mathematical constructions are in other words empirical statements involving the least possible risk of error. This is the reason why in discussing the process of proof—one of the principal subjects of the science of formalisms—Curry says, very naturally, that it is 'difficult to imagine a process more clear cut and objective'.

For Hilbert the *raison d'être* of formal systems is to save and safeguard the pre-existing—albeit somewhat modified—classical theories, in particular Cantor's theory of sets. For Curry formal systems are the substitutes of classical mathematics. From these fundamental differences between moderate and strict formalism others follow. For Hilbert, who intends to establish the (logical) consistency of theories *via* the (formal) consistency of formal systems, a (formally) inconsistent formal system is useless. Not so for Curry. He maintains that for the acceptability or usefulness of a formal system 'a proof of consistency is neither necessary nor sufficient'.[1] Indeed inconsistent formal systems, he argues, have in the past proved of the greatest importance, *e.g.* to physics.

Both Hilbert and Curry deny the possibility of deducing mathematics from logic. Yet whereas Hilbert regards principles of reasoning which are sufficient for elementary arithmetic as logical principles of a finite and, as it were, minimal logic, Curry separates logic and mathematics even more drastically. It all hinges, he says,[2] 'on what one means by "logic"—"mathematics" we have already defined. . . . On the one hand logic is that branch of philosophy in which we discuss the nature and criteria of reasoning; in this sense let us call it logic (1). On the other hand in the study of logic (1) we may construct formal systems having an application therein; such systems and some others we often call "logics". We thus have two-valued, three-valued, modal, Brouwerian, etc. "logics", some of which are connected with logic (1) only indirectly. The study of these systems I shall call logic (2). The first point regarding the connection of mathematics and logic is that mathematics is independent of logic (1). . . . Whether or not there are *a priori* principles of reasoning in logic (1), we at least do not need them for mathematics.'

[1] *Op. cit.*, p. 61. [2] *Op. cit.*, p. 65.

Hilbert has never explicitly and at any length dealt with the philosophical problem of applied mathematics. He seems to favour the view that there is a partial isomorphism between pure mathematics and the realm of experience to which it is applied. Elementary arithmetic, that is to say, either is itself the empirical subject-matter of our study—a 'physics' of stroke-symbols and stroke-operations—or else can be brought into one-one correspondence with some other empirical subject-matter; for example, to take a trivial case, apples and apple-operations. The non-elementary parts of the amplified arithmetic, on the other hand, have no empirical correlates. Their purpose is to complete, systematize and safeguard the elementary core which alone either is empirical or has empirical correlates.

According to Curry, who is quite explicit on this question, we must distinguish between the *truth* of a formula within a formal system—*i.e.* the statement that it is derivable within the system—and the *acceptability* of the system as a whole. The former is 'an objective matter about which we can all agree; while the latter may involve extraneous considerations'.[1] Thus he holds that 'the acceptability of classical analysis for the purposes of application in physics is . . . established on pragmatic grounds and neither the question of intuitive evidence nor that of a consistency proof has any bearing on this matter. The primary criterion of acceptability is empirical; and the most important considerations are adequacy and simplicity.'[2] When it comes to the application of mathematics Curry is a pragmatist. He does not go so far as the pragmatic logicist whose view of pure mathematics is also pragmatist and who denies that logical, mathematical and empirical propositions can be distinguished by any sharp criteria. (See p. 57.) The domain of formal theories and the propositions about their formal properties are, Curry holds, clearly demarcated.

[1] *Op. cit.*, p. 60. [2] *Op. cit.*, p. 62.

Before describing some formal systems in outline, we may perhaps be allowed an imprecise, metaphorical characterization of the basic ideas of formalism. According to most philosophers, from Plato to Frege, the truths of mathematics exist (or 'subsist') independently of their being known and independently of their embodiments in sentences or formulae, even if these are needed for the truths to be grasped. It was Hilbert's ingenious programme—foreshadowed to some extent by Leibniz—so to embody the truths of classical mathematics that the perceptual features of the bodies or of the processes by which they are produced correspond to logical features of mathematical propositions. The theorem-formulae are, as it were, the bodies and the disembodied truths the souls—every soul having at least one body. This programme, as will be explained a little more precisely later, cannot be carried out. It has been demonstrated by Gödel that every embodiment of classical mathematics in a formalism must be incomplete; there are always mathematical truths which are not embodied in theorem-formulae.

In order to appreciate this result we must be a little more specific about the nature of formalisms. Hilbert remarks on a kind of pre-established harmony which favours the progress of mathematics and the natural sciences. Results which are achieved in the pursuit of quite diverse purposes often provide the much needed instrument for a new scientific aim. The logical apparatus of *Principia Mathematica*, which, on the basis of previous researches with still different aims, was devised for the purpose of reducing mathematics to logic, provided, in Hilbert's own particular case, the *almost* finished tool for executing his quite different programme. Where *Principia Mathematica* falls short is in its incomplete formalization. It is not wholly a system of rules for manipulating marks and formulae, in particular theorem-formulae in total independence of the fact that they can be interpreted as propositions of classical mathematics. But *Principia Mathematica* is an *almost* perfect foundation for the rigorous formalization of classical mathematics.

Indeed, of the formal systems, those outlined in discussing the logicist philosophy of mathematics are as good examples as any. This applies in particular to the propositional calculus and the formal system of Boolean class-logic. Here we shall do no more than describe the general nature of formal systems. They are machines for the production of physical objects of various kinds, machines whose properties have been made the subject of extensive and detailed inquiries by Hilbert, Bernays, Post, Carnap, Quine, Church, Turing, Kleene and many others. As the result of the work done by these

authors the terms 'machine' and 'mechanical properties' have in logical contexts long ceased to be metaphorical. (Indeed, most important insights into the nature of formalisms, that is to say the most important theorems of metamathematics or, as it is also called, proof-theory, can most simply and clearly be formulated as statements to the effect that certain formula-producing machines can, and certain others cannot, be constructed.)

Strict formalism regards, as we have seen, all mathematics as the science of formal systems, whether they are formally consistent or not, and whether or not they are intended to be formalizations of pre-existing theories; and it has made the nature of formalisms *per se* easier to grasp. To do this has become necessary for any philosophy of mathematics. For there can be no doubt that whatever else mathematics may mean, either now or in the future, it must always include the science of formal systems.

A very clear characterization of formal systems in general is given by Curry.[1] Each is defined by a set of conventions, its so-called primitive frame. By indicating the primitive frame we are providing an engineer with all the data he needs (apart from his knowledge of engineering) for constructing the required formula-producing machine. Curry distinguishes the following features in any primitive frame:

(i) Terms

These are (a) *Tokens*, which are specified by giving a list of objects of different types, *e.g.* marks on paper, stones or other physical objects. (b) *Operations*, *i.e.* modes of combination for forming new terms. (c) *Rules of formation* specifying how new terms are to be constructed. For example, if marbles and boxes are among our terms and the enclosing of marbles in boxes among our operations, we might adopt the rule of formation permitting the enclosure of each marble in a box, and stipulate that the enclosed marbles belong to the same kind of term as the loose ones.

(ii) Elementary Propositions

These are specified by giving a list of 'predicates' with the number and kind of 'arguments' for each. For example, we may specify as predicates pieces of wood with *n* holes into which both enclosed and loose marbles can be fitted and then determine that our elementary propositions are all those pieces of wood the holes of which have been duly filled in by enclosed or loose marbles.

[1] *Op. cit.*, chapter IV.

(iii) **Elementary Theorems**

(a) *Axioms, i.e.* elementary 'propositions' which are stated to be 'true' unconditionally. (b) *Rules of Procedure* which are of the following form: 'If $P_1, P_2, \ldots, P_m$ are elementary theorems subject to such and such conditions, and if Q is an elementary proposition having such and such a relation to $P_1, P_2, \ldots, P_m$, then Q is true.' For example, if two pieces of wood with holes filled in by marbles are elementary theorems, then any piece of wood which is produced from the former by gluing them together is also 'true'.

