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A COURSE
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PURE MATHEMATICS

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55. **The phrase 'n tends to infinity'.** There is a somewhat different way of looking at the matter which it is natural to adopt. Suppose that n assumes successively the values $1, 2, 3, \ldots$. The word 'successively' naturally suggests succession in time, and we may suppose n, if we like, to assume these values at successive moments of time (*e.g.* at the beginnings of successive seconds). Then as the seconds pass n gets larger and larger and there is no limit to the extent of its increase. However large a number we may think of (*e.g.* 2147483647), a time will come when n has become larger than this number.

It is convenient to have a short phrase to express this unending growth of n, and we shall say that n **tends to infinity**, or $n \to \infty$, this last symbol being usually employed as an abbreviation for 'infinity'. The phrase 'tends to' like the word 'successively' naturally suggests the idea of change in time, and it is convenient to think of the variation of n as accomplished in time in the manner described above. This however is a mere matter of convenience. The variable n is a purely logical entity which has in itself nothing to do with time.

The reader cannot too strongly impress upon himself that when we say that n 'tends to ∞' we mean simply that n is supposed to assume a series of values which increase continually and without limit. **There is no number 'infinity'**: such an equation as

$$n = \infty$$

is as it stands *absolutely meaningless*: n cannot be equal to ∞, because 'equal to ∞' means nothing. So far in fact the symbol ∞ means nothing at all except in the one phrase 'tends to ∞', the meaning of which we have explained above. Later on we shall learn how to attach a meaning to other phrases involving the symbol ∞, but the reader will always have to bear in mind

(1) that ∞ *by itself* means nothing, although *phrases containing it* sometimes mean something,

(2) that in every case in which a phrase containing the symbol ∞ means something it will do so simply because we have previously attached a meaning to this particular phrase by means of a special definition.

Now it is clear that if $\phi(n)$ has the property P for large values of n, and if n 'tends to ∞', in the sense which we have just explained, then n will ultimately assume values large enough to ensure that $\phi(n)$ has the property P. And so another way of putting the question 'what properties has $\phi(n)$ for sufficiently large values of n?' is 'how does $\phi(n)$ behave as n tends to ∞?'

56. **The behaviour of a function of n as n tends to infinity.** We shall now proceed, in the light of the remarks made in the preceding sections, to consider the meaning of some kinds of statements which are perpetually occurring in higher mathematics. Let us consider, for example, the two following statements: (*a*) $1/n$ *is small for large values of* n, (*b*) $1-(1/n)$ *is nearly equal to* 1 *for large values of* n. Obvious as they may seem, there is a good deal in them which will repay the reader's attention. Let us take (*a*) first, as being slightly the simpler.

We have already considered the statement '$1/n$ *is less than* ·01 *for large values of* n'. This, we saw, means that the inequality $1/n < \cdot 01$ is true for all values of n greater than some definite value, in fact greater than 100. Similarly it is true that '$1/n$ *is less than* ·0001 *for large values of* n': in fact $1/n < \cdot 0001$ if $n > 10000$. And instead of ·01 or ·0001 we might take ·000001 or ·00000001, or indeed any positive number we like.

It is obviously convenient to have some way of expressing the fact that *any* such statement as '$1/n$ *is less than* ·01 *for large values of* n' is true, when we substitute for ·01 any smaller number, such as ·0001 or ·000001 or any other number we care to choose. And clearly we can do this by saying that '*however small* δ *may be* (provided of course it is positive), *then* $1/n < \delta$ *for sufficiently large values of* n'. That this is true is obvious. For $1/n < \delta$ if $n > 1/\delta$, so that our 'sufficiently large' values of n need only all be greater than $1/\delta$. The assertion is however a complex one, in that it really stands for the whole class of assertions which we obtain by giving to δ special values such as ·01. And of course the smaller δ is, and the larger $1/\delta$, the larger must be the least of the 'sufficiently large' values of n: values which are sufficiently large when δ has one value are inadequate when it has a smaller.

The last statement italicised is what is really meant by the statement (*a*), that $1/n$ is small when n is large. Similarly (*b*) really means "*if* $\phi(n)=1-(1/n)$, *then the statement* '$1-\phi(n)<\delta$ *for sufficiently large values of n' is true whatever positive value (such as* ·01 *or* ·0001) *we attribute to* δ". That the statement (*b*) is true is obvious from the fact that $1-\phi(n)=1/n$.

There is another way in which it is common to state the facts expressed by the assertions (*a*) and (*b*). This is suggested at once by § 55. Instead of saying '$1/n$ is small for large values of n' we say '$1/n$ tends to 0 as n tends to ∞'. Similarly we say that '$1-(1/n)$ tends to 1 as n tends to ∞': and these statements are to be regarded as precisely equivalent to (*a*) and (*b*). Thus the statements

'$1/n$ is small when n is large',

'$1/n$ tends to 0 as n tends to ∞',

are equivalent to one another and to the more formal statement

'if δ is any positive number, however small, then $1/n<\delta$ for sufficiently large values of n',

or to the still more formal statement

'if δ is any positive number, however small, then we can find a number n_0 such that $1/n<\delta$ for all values of n greater than or equal to n_0'.

The number n_0 which occurs in the last statement is of course a function of δ. We shall sometimes emphasize this fact by writing n_0 in the form $n_0(\delta)$.

The reader should imagine himself confronted by an opponent who questions the truth of the statement. He would name a series of numbers growing smaller and smaller. He might begin with ·001. The reader would reply that $1/n<{}$·001 as soon as $n>1000$. The opponent would be bound to admit this, but would try again with some smaller number, such as ·0000001. The reader would reply that $1/n<{}$·0000001 as soon as $n>10000000$: and so on. In this simple case it is evident that the reader would always have the better of the argument.

We shall now introduce yet another way of expressing this property of the function $1/n$. We shall say that '*the* **limit** *of* $1/n$ *as n tends to* ∞ *is* 0', a statement which we may express symbolically in the form

$$\lim_{n\to\infty}\frac{1}{n}=0,$$

or simply $\lim(1/n)=0$. We shall also sometimes write

$$\text{'}1/n\to 0$$

as $n\to\infty$', which may be read '$1/n$ tends to 0 as n tends to ∞'; or simply '$1/n\to 0$'. In the same way we shall write

$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)=1,\quad \lim\left(1-\frac{1}{n}\right)=1,$$

or $1-(1/n)\to 1$.

57. Now let us consider a different example: let $\phi(n)=n^2$. Then '*n^2 is large when n is large*'. This statement is equivalent to the more formal statements

'if Δ is any positive number, however large, then $n^2>\Delta$ for sufficiently large values of n',

'we can find a number $n_0(\Delta)$ such that $n^2>\Delta$ for all values of n greater than or equal to $n_0(\Delta)$'.

And it is natural in this case to say that 'n^2 tends to ∞ as n tends to ∞', or 'n^2 tends to ∞ with n', and to write

$$n^2\to\infty.$$

Finally consider the function $\phi(n)=-n^2$. In this case $\phi(n)$ is large, but negative, when n is large, and we naturally say that '$-n^2$ tends to $-\infty$ as n tends to ∞' and write

$$-n^2\to-\infty.$$

And the use of the symbol $-\infty$ in this sense suggests that it will sometimes be convenient to write $n^2\to+\infty$ for $n^2\to\infty$ and generally to use $+\infty$ instead of ∞, in order to secure greater uniformity of notation.

But we must once more repeat that in all these statements the symbols ∞, $+\infty$, $-\infty$ mean nothing whatever by themselves, and only acquire a meaning when they occur in certain special connections in virtue of the explanations which we have just given.