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A COURSE OF PURE MATHEMATICS

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CHAPTER II

FUNCTIONS OF REAL VARIABLES

20. The idea of a function. Suppose that x and y are two continuous real variables, which we may suppose to be represented geometrically by distances $A_0P = x$, $B_0Q = y$ measured from fixed points A_0, B_0 along two straight lines Λ, M. And let us suppose that the positions of the points P and Q are not independent, but connected by a relation which we can imagine to be expressed as a relation between x and y: so that, when P and x are known, Q and y are also known. We might, for example, suppose that $y = x$, or $y = 2x$, or $\frac{1}{2}x$, or $x^2 + 1$. In all of these cases the value of x determines that of y. Or again, we might suppose that the relation between x and y is given, not by means of an explicit formula for y in terms of x, but by means of a geometrical construction which enables us to determine Q when P is known.

In these circumstances y is said to be a *function* of x. This notion of functional dependence of one variable upon another is perhaps the most important in the whole range of higher mathematics. In order to enable the reader to be certain that he understands it clearly, we shall, in this chapter, illustrate it by means of a large number of examples.

But before we proceed to do this, we must point out that the simple examples of functions mentioned above possess three characteristics which are by no means involved in the general idea of a function, viz.:

(1) y is determined *for every value of* x;

(2) to each value of x for which y is given corresponds *one and only one value of* y;

(3) the relation between x and y is expressed by means of *an analytical formula*, from which the value of y corresponding to a given value of x can be calculated by direct substitution of the latter.

It is indeed the case that these particular characteristics are possessed by many of the most important functions. But the consideration of the following examples will make it clear that they are by no means essential to a function. All that is essential is that there should be some relation between x and y such that to some values of x at any rate correspond values of y.

Examples X. 1. Let $y = x$ or $2x$ or $\frac{1}{2}x$ or $x^2 + 1$ Nothing further need be said at present about cases such as these.

2. Let $y = 0$ whatever be the value of x. Then y is a function of x, for we can give x any value, and the corresponding value of y (viz. 0) is known. In this case the functional relation makes the same value of y correspond to all values of x. The same would be true were y equal to 1 or $-\frac{1}{2}$ or $\sqrt{2}$ instead of 0. Such a function of x is called *a constant*.

3. Let $y^2 = x$. Then if x is positive this equation defines *two* values of y corresponding to each value of x, viz. $\pm\sqrt{x}$. If $x = 0$, $y = 0$. Hence to the particular value 0 of x corresponds *one* and only one value of y. But if x is negative there is *no* value of y which satisfies the equation. That is to say, the function y is not defined for negative values of x. This function therefore possesses the characteristic (3), but neither (1) nor (2).

4. Consider a volume of gas maintained at a constant temperature and contained in a cylinder closed by a sliding piston*.

Let A be the area of the cross section of the piston and W its weight. The gas, held in a state of compression by the piston, exerts a certain pressure p_0 per unit of area on the piston, which balances the weight W, so that

$$W = Ap_0.$$

Let v_0 be the volume of the gas when the system is thus in equilibrium. If additional weight is placed upon the piston the latter is forced downwards. The volume (v) of the gas diminishes; the pressure (p) which it exerts upon unit area of the piston increases. Boyle's experimental law asserts that the product of p and v is very nearly constant, a correspondence which, if exact, would be represented by an equation of the type

$$pv = a \quad \text{.....................................(i)},$$

where a is a number which can be determined approximately by experiment.

Boyle's law, however, only gives a reasonable approximation to the facts provided the gas is not compressed too much. When v is decreased and p increased beyond a certain point, the relation between them is no longer expressed with tolerable exactness by the equation (i). It is known that a

* I borrow this instructive example from Prof. H. S. Carslaw's *Introduction to the Calculus*.

much better approximation to the true relation can then be found by means of what is known as 'van der Waals' law', expressed by the equation

$$\left(p+\frac{a}{v^2}\right)(v-\beta)=\gamma \quad \text{..............................(ii)},$$

where a, β, γ are numbers which can also be determined approximately by experiment.