In order to be able to speak of the primitive frame we must have names for the tokens, operations and predicates and also indications of the way in which the predicates are applied to terms. Specification of the features which constitute the primitive frame of a formal system must be effective or *definite* (a term used by Carnap). This means that it must be possible to determine after a finite number of steps whether an object has or has not this feature. Indeed if a formal system is to be capable of being treated by finite methods (à la Hilbert), if in other words what is to be proved about it can be proved by demonstrations *ad oculos*, then the properties of being a formal predicate, of being a formal axiom, of a formula's being formally derived from another in accordance with a rule of procedure, must all be definite.

The property of being a theorem-formula may be but it need not be definite; but the formal relation between a formula and the sequence of formulae constituting its proof must, of course, be definite. In most mathematical theories a formula does, so to speak, not bear on its forehead the mark of being a theorem, but the proof of it, once given, must be capable of being checked in a finite number of steps.

Many formal systems have been constructed by mathematicians in the present century. The motive of the activity has usually been the need so to embody propositions into formulae that the formal properties and relations *of the formulae* guarantee corresponding logical properties and relations of the propositions. Indeed, as we have seen, the ultimate purpose of Hilbert's programme, and what would be its consummation, is a proof of the logical consistency of the main body of classical mathematics reached *via* a proof of the formal consistency of a suitable formal system.

As has often happened before in other branches of mathematics, the study of formal systems led to unexpected results, to new problems, new techniques and to at least one new branch of pure mathematics, namely the theory of recursive functions. The importance of this theory is considered by the experts very great. Thus E. L. Post who has not only made important contributions to this subject, but who also

has expressed its main ideas in a manner which makes them accessible to non-experts, expresses the view that the formulation of the notion of recursive functions 'may play a role in the history of combinatory mathematics second to that of the formulation of natural number'.[1]

The reader of a book on the philosophy of mathematics cannot expect that a full knowledge of these new ideas and techniques will be conveyed in it. Yet he will readily see that the question how far the correspondence between pre-existing theory and formal system can be established is of great philosophical relevance; and he will expect a report of results achieved by the mathematicians. *Prima facie* the complete embodiment of mathematical theories in formalisms may seem possible; and then it will at least be arguable that the pre-existing theories are merely 'intuitive' in the somewhat disparaging sense in which the term is used by mathematicians on the first few pages of their treatises before they get down to business, and that the said theories are merely heuristic preliminaries for the construction of formalisms and statements about them.

We must, therefore, attempt to give an account of some results in the science of formal systems, trusting the mathematicians—as we have always done so far—to have done their job efficiently.

4. *Some results of metamathematics*

Only a very brief and very rough outline of Gödel's main result and of some new developments connected with it can be given.[2] Suppression of 'technicalities' must here inevitably mean suppression of essential arguments and insights. To whet the appetite of the reader without crass misstatements is perhaps the best that can be done.

We assume with Hilbert that the method and results of elementary arithmetic (see p. 77) need no justification; and we consider a consistent formal system F which is sufficiently expressive to permit the formalization of elementary arithmetic in it. This implies the requirement

[1] *Bulletin of the American Mathematical Society*, 1944, vol. 50, no. 5.

[2] The fundamental paper is Gödel's 'Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme, I' in *Monatshefte für Mathematik und Physik*, 1931, vol. 38. For 'an informal exposition of Gödel's theorem and Church's theorem' see J. B. Rosser's article of this title, *Journal of Symbolic Logic*, 1939, vol. IV, no. 2. An informal and formal account of Gödel's theory is found in *Sentences Undecidable in Formalized Arithmetic* by Mostowski, Amsterdam, 1952; also in Kleene, *op. cit.*, and Hilbert-Bernays, *op. cit.*, vol. 2. The theory of recursive functions is developed from first principles and without a specialized logical symbolism in R. Péter's *Rekursive Funktionen*, 2nd ed., Budapest, 1958. For an excellent general survey of the present state of the theory, see John Myhill, *Philosophy in Mid-Century*, Florence, 1958.

that all arithmetical expressions correspond to formal expressions in such a fashion that no formal theorem of F corresponds to a false arithmetical proposition. If a formal statement, say f, is the formalization of an arithmetical proposition a, a is also said to be an (arithmetical) interpretation of, or the intuitive meaning of, f.

Let us say that F *completely* formalizes elementary arithmetic provided that in the case of every formal statement f which is the formalization of an arithmetical statement either f or $\sim f$ is a formal theorem of F; or briefly, provided that f is decidable. Hilbert aimed at the complete formalization of (substantially) the whole of classical mathematics. Gödel has shown that even a formal system which formalizes no more than elementary arithmetic does *not* formalize it completely.

The incompleteness of F is established by the actual construction of a formal statement f which formalizes an arithmetical proposition while yet neither f nor $\sim f$ is a formal theorem of F, *i.e.* while f is undecidable. The interpretation of f reminds one of the liar-paradox: 'The proposition which I am now asserting is false.' If the assertion of the proposition is correct then the proposition is false, from which it follows that the assertion is incorrect. The statement is 'about' itself. It states its own falsehood, and states no more. It is this kind of self-reference which Gödel's formal proposition possesses. But whereas in the liar-paradox the relation between linguistic expression and its meaning is far from clear, Gödel's formal proposition is as clear as F and arithmetic.

We now turn to the construction of the undecidable f (following Mostowski's exposition). Since F formalizes elementary arithmetic, the integers and properties of integers must have formal counterparts in F. The formal integers or numerals will be printed in bold-faced type so that, *e.g.*, **1** corresponds to 1. The formal properties of integers will be expressed by $W(.)$, different formal properties being distinguished by different subscripts. If $W_0(.)$ is the formal counterpart of 'x is a prime number', then $W_0(\mathbf{5})$ is the formal counterpart of the arithmetical proposition that 5 is a prime number. The set of all formal properties of integers can be ordered in many ways into a sequence and we consider one of these sequences, say,

$$(1) \quad W_1(.), W_2(.), W_3(.), \ldots$$

In order now to construct the self-referring formal proposition let us formulate first any formal proposition arrived at by 'saturating' some formal property with the numeral corresponding to its subscript. Such formal propositions are $W_1(\mathbf{1})$, $W_2(\mathbf{2})$, $W_3(\mathbf{3})$, ... We next pick

out, say, $W_5(\mathbf{5})$. This formal proposition may or may not be a formal theorem of F. Let us assume that it is not, *i.e.* that

$W_5(\mathbf{5})$ is not a formal theorem of F.

This proposition is on the face of it not a formal proposition of F, but is a real proposition about a formal proposition, namely about the formal proposition $W_5(\mathbf{5})$. It is in Hilbert's sense a metastatement, belonging to the metalanguage in which we talk about F. Similarly the property:

(2) $W_n(\mathbf{n})$ is not a formal theorem of F

is on the face of it not a formal property belonging to F but a meta-property belonging to the metalanguage. It seems implausible that this property has a formal counterpart among the formal properties of F, in particular among the members of the sequence (1).

But Gödel shows that (2) must have such a counterpart in (1)—that a member of the sequence (1) formalizes the metaproperty (2) or, which amounts to the same thing, that this metaproperty is the interpretation or intuitive meaning of a member of the sequence (1). The method by which he shows this is known as the arithmetization (also the 'Gödelization') of the metalanguage or metamathematics, a procedure which is quite analogous to Descartes' arithmetization of Euclidean geometry—the provision of numerical coordinates for non-numerical objects, and of numerical relations for the non-numerical relations between these objects.

To each of the signs of F—*e.g.* $\sim$, v, (—an integer is assigned so that every finite sequence of signs corresponds to a finite sequence of integers. It is easy to find functions which will establish a one-one correspondence between finite sequences of numbers and numbers. (For example, if we agree to assign to a sequence $n_1, n_2, \ldots, n_m$ the product $p_1^{n_1}.p_2^{n_2}\ldots p_m^{n_m}$, where the p's are the prime numbers in their natural order, it is always possible to reconstruct the sequence from the number by factorization.) In this way every sign, every sequence of signs (*e.g.* every formal proposition) and every sequence of sequences of signs is assigned its numerical coordinate or Gödel number. Statements about formal expressions can thus be replaced by statements about integers.

Again, to every class of expressions there corresponds a class of Gödel numbers. The classes of Gödel numbers needed for the incompleteness theorem are all defined recursively, *i.e.* each element can be actually calculated from the previous ones. The same is true of the required relations between Gödel numbers and of the functions which take Gödel numbers for their arguments and values. It is in particular possible to demarcate in this manner a class T, the class of all formal propositions which are formal theorems in F. (The statement that $p \vee \sim p$ is a formal theorem of F is then equivalently expressed by $c \in T$, where c is the Gödel number of $p \vee \sim p$ in F.) It is equally possible to indicate in this manner a recursive function $\phi(n, p)$ of two integral arguments whose value is the Gödel number of the formal proposition $W_n(\mathbf{p})$, *i.e.* the formal proposition which we get by 'saturating' the nth member of the sequence (1) with the numeral $\mathbf{p}$. After these preparations (which, in the actual proof, naturally take more time, space and effort, and give accordingly more insight into its nature) we can give the Gödel translation of (2), *i.e.* of

$$W_n(\mathbf{n}) \text{ is not a formal theorem of } F$$

as

$$(3) \quad \phi(n, n) \text{ non} \in T,$$

i.e. the value of $\phi(n, n)$ is a Gödel number which is not a member of the class T of the Gödel numbers of formal theorems of F.

Now (3) is a property of integers belonging to elementary arithmetic. It must, therefore, have a formalization in F, which must moreover be found in the sequence (1) of $W(.)$'s; for this sequence contains every formal property of numerals. Assume then that we have found that (3) is formalized by the qth member of the sequence, *i.e.* by $W_q(.)$.

The formal property $W_q(.)$ takes numerals as its arguments, among them also the numeral $\mathbf{q}$. We consider therefore the formal proposition $W_q(\mathbf{q})$, which is the undecidable formal proposition we wished to construct. The interpretation of $W_q(\mathbf{q})$ is: the integer q has the property formalized by $W_q(.)$, *i.e.* the arithmetical property: $\phi(n, n)$ non $\in T$; or equivalently: $W_q(\mathbf{q})$ is not a theorem of F.

If $W_q(\mathbf{q})$ were a formal theorem of F it would formalize a false arithmetical proposition. If $\sim W_q(\mathbf{q})$ were a formal theorem of F, then $W_q(\mathbf{q})$ would formalize a true arithmetical proposition. But then a false arithmetical proposition, namely $\sim W_q(\mathbf{q})$, would be formalized by a formal theorem of F. Since *ex hypothesi* F is a consistent formalization of elementary arithmetic, neither case can arise. $W_q(\mathbf{q})$ is undecidable and F is incomplete.

Variants of Gödel's result are obtained by varying the assumptions concerning F, and the methods of proof—all of which, however, allow the actual construction of the desired formal propositions.

The ideas and techniques, especially the arithmetization of metamathematics, which yield the incompleteness theorem and its variants

also yield Gödel's second theorem concerning formalisms of type F. If F is consistent and if f is a formalization of the statement that F is consistent, then f is not a formal theorem of F. Briefly, the consistency of F is not provable in F.

The second theorem implies the impossibility of proving the consistency of formalized classical mathematics by finitist methods. For in spite of a certain vagueness in demarcating the notion of finitist proofs, any such proof can be arithmetized and incorporated into F. To prove the consistency of F by finite or 'finitary' means is thus equivalent to proving the consistency of F in F—which by Gödel's second theorem is impossible. The original programme for a consistency proof has to be abandoned, or it has to be relaxed by redefining 'finitist proof'.

We may now make some brief remarks on the theory of recursive functions which was the main instrument of Gödel's proofs. (The remarks follow in the main R. Péter's treatment.) A recursive function is a function which takes non-negative integers as arguments, whose values are again non-negative integers and which is so defined that its values can be 'effectively' calculated. The meaning of 'effective calculation' or 'computability' itself is clarified in developing the theory. The definition of a recursive function does not depend on any assumption either that *there exists* among the totality of integers one which is specified only as having a certain property, or that *all members* of this totality have a certain property. The theory of recursive functions can thus be developed without the universal or existential quantifier. That a large part of arithmetic and logic can be developed in this manner was recognized by Skolem as early as 1923.[1] A main motive for developing this theory was the fact that by abandoning unrestricted quantification, the set-theoretical antinomies can be avoided—'existence of a set' becoming equivalent with computability of its members.[2]

One of the simplest recursive functions can serve as the definition of adding to a fixed non-negative integer a another integer n. Consider

$$\phi(0, a) = a$$
$$\phi(n+1, a) = \phi(n, a)+1.$$

The first equation, here, tells us the value of the addition of 0 to a. The second tells us how to find the value of the addition of $n+1$ to a when the value of the addition of n to a has already been found. We can

[1] *Begründung der elementaren Arithmetik durch die rekurrierende Denkweise ohne Anwendung scheinbarer Veränderlichen mit unendlichen Ausdehnungsbereich*, Videnskapsselskapets Skrifter 1, Math.—Naturw. Kl. 6, 1923.

[2] See also R. L. Goodstein, *Recursive Number Theory*, Amsterdam, 1957.

thus find the values of the function for $n=0$, $n=1$, $n=2$, $n=3$, etc. They are a, $a+1$, $a+2$, $a+3$, etc. If we write $\beta(a)$ for $a+1$, then $\beta(a)$ expresses the operation of forming the immediate successor of a non-negative integer. Our recursive function can then be written

$$\phi(0, a) = a$$
$$\phi(\beta(n), a) = \beta(\phi(n, a)).$$

In similar fashion we can define multiplication of a fixed positive integer a by a positive integer n. If $\phi\ (n, a)=n.a$, we have

$$\phi(0, a) = 0$$
$$\phi(n+1, a) = \phi(n, a)+a.$$

In the same way we can define exponentiation and other functions of arithmetic.

The form of these recursive functions is:

$$\phi(0) = K$$
$$\phi(n+1) = \beta(n, \phi(n))$$

Here ϕ is a function of one variable, β a function of two variables, and K a constant or function with no variable. The variable n for which successively 0, 1, 2, etc. are substituted is called the recursion variable. But the values of ϕ and, therefore, β may depend also on other variables which, however, do not enter into the process of recursion, during which they are treated as constants—different values being substituted for them either before or after the recursion, *i.e.* the calculation consisting in the successive substitutions for n. These other variables are, in accordance with the usual terminology of mathematics, called 'parameters'. A definition of the form

$$\phi(0, a_1, a_2, \ldots, a_r) = \alpha(a_1, a_2, \ldots, a_r)$$
$$\phi(n+1, a_1, \ldots, a_r) = \beta(n, a_1, a_2, \ldots, a_r, \phi(n_1, a_1, a_2, \ldots, a_r))$$

is called a *primitive recursion.*

If two functions are given we may form a new function by substituting one function for one variable in the other, *e.g.* from $\phi(x, y, z)$ and $\psi(u)$ we can get by substitution $\phi(\psi(u), y, z)$, $\phi(x, y, \psi(u))$, $\psi(\phi(x, y, z))$, etc. Primitive recursions and substitutions yield a large and important class of functions called *primitive recursive functions* characterized[1] as those functions whose arguments and values are non-negative integers and which starting from 0 and $n+1$ are defined by a finite number of substitutions and primitive recursions.