Of course the two equations, even taken together, do not give anything like a complete account of the relation between p and v. This relation is no doubt in reality much more complicated, and its form changes, as v varies, from a form nearly equivalent to (i) to a form nearly equivalent to (ii). But, from a mathematical point of view, there is nothing to prevent us from contemplating an ideal state of things in which, for all values of v not less than a certain value V, (i) would be exactly true, and (ii) exactly true for all values of v less than V. And then we might regard the two equations as together defining p as a function of v. It is an example of a function which for some values of v is defined by one formula and for other values of v is defined by another.

This function possesses the characteristic (2): to any value of v only one value of p corresponds: but it does not possess (1). For p is not defined as a function of v for negative values of v; a 'negative volume' means nothing, and so negative values of v do not present themselves for consideration at all.

5. Suppose that a perfectly elastic ball is dropped (without rotation) from a height $\frac{1}{2}g\tau^2$ on to a fixed horizontal plane, and rebounds continually.

The ordinary formulae of elementary dynamics, with which the reader is probably familiar, show that $h=\frac{1}{2}gt^2$ if $0\leqq t\leqq\tau$, $h=\frac{1}{2}g(2\tau-t)^2$ if $\tau\leqq t\leqq 3\tau$, and generally

$$h=\tfrac{1}{2}g(2n\tau-t)^2$$

if $(2n-1)\tau\leqq t\leqq(2n+1)\tau$, h being the depth of the ball, at time t, below its original position. Obviously h is a function of t which is only defined for positive values of t.

6. Suppose that y is defined as being *the largest prime factor of* x. This is an instance of a definition which only applies to a particular class of values of x, viz. *integral* values. 'The largest prime factor of $\frac{11}{3}$ or of $\sqrt{2}$ or of π' means nothing, and so our defining relation fails to define for such values of x as these. Thus this function does not possess the characteristic (1). It does possess (2), but not (3), as there is no simple formula which expresses y in terms of x.

7. Let y be defined as *the denominator of x when x is expressed in its lowest terms*. This is an example of a function which is defined if and only if x is *rational*. Thus $y=7$ if $x=-11/7$: but y is not defined for $x=\sqrt{2}$, 'the denominator of $\sqrt{2}$' being a meaningless form of words.

8. Let y be defined as *the height in inches of policeman Cx, in the Metropolitan Police, at* 5.30 *p.m. on* 8 *Aug.* 1907. Then y is defined for a certain number of integral values of x, viz. 1, 2, ..., N, where N is the total number of policemen in division C at that particular moment of time.

21. The graphical representation of functions.

Suppose that the variable y is a function of the variable x. It will generally be open to us also to regard x as a function of y, in virtue of the functional relation between x and y. But for the present we shall look at this relation from the first point of view. We shall then call x the *independent variable* and y the *dependent variable*; and, when the particular form of the functional relation is not specified, we shall express it by writing

$$y=f(x)$$

(or $F(x)$, $\phi(x)$, $\psi(x)$, ..., as the case may be).

The nature of particular functions may, in very many cases, be illustrated and made easily intelligible as follows. Draw two lines OX, OY at right angles to one another and produced indefinitely in both directions. We can represent values of x and y by distances measured from O along the lines OX, OY respectively, regard being paid, of course, to sign, and the positive directions of measurement being those indicated by arrows in Fig. 6.

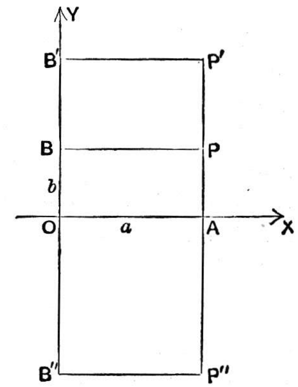

Fig. 6.