1 Péter, *op. cit.*, p. 32.

In his proofs Gödel used only primitive recursive functions. To see how formal properties can be arithmetized we consider the definition of recursive relations. A relation $B(a_1, \ldots, a_r)$ is primitive recursive if there exists a primitive recursive function $\beta(a_1, \ldots, a_r)$, such that it equals 0, if and only if the relation B holds between $a_1, \ldots, a_r$. If $W(a)$ is a property, it is primitive recursive provided there exists a primitive recursive function which equals 0, if and only if a has W. The complementary relation $B'(a_1, \ldots, a_r)$ of $B(a_1, \ldots, a_r)$ is also primitive recursive and holds only if $\beta(a_1, \ldots, a_r) \neq 0$. In this way the notions 'being a complement', 'being a conjunction' and more complex notions of methamathematics including 'being a formal theorem of F' become expressible as primitive recursive functions, and relations between Gödel numbers.

It follows from a theorem of Turing (1937) that the computation of any primitive recursive function can be left to a machine. In fact he showed that a wider class of functions, the so-called *general recursive functions*, are computable by Turing-machines. Before this was shown, Church had proposed that the rather vague notion of effective computability should be analysed as solvability by general recursive functions. This proposal was justified by Church's own results and by other results which, though at first sight unconnected, all proved equivalent. As regards this problem of identifying effective computability with solvability by general recursive functions, expert opinion is no longer undivided.[1] On this question nothing can profitably be said in the present context by the present author. The theory is developing into a new branch of pure mathematics whose relevance to the problems raised by Hilbert is merely one of its important aspects, and perhaps no longer the most important.[2]

1 See Péter, *op. cit.*, §§ 20–22.
2 See Myhill, *op. cit.*, p. 136.

statement of the logical implication, is therefore of little importance. But this is far from true. There are indeed trivial logical implications, *e.g.*: 'Construction x possesses C' logically implies 'construction x conforms to r which prescribes that x should possess C'. But there are others which are not trivial, *e.g.*: 'Construction x possesses C' logically implies 'construction x conforms to r which prescribes that x should possess D'—where the question whether the possession of C by a construction logically implies possession of D turns on the validity of a complicated deduction from 'x has C' to 'x has D', employing *certain admissible principles of inference*. (So-called constructive proofs are on the whole more, and not less, complex than non-constructive.)

The situation then is this: *Prima facie* the formalist does not rely on logical principles but merely on perceptual statements such as 'a given construction of perceptual objects with perceptual characteristics C *ipso facto* possesses characteristics D'. To this the qualification has to be added that the construction has to be correct. The proposition, however, that a construction is correct, *i.e.* that it conforms to an adopted rule, is no longer perceptual but involves a logical implication or an inference the validity of which depends on logical principles. These principles must be adopted before we can decide the correctness of a construction.

In deducing statements about constructions from other such statements one employs fewer logical principles than in classical mathematics. But these principles though suggested by constructions—*e.g.* of strokes and stroke-expressions—are not perceptual judgements. Only if we were to assume that the medium in which we make our constructions is of a special kind so that they can be immediately described by general and necessary propositions without raising the question as to whether a particular construction is correct or incorrect, could we dispense with logical principles. The intuitionists are aware of the fact that ordinary perception is not the medium for such constructions and claim therefore that the general principles of reasoning in mathematics are validated not by constructions in ordinary perception, but in a *sui generis* intuition.

The formalist logic is a minimal *logic*—or better the minimum logic needed for metamathematical reasoning. It is *not* a system of statements describing perceptual features of various constructions. This conclusion is independent of the point urged earlier that mathematical concepts, being exact, differ from perceptual characteristics which are inexact or admit of border-line cases.

VI

MATHEMATICS AS THE ACTIVITY OF INTUITIVE CONSTRUCTIONS: EXPOSITION

It is one of the fundamental convictions of the intuitionist school, whose doctrine is the subject of this chapter, that mathematics—if properly understood and practised—is a wholly autonomous and self-sufficient activity. Its methods and insights are regarded as being neither capable of nor in need of the guarantees which the logicists and the formalists each profess to provide. According to the intuitionists the impression that mathematics needs the support of an extended logic or of rigorous formalization has arisen only where mathematics has not been properly pursued.

Logicism and formalism have treated the antinomies of classical mathematics as a malady capable of a cure which would leave classical mathematics substantially intact. The intuitionists consider the antinomies as merely a symptom that mathematics has in many of its branches not been true to itself. Logicism and formalism tried so to reconstruct the building or to secure its foundation that the mathematical work could go on in the upper storeys without much disturbance. The intuitionists attempt to build a new mathematics at all levels by what they regard as the truly mathematical methods.

Both formalists and intuitionists and in particular their modern leaders, Hilbert and Brouwer, acknowledge, as we saw, the influence of Kant's philosophy of mathematics and reject the Leibnizian tradition according to which all mathematical propositions are analytic in the sense that their truth can be demonstrated merely by an application of the principles of logic. Both Brouwer and Hilbert regard mathematical theories as synthetic, in a sense of the term which is based on a mutually exclusive and jointly exhaustive classification of propositions into analytic and synthetic.

Yet Brouwer's conception of the synthetic character of mathematics is very different from Hilbert's, and nearer to Kant. According to Kant, it will be remembered, the axioms and theorems of arithmetic and geometry are synthetic *a priori*—*i.e.* they are descriptive of the pure intuition of space and time and of constructions in it. Brouwer accepts without reservation Kant's doctrine of the pure intuition of time—time apart from any perceptual content—and regards this as the substratum of mathematics. Like Kant he regards such intuition as independent of sense-perception, including in sense-perception in particular the perception of such symbols and operations upon them, as are the strokes and stroke-operations of Hilbert which, together with other marks and operations, constitute the subject-matter of formalist metamathematics.

The subject-matter of metamathematics is perceptual objects and constructions, of so simple and transparent a structure that we can be certain of the truth of the synthetic empirical judgements which are descriptive of them. The subject-matter of intuitionist mathematics, on the other hand, is intuited non-perceptual objects and constructions which are introspectively self-evident. Brouwer does appeal, not indeed to the inspection of external objects, but to 'close introspection'.[1] The distinction between perceptual and intuitive constructions is of some philosophical importance since we can with more plausibility claim that the latter can be apprehended as universal and necessary without the application of the notion of correctness and thus without employing logical principles. (This point was discussed at the end of the last chapter.)

In spite of the differences between the inspectible data of metamathematics and the introspectible data of intuitionist mathematics, they have much in common. The most important common feature is that a completed infinite totality can neither be inspected nor introspected. In other words neither metamathematics nor intuitionist mathematics can admit statements about actual infinities, only about potential ones.

For a better understanding of intuitionism it is worth asking whether it would reduce to formalist metamathematics if one were to ignore the difference of the substrata, real or alleged, between the two activities. As one would expect, both would employ on the whole the same finite methods—methods such as were described earlier, in our exposition of formalism. However, the formalist would not use them beyond the point at which, having established the consistency of a formal system, he could start using it. For the intuitionist, on the other hand, since he cannot find, or hope for, refuge in a formal system, the incentive to use finite methods even in spite of increasing complexity

[1] See, *e.g.*, 'Historical Background, Principles and Methods of Intuitionism' in *South African Journal of Science*, Oct.–Nov., 1952, p. 142, footnote.

and difficulty is much greater. Finitist intuitionist mathematics has in fact been developed much further than finitist metamathematics.

Contained in the first chapter of Heyting's *Intuitionism—An Introduction*[1] is a disputation in which one disputant called 'Int' addresses another called 'Form' in the following words: '... you also use meaningful reasoning in what Hilbert called metamathematics, but your purpose is to separate these reasonings from purely formal mathematics, and to confine yourselves to the most simple reasonings possible. We, on the contrary, are interested not in the formal side of mathematics, but exactly in that type of reasoning which appears in metamathematics; we try to develop it to its farthest consequences. This preference arises from the conviction that we find there one of the most fundamental faculties of the human mind.'

For a brief exposition of intuitionism, it will be well first to explain its conception of pure mathematics and the programme based upon this conception; and then to give some examples of the intuitionist method at work especially in dealing with the notion of potential infinity. As to the problem of applied mathematics, the intuitionists have shown even less interest in it than either the logicists or the formalists.

1. *The programme*

Brouwer in one of his more recent English papers[2] describes the situation of the philosophy of mathematics as formulated by the old and new formalists and pre-intuitionists, as he calls those thinkers who in some ways anticipated him, in particular Poincaré, Borel and Lebesgue.

As it presented itself to Brouwer, the situation was this: mathematics, as practised by the pre-intuitionists and formalists, consisted of two separate parts—an autonomous mathematics and a mathematics dependent for its trustworthiness on language and logic. For the autonomous mathematics, 'exact existence, absolute reliability, and non-contradictority were universally acknowledged, independently of language and without proof'. It embraced 'the elementary theory of natural numbers, the principle of complete induction, and more or less considerable parts of algebra and theory of numbers'. The non-autonomous mathematics embraced the theory of the continuum of real numbers. For this a proof of non-contradictory existence was lacking and, as was more or less generally agreed, was needed.

The fundamental theses of the intuitionist philosophy of mathematics are clearly formulated by Brouwer. He describes them as 'two

[1] Amsterdam, 1956. [2] *Op. cit.*

acts' by which intuitionism 'intervened' in the situation created by its predecessors and the formalists. The acts could also be called 'insights' —a term used frequently by Brouwer. It is best to quote here *verbatim* and at length from his paper.[1]

'The *first act of intuitionism* completely separates mathematics from mathematical language, in particular from the phenomena of language which are described by theoretical logic, and recognizes that intuitionist mathematics is an essentially languageless activity of the mind having its origin in the perception of *a move of time*, *i.e.* of the falling apart of a life moment into two distinct things, one of which gives way to the other, but is retained by memory. If the two-ity thus born is divested of all quality, there remains the *empty form of the common substratum of all two-ities*. It is this common substratum, this empty form, which is the *basic intuition of mathematics*.'

The doctrine of this and similar passages in Brouwer's writings is substantially that of *The Critique of Pure Reason*—the main difference being that according to Brouwer Kant's intuition of space and the (Euclidean) constructions in it are not part of the intuition which underlies mathematics (see chapter I). Mathematics according to Kant and Brouwer presupposes an intuition which is different on the one hand from sense-perception, of which it is the invariant form, and on the other hand from the apprehension of logical connections between concepts or statements. Just as the experience of, say, climbing a mountain is not to be confused with its linguistic description and communication to others, so the experience of mathematical intuitions and constructions must not be confused with *its* linguistic description and communication (although such linguistic formulation may be of great help to the climber or mathematician and to those who wish to follow his example).

In the same sense in which climbing is not dependent on language, the mathematical activity, with its intuitive insights and constructions, is languageless. According to Brouwer the principles of classical logic are linguistic rules in that those who 'linguistically follow' them may but need not 'be guided by experience'. This means that the rules of classical logic are employed in description and communication but not in the activity itself of constructing; as they are not employed, except as inessential aids, in the activity of mountain climbing. Mathematics is essentially independent, in this sense, not only of language but also of logic.

We must thus according to Brouwer distinguish sharply between two different activities: on the one hand the mathematical construction; and on the other the linguistic activity, *i.e.* all statements of the results of construction and all application of logical principles of reasoning to these statements. In view of the fundamental difference between the two it makes very good sense to ask whether the logico-linguistic representation is always adequate to the construction; in particular whether the representation does not outrun the construction. That language sometimes outruns its subject-matter is a familiar fact. Usually the danger of its doing so had been regarded as very great in the case of philosophical language and very small in mathematical. But according to Brouwer there is much of it in mathematics too. Thus in the case of all mathematicians who employ the law of excluded middle in reasoning about infinite systems of mathematical objects, language is outrunning and misrepresenting the mathematical reality.

[1] *Op. cit.*

It is again convenient here to quote part of Brouwer's own clear formulation, *verbatim*: 'Suppose that an intuitionist mathematical construction has been carefully described by means of words, and then, the introspective character of the mathematical construction being ignored for a moment, its linguistic description is considered by itself and submitted to a linguistic application of a principle of classical logic. Is it then always possible to perform a languageless mathematical construction finding its expression in the logico-linguistic figure in question?

'After a careful examination one answers the question in the *affirmative* (if one allows for the inevitable inadequacy of language as a mode of description) as far as the principles of contradiction and syllogism are concerned; but in the *negative* (except in special cases) with regard to the principle of excluded third, so that the latter principle, as an instrument for discovering new mathematical truths must be rejected.'

We shall presently consider some mathematical constructions, the examination of which led Brouwer and his followers to reject the law of excluded middle and certain other principles of reasoning for infinite sets of objects. The same rejection we have found in the original limitation of concrete metamathematics by the formalists, who however admit the *formal* application of these principles within the formalized theories of classical mathematics. This way of saving classical mathematics is not open to the intuitionists since it is in conflict with their conception of mathematics as languageless construction.

The limitation of mathematics to the finite methods of formalist metamathematics—whether these be applied to objects of ordinary perception or of intuition—would be a crippling blow to the structure of classical mathematics. But, and this is the second insight of intuitionism, there is a mathematics of the potential infinite, which while avoiding the perceptually and intuitively empty notion of actual, pre-existing infinite totalities, constitutes a firm, intuitive foundation of a new analysis and opens a field of development which 'in several places far exceeds the frontiers of classical mathematics. . . .'

This field of a new autonomous mathematics of the potential infinite is opened by '*the second act of intuitionism* which recognizes the possibility of generating new mathematical entities: first in the form of *infinitely proceeding sequences* $p_1, p_2, \ldots$ whose terms are *chosen more or less freely from mathematical entities previously acquired*, in such a way that the freedom of choice existing perhaps for the first element p_1 may be subjected to a lasting restriction at some following p_r, and again and again to sharper lasting restrictions or even abolition at further subsequent p_r's, while all these restricting interventions, as well as the choice of the p_r's themselves, at any stage may be made to depend on future mathematical experiences of the creating subject; secondly in the form of mathematical species, *i.e. properties supposable for mathematical entities previously acquired*, and satisfying the condition that, if they hold for a certain mathematical entity, they also hold for all mathematical entities which have been defined to be equal to it, relations of equality having to be symmetric, reflexive and transitive; mathematical entities previously acquired for which the property holds are called elements of the species.'

As we shall see in more detail, intuitionist mathematics differs greatly from classical, whether as practised 'naively', as supported by a logicist substructure, or as safeguarded by formalization. Its programme is formulated simply enough, even if its execution involves difficult, or at least very unfamiliar, procedures and concepts and even if the nature of intuitionist construction may not be *prima facie* clear to the non-intuitionist. It is to make mathematical constructions in the medium of pure intuition and then to communicate them to others as clearly as possible so that they can repeat them.

Not every mathematical construction is of equal interest and importance. But there is never much doubt as to which constructions are important, since the motives for finding constructions arise, as in non-intuitionist mathematics, from the curiosity of pure mathematicians and the needs of those who employ mathematics for other purposes. The programme of the intuitionist is to practise intuitionist mathematics, *i.e.* to create or construct mathematical objects since only constructed objects have mathematical existence. It is not to

show the legitimacy of these constructions by either logic or formalization. For they are legitimate in themselves, they are self-validating.

2. *Intuitionist mathematics*

To the intuitionist mathematics is the construction of entities in pure intuition, not the promise of such a construction or the enquiry whether it is logically possible.