Let a be any value of x for which y is defined and has (let us suppose) the single value b. Take $OA=a$, $OB=b$, and complete the rectangle $OAPB$. Imagine the point P marked on the diagram. This marking of the point P may be regarded as showing that the value of y for $x=a$ is b.

If to the value a of x correspond several values of y (say b, b', b''), we have, instead of the single point P, a number of points P, P', P''.

We shall call P the *point* (a, b); a and b the *coordinates of* P *referred to the axes* OX, OY; a the *abscissa*, b the *ordinate of* P; OX and OY the *axis of* x and the *axis of* y, or together the *axes of coordinates*, and O the *origin of coordinates*, or simply the *origin*.

Let us now suppose that for all values a of x for which y is defined, the value b (or values $b, b', b'', \ldots$) of y, and the corresponding point P (or points $P, P', P'', \ldots$), have been determined. We call the aggregate of all these points the **graph** of the function y.

To take a very simple example, suppose that y is defined as a function of x by the equation

$$Ax + By + C = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where A, B, C are any fixed numbers*. Then y is a function of x which possesses all the characteristics (1), (2), (3) of § 20. It is easy to show that *the graph of y is a straight line.* The reader is in all probability familiar with one or other of the various proofs of this proposition which are given in text-books of Analytical Geometry.

We shall sometimes use another mode of expression. We shall say that when x and y vary in such a way that equation (1) is always true, *the locus of the point (x, y) is a straight line*, and we shall call (1) *the equation of the locus*, and say that the equation *represents* the locus. This use of the terms 'locus', 'equation of the locus' is quite general, and may be applied whenever the relation between x and y is capable of being represented by an analytical formula.

The equation $Ax + By + C = 0$ is *the general equation of the first degree*, for $Ax + By + C$ is the most general polynomial in x and y which does not involve any terms of degree higher than the first in x and y. Hence *the general equation of the first degree represents a straight line.* It is equally easy to prove the converse proposition that *the equation of any straight line is of the first degree.*

We may mention a few further examples of interesting geometrical loci defined by equations. An equation of the form

$$(x - \alpha)^2 + (y - \beta)^2 = \rho^2,$$

* If $B = 0$, y does not occur in the equation. We must then regard y as a function of x defined for one value only of x, viz. $x = -C/A$, and then having *all* values.

or

$$x^2 + y^2 + 2Gx + 2Fy + C = 0,$$

where $G^2 + F^2 - C > 0$, represents a circle. The equation

$$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0$$

(*the general equation of the second degree*) represents, assuming that the coefficients satisfy certain inequalities, a conic section, *i.e.* an ellipse, parabola, or hyperbola. For further discussion of these loci we must refer to books on Analytical Geometry.

22. Polar coordinates. In what precedes we have determined the position of P by the lengths of its coordinates $OM = x$, $MP = y$. If $OP = r$ and $MOP = \theta$, θ being an angle between 0 and 2π (measured in the positive direction), it is evident that

$$x = r\cos\theta, \qquad y = r\sin\theta,$$
$$r = \sqrt{(x^2 + y^2)}, \quad \cos\theta : \sin\theta : 1 :: x : y : r,$$

and that the position of P is equally well determined by a knowledge of r and θ. We call r and θ the *polar coordinates* of P. The former, it should be observed, is essentially positive*.

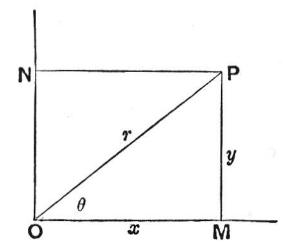

Fig. 7.

If P moves on a locus there will be some relation between r and θ, say $r = f(\theta)$ or $\theta = F(r)$. This we call the *polar equation* of the locus. The polar equation may be deduced from the (x, y) equation (or *vice versa*) by means of the formulae above.

Thus the polar equation of a straight