The classical mathematician, the logicist and formalist allow as legitimate statements to the effect that 'there exists' a number with certain properties although so far no method for constructing this number is known. Such statements—pure existence-theorems—the intuitionist does not allow into his mathematics. He is consequently quite unworried if one finds it odd that a mathematical theorem showing the actual constructibility of some number should only become true after it has been (by his methods) proven. There is no oddity in it to him nor should there be to anybody who understands the intuitionist position, for which 'mathematical existence' means the same as 'actual constructibility'. What is to count as actual constructibility is indeed never quite precisely defined in general terms, but—the intuitionist asserts—it is made clear in practice.

In explaining some of the elementary ideas of intuitionist mathematics—which is all that can be attempted here—I shall be following closely the exposition of Heyting's *Intuitionism—An Introduction.* Heyting leads his reader very much further by explaining the intuitionist approach to special topics of advanced mathematics, such as the theories of algebraic fields and the theory of measure and integration.

Intuitionist mathematics starts, then, with the notion of an abstract entity and of the sequence of such entities. It starts in other words with the sequence of natural numbers. There is no need to formulate a deductive system of elementary arithmetic—for such formulation would be adequate only if it formulated what is self-evident without it. It confers neither self-evidence nor security. It only, at best, reflects it linguistically. For the intuitionist Peano's axioms (see Appendix A) merely formulate self-evident results of the process of generating the natural numbers.

The difference between classical mathematics (equally in its 'naive' and in its logicized or formalized form) and the intuitionist shows itself very clearly when it comes to defining real numbers. In classical mathematics the notion of a real number can be defined in terms of a so-called Cauchy sequence of rational numbers. A classical Cauchy sequence is defined as follows: $a_1, a_2, a_3, \ldots$ or, briefly, $\{a_n\}$ or a, where every term is a rational number, is a Cauchy sequence if for every natural number k (and therefore for every fraction, however small, $1/k$) *there exists* a natural number $n=n(k)$ such that, for every natural number $p, |a_{n+p}-a_n|<1/k$.

Roughly speaking this means that if we consider any fraction $1/k$ there always *exists* a term, say the nth, such that on subtraction of it from any of its successors, the absolute value of the difference is smaller than $1/k$. (The absolute value of a non-negative number is this number itself, the absolute value of a negative number is that number which results from changing its minus sign into a plus sign.) The absolute value of the difference of pairs of rational numbers thus becomes smaller as we choose them from 'later' members of the sequence.

The definition of the notion of an intuitionist Cauchy sequence can be formulated in almost the same words. The only difference consists in replacing the phrase 'there exists' by the phrase '*there can effectively be found*' or 'there can effectively be constructed'. It is worthwhile to attend to the difference of meaning between these two phrases since it leads to the core of intuitionist mathematics.

Heyting brings it out by means of the following example. Consider the following definitions of classical Cauchy sequences. The first sequence (a_n) is: 2/1, 2/2, 2/3, . . . or $\{2/n\}$. In this series each component can be effectively constructed, *e.g.* the thousandth member is 2/1000. Consider now a second sequence $\{b_n\}$ defined as follows: if the nth digit after the decimal point in the decimal expansion of $\pi=3{\cdot}1415\ldots$ is the 9 of the first sequence 0123456789 in this expansion, $b_n=1$; in every other case $b_n=2/n=a_n$.

Since the sequence $\{b_n\}$ differs from $\{a_n\}$ in at most one term, it is a Cauchy sequence in the classical sense. But since we do not know of any construction which would show whether or not the critical term occurs in $\{b_n\}$—whether a sequence 0123456789 occurs in π—we have no right to assert that $\{b_n\}$ is a Cauchy sequence in the intuitionist sense. An intuitionist Cauchy sequence, which like $\{a_n\}$ must be constructible, is also called a '(real) number generator'. It is clear that the intuitionist cannot allow the idea of all number-generators into his mathematics—even if it could be shown to lead to no inconsistency in a given formal system.

The identification of the existence with the actual constructibility of number-generators must lead to a thorough modification of the classical notion of the equality and difference of two real numbers. Heyting defines two equality-relations between real number generators, namely 'identity' and (the more important relation of) 'coincidence'. Two number generators $\{a_n\}$ and $\{b_n\}$ are identical—in symbols $a\equiv b$—if for every n, $a_n=b_n$. They coincide—in symbols $a=b$—if for every k we can find an integer $n=n(k)$ such that $|a_{n+p}-b_{n+p}|<1/k$ for every p.

That we cannot find the required $n=n(k)$ for every k, does not entitle us to say that a and b do not coincide: for an intuitionist negation, just as an intuitionist affirmation, must be based on a construction—not on the absence of a construction. Only if $a=b$ is contradictory, *i.e.* '*only if we can effect a construction which deduces a contradiction from the supposition that* $a=b$', are we entitled to assert that a and b do not coincide, *i.e.* $a\neq b$.

It might be thought that proving in turn that $a\neq b$ is contradictory (impossible) is *ipso facto* a proof that $a=b$. As a matter of fact it is a theorem of intuitionist mathematics that the contradictoriness (impossibility) of $a\neq b$ does amount to $a=b$.[1] But—and this is a very mportant feature of intuitionist mathematics—'a proof of the impossibility of the impossibility of a property is not in every case a proof of the property itself'. In other words if we write '$\neg$' for 'is contradictory' or 'is impossible'—in the sense in which this notion must be backed by constructive proof—and 'p' for any mathematical affirmation (which is not the affirmation of an impossibility!), then $\neg\neg p$ does not as in classical logic in general imply p. The following example, which shows that this principle is not valid in intuitionist logic, has been given by Brouwer and is also found in Heyting's recent book.

'I write the decimal expansion of π and under it the decimal fraction $\rho=0{\cdot}333\ldots$, which I break off as soon as a sequence of digits 0123456789 has appeared in π. If the 9 of the first sequence 0123456789 in π is the kth digit after the decimal point, $\rho=10^k-1/3.10^k$. Now suppose that ρ could not be rational; then $\rho=10^k-1/3.10^k$ would be impossible and no sequence could appear in π; but then $\rho=\frac{1}{3}$, which is also impossible. The assumption that ρ cannot be rational has led to a contradiction; yet we have no right to assert that ρ is rational, for this would mean that we could calculate integers p and q so that $\rho=\dfrac{p}{q}$; this evidently requires that we can either indicate a sequence 0123456789 in π or demonstrate that no such sequence can appear.'

If two number-generators do not coincide (*i.e.* if $a\neq b$) a stronger inequality relation may hold between them. This is the relation of apartness. That 'a lies apart from b'—in symbols $a \# b$—means that 'n and k can be found such that $|a_{n+p}-b_{n+p}|\geq 1/k$ for every p'. It is evident that whereas $a \# b$ entails in general that $a\neq b$, the converse is not true. To the classical mathematician a mathematics which

[1] For the proof see Heyting, *op. cit.*, p. 17.

distinguishes between non-coincidence and apartness in this way would very likely seem unnecessarily complicated and prolix. But this prolixity may be due to mere unfamiliarity. Just as, in philosophy, apparently lucid writers are sometimes confused thinkers, so classical mathematicians may for all their apparent lucidity be fundamentally unclear. (Indeed no antinomies have so far been discovered in intuitionist mathematics.)

The fundamental operations with real number-generators can be explained in a perfectly straightforward manner. But it must be noted that a real number-generator is not a real number. In classical mathematics one might, having defined a certain number-generator, proceed to define a corresponding real number as '*the set of all* number-generators which coincide with the given number-generator'. But the phrase 'the set of all . . .' does not here refer to a constructible entity and has to be given a new intuitionist content. Indeed to the classical notion of a set there correspond two intuitionist notions, that of a spread and that of a species—a spread being defined by a common mode of generating its (constructible) elements, and a species being defined by a characteristic property which can be assigned to mathematical entities, which have been or could have been constructed *before* defining the species.

In defining a spread the first step consists in conceiving the very general notion of an *infinitely proceeding sequence*, *i.e.* a sequence which can be continued *ad infinitum* no matter how the components of the sequence are determined, whether by law, free choice or what you will. Of such sequences the above defined Cauchy sequences or number-generators are special cases. The intuition of them, and the insight which reveals their mathematical usefulness is—as we have seen (section 1)—claimed to be one of the basic 'acts' of intuitionism.

To the intuitionist the continuum of real numbers is not the completed totality of dimensionless points on a line, but rather the 'possibility of a gradual determination of points'—points describable in terms of the notions of infinitely proceeding sequence and of spread. A spread M is defined by two laws which Heyting[1] whose definitions I am almost literally repeating calls '*spread-law* Λ_M, and '*complementary law* Γ_M'.

A spread law is a rule Λ which divides the finite sequences of natural numbers into admissible and inadmissible sequences according to the following four prescriptions, namely

(i) By the rule Λ it can be decided for every natural number k. whether it is a one-member admissible sequence or not.

[1] *Op. cit.*, pp. 34 ff.

(A one-member sequence consists of one natural number, and an n-member sequence of n such numbers. The sequence a_1, a_2, a_3 is called an immediate descendant of the sequence a_1, a_2 and a_1, a_2 an immediate ascendant of a_1, a_2, a_3. And the same terminology is used in the general case of $a_1, a_2, \ldots, a_n, a_{n+1}$ and $a_1, a_2, \ldots, a_n$.)

(ii) Every admissible sequence $a_1, a_2, \ldots, a_n, a_{n+1}$ is an immediate descendant of an admissible sequence $a_1, a_2, \ldots, a_n$.

(iii) If an admissible sequence $a_1, \ldots, a_n$ is given, the rule Λ allows us to decide for every natural number k whether $a_1, \ldots, a_n, k$ is an admissible sequence or not.

(iv) To any admissible sequence $a_1, \ldots, a_n$ at least one natural number k can be found such that $a_1, \ldots, a_n, k$ is an admissible sequence.

The *complementary law* Γ_M of a spread M assigns a definite mathematical entity to any finite sequence which is admissible according to the spread law of M.

Consider now an infinitely proceeding sequence, and subject it to the restriction that, for every n, $a_1, a_2, \ldots, a_n$ must be an admissible sequence in accordance with a spread law Λ_M. Such an infinitely proceeding sequence—briefly *ips*—is no longer a free *ips*; but an admissible *ips* (admissible by Λ_M). The complementary law assigns to each admissible sequence a_1; a_1, a_2; a_1, a_2, a_3; . . . a mathematical entity—it assigns, say, b_1 to a_1; b_2 to a_1, a_2; . . . ; b_n to $a_1, a_2, \ldots, a_n$. Each of these infinitely proceeding sequences of assigned entities such as $b_1, b_2, b_3, \ldots, b_n$ is called *an element of the spread M*—with b_n as its nth component. Two elements of spreads are equal if their nth components are equal; and two spreads are equal if to every element of one of them, an equal element of the other can be found.

If we understand the notion of spread we can understand the intuitionist notion of the continuum as a possibility of certain actual constructions. Let us—closely following Heyting's exposition as before—consider an enumeration of rational numbers: $r_1, r_2, \ldots$ (*i.e.* we assign to every natural number 1, 2, 3, . . . —after its construction—a rational number, in a manner which guarantees that no rational number is left out).

We now define the spread M, which represents the intuitionist continuum, as follows: its *spread-law* Λ_M determines that every natural number shall form an admissible one-member sequence; and if $a_1, \ldots, a_n$ is an admissible sequence, then $a_1, a_2, \ldots, a_n, a_{n+1}$ is an admissible sequence if and only if $|r_{a_n} - r_{a_{n+1}}| < \frac{1}{2^n}$ ($r_{a_n}, r_{a_{n+1}}$ are the rational numbers which, in our enumeration of rational numbers, have the indices a_n and a_{n+1} respectively). The *complementary law* Γ_M assigns to every admissible sequence the rational number r_{a_n}.

Γ_M thus generates infinitely proceeding sequences of rational numbers. Every such *ips* is an element of M and a real number-generator. Indeed, to any real number-generator c, an element m of M can be found, such that $c = m$. It is worth emphasizing again that nowhere in all this chain of definitions have we assumed an actually given infinity or relinquished the principle that only constructible entities exist.

Just as the notion of a spread does not allow us to assume a completed infinite totality of mathematical entities—being, as it were, a set always in the making but never made—so the notion of a species (a mathematical property) does not allow us to assume actually infinite sets. Obviously the exclusion of 'infinite totality' from mathematics implies the prohibition of properties of infinite totalities.

A *species* is a property which mathematical entities can be supposed to possess. After a species S has been defined, any mathematical entity which has been or might have been defined *before* S was defined, and satisfies the condition S, is a member of the species S.[1] For example, the property of coinciding with a real number-generator is the species 'real number'.

It is important to emphasize with Heyting that the vicious-circle-antinomy (of the set of all sets which do not contain themselves as elements) cannot arise in intuitionist mathematics. For the intuitionist so defines 'species' that only entities which are definable independently of the definition of any given species can be members of that species.

The identification of intuitionist existence with actual constructibility also accounts for fundamental differences between the classical theory of sets or classes on the one hand and the intuitionist theory of species on the other. Thus whereas '$a \in S$' means that a is an element of S—if a is definable independently of S—'$a \notin S$' means that it is impossible for a to be a member of S, in other words that the assumption $a \in S$ leads to a contradiction. Again if T is a subspecies of S (every member of T being a member of S) $S - T$ is not the species of those members of S which are not members of T but of those members of S *which cannot possibly* be members of T. In classical set theory '$T \cup (S - T)$' means the class of all entities which are members of T or of $S - T$ or both and this class is equal to S. In view of the stronger, constructive, definition of $S - T$, the species $T \cup (S - T)$ may but need not be equal to S. (In the former case T is called a detachable species of S.)

[1] Heyting, *op. cit.*, p. 37.

It is clear that the intuitionist theory of cardinal numbers will differ greatly from the classical theory. Thus the requirement of constructibility and the intuitionist conception of negation, as requiring together to be backed up by the actual construction of a contradiction, leads to the denial that a species which is not finite is therefore infinite. (An 'infinite species' is one which has denumerably infinite subspecies, 'denumerable' meaning constructible one-one correspondence with the species of natural numbers.)

3. *Intuitionist logic*

The intuitionist logic is a *post factum* record of the principles of reasoning which have been employed in mathematical constructions. Whereas the logicist formulates these principles in order to abide by them, the intuitionist admits that future mathematical constructions—a notion which to him is unproblematic—might embody principles so far unformulated and unforeseen. Whereas the logicist justifies his mathematics by an appeal to logic, the intuitionist justifies his logic by an appeal to mathematical constructions.

The intuitionist is not concerned with logic in general but only with the logic of mathematics, *i.e.* with 'mathematical logic' in the sense not of a mathematized general logic, but of a formulation of the principles employed in the activity of mathematical construction. Although intuitionists have produced formal systems, which can be made and have been made objects of metamathematical investigation, these systems are regarded by them as linguistic by-products of the 'essentially languageless' activity of mathematics; and as being mainly of pedagogical value.

From a purely formal point of view—that is to say apart from any intended interpretation of the symbols, formulae and transformation rules—intuitionist logic appears as a subsystem of the classical logic. This is particularly obvious in the case of certain formal systems which have been constructed for the purpose *inter alia* of separating intuitionist principles and rules of inference from the wider class of principles and rules which have been adopted by classical and non-intuitionist logicians.[1]

Every intuitionist proposition p, whether or not the (intuitionist) negation occurs in it, is the record of a construction. As Heyting in effect puts it, it says: 'I have effected a construction A in my mind.' An intuitionist negation $\neg p$ is also the record of a construction, and

[1] See for example the formal system of Kleene's *Metamathematics*, §§ 19–23, where intuitionistically valid principles, rules of inference and proofs are clearly distinguished from those that are only classically valid.

thus really an affirmation. It says: 'I have effected in my mind a construction B which deduces a contradiction from the supposition that the construction A were brought to an end.' The proposition 'I have not effected a construction . . .' is of no interest to either the intuitionist or the classical mathematician. But whereas the classical mathematician admits 'there exists a mathematical construction . . .', even if nobody has so far been able to effect it, such a proposition could from the intuitionist point of view only be an empty promise—perhaps an incitement to research, but not a piece of mathematics.

Considering the intuitionist meaning of p and $\neg p$ we can see at once that if, with the intuitionist, we are to regard mathematics as the science of intuitive constructions then, taking '$\neg$' in its required meaning, the proposition (p or $\neg p$) is not a universally valid principle of the logic of mathematics. By the meaning of the various intuitionist symbols and by the examples of the previous section we see that *if* we adopt the conception and programme of intuitionist mathematics there is nothing at all strange in intuitionist logic. In what follows we shall briefly consider the vocabulary and some theorems of intuitionist logic without attempting a rigid systematization such as would be, in any case, foreign to its spirit.

$p \wedge q$ (p and q) can be asserted if, and only if, both can be asserted; $p \vee q$ (p or q) if, and only if, p or q or both can be asserted. The meaning of '$\neg p$' has been explained already. It is worth noting here that even the strong negation of intuitionist logic has been rejected by some intuitionists as too weak—the reason being that proof of the impossibility of a construction does not seem to them to amount to an actual construction which according to a more radical programme, is alone mathematical. The radical intuitionist requires a completely negationless mathematics and logic. He seems to agree with Goethe's Faust that 'a perfect contradiction remains as mysterious to wise men as it does to fools'.[1]

The intuitionist implication $p \rightarrow q$ is not a truth-function. Heyting interprets it thus: $p \rightarrow q$ can be asserted if, and only if, we possess a construction W which joined to any construction proving p (supposing that the latter be effected) would automatically effect a construction proving q. Or, as he puts it more concisely, a proof of p, together with W, would form a proof of q. We may now put down some intuitionist theorems and non-theorems placing the usual assertion sign $\vdash$ in front of the former and $*$ in front of the latter. Reflection and the meaning of the symbols should ultimately justify the distinction.

[1] For details of this view and references see Heyting, *op. cit.*

(i) $\vdash p \rightarrow \neg\neg p$
$* \neg\neg p \rightarrow p$

(ii) $\vdash (p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)$
$* (\neg q \rightarrow \neg p) \rightarrow (p \rightarrow q)$

(iii) $\vdash \neg p \rightarrow \neg\neg\neg p$
$\vdash \neg\neg\neg p \rightarrow \neg p$

(In other words, the assertion of the impossibility of p is equivalent to the assertion of the impossibility *of the impossibility of the impossibility of p*. Three intuitionist negations can always be contracted into one.)

(iv) $* p \vee \neg p$
$\vdash \neg\neg (p \vee \neg p)$

(v) $\vdash \neg (p \vee q) \rightarrow \neg p \wedge \neg q$
$* \neg (p \wedge q) \rightarrow \neg p \vee \neg q$

In Heyting's formal system $q \rightarrow (p \rightarrow q)$ is an axiom and he gives reasons[1] why he considers it to be intuitively clear. We may observe at this point that at least one intuitionist or near-intuitionist logician denies intuitive clarity to this proposition. Such disagreement about the nature of mathematical intuition is philosophically important and will occupy us in the next chapter.

In developing the usual theory of quantification it is, we have seen, a useful heuristic consideration to regard the universal quantifier as *a kind of* conjunction—and the existential quantifier as *a kind of* alternation-sign. If the members of the conjunction or alternation are finite in number the quantifiers are merely abbreviative devices for the formulation of truth-functional propositions. If the transition to infinite conjunctions and alternations is made, the analogy between universally or existentially quantified propositions on the one hand and conjunctions or alternations on the other, though helpful in some cases, may be very misleading. An 'infinite conjunction' or an 'infinite alternation' are even in the usual theory quite different from a finite conjunction or finite alternation. (See p. 48.)

In developing the intuitionist theory of quantification the heuristic derivation of the principles of quantification from the propositional calculus must be used with even greater care. It must be constantly checked against the principle that mathematical existence is from the intuitionist point of view actual constructibility; and against the particular notions of infinitely proceeding sequences and of spreads, which two notions embody the intuitionist conception of potential

[1] *Op. cit.* p. 102.

infinity. We may again set down the meaning of some of the intuitionist key terms and some theorems and non-theorems.

If $P(x)$ is a predicate of one variable ranging over a certain mathematical species α then

'$(x)\,P(x)$' means that we possess a general method of construction which, if any element a of α is chosen, yields the construction $P(a)$, and

'$(\exists x)\,P(x)$' means that for some particular element a of α $P(a)$ has actually been constructed. By these definitions the following formulae show themselves as theorems or non-theorems respectively.

(vi) $\vdash (x)\,P(x) \rightarrow \neg\,(\exists x) \neg\,P(x)$

$* \neg\,(\exists x) \neg\,P(x) \rightarrow (x)\,P(x)$

(vii) $\vdash (\exists x)\,P(x) \rightarrow \neg\,(x) \neg\,P(x)$

$* \neg\,(x) \neg\,P(x) \rightarrow (\exists x)\,P(x)$

(viii) $\vdash (\exists x) \neg\,P(x) \rightarrow \neg\,(x)\,P(x)$

$* \neg\,(x)\,P(x) \rightarrow (\exists x) \neg\,P(x)$

(ix) $\vdash (x) \neg\neg\,P(x) \rightarrow \neg\,(\exists x) \neg\,P(x)$

(x) $\vdash \neg\,(\exists x) \neg\,P(x) \rightarrow (x) \neg\neg\,P(x)$

These sections on intuitionist logic and intuitionist mathematics are of course schematic and incomplete. They can at best convey some of the spirit of intuitionist mathematics. Those interested in making closer contact with its substance are advised to master Heyting's work and refer to its (extensive) bibliography. As to the relation between formalism and intuitionism from the point of view of logic and mathematics readers will find most of the available results in Kleene's *Metamathematics*.

VII

MATHEMATICS AS THE ACTIVITY OF INTUITIVE CONSTRUCTIONS: CRITICISM

In accordance with the plan of this essay we must now examine the intuitionist philosophy of pure and of applied mathematics, and its distinctive theory of mathematical infinity. To the problem of the nature of applied mathematics modern intuitionists have, however, given even less attention than have either the logicists or the formalists. Indeed their philosophy of applied mathematics is something we have largely to conjecture—the basis of the conjecture being chiefly, (i) certain remarks of Brouwer and Weyl (of Brouwer on the affinity of his philosophy to Kant's, of Weyl on the relation between intuitionist mathematics and the natural sciences) and (ii) the reasonable presumption that the intuitionist philosophy of applied mathematics and its philosophy of pure mathematics are consistent with each other. These theories will be treated in the order indicated.

A concluding section will note some indications of new developments springing mainly from a fruitful clash between the formalist and the intuitionist points of view. This section, though expository in character, is best placed at the end of our discussion of formalism and intuitionism as separate points of view.

1. *Mathematical theorems as reports on intuitive constructions*

We have seen that the formalist metamathematician and the intuitionist mathematician make the same claim, that their statements are not statements of logic. They are about a subject matter which is first produced (constructed) and then described. Consequently they are not 'analytic' but 'synthetic'. The constructions of the formalist are made, or can be made, in the physical world; those of the intuitionist in the mind, a medium which is different from sense-perception and open to introspection only. The formalist's statements are synthetic and empirical, the intuitionist's synthetic and non-empirical, *i.e. a priori*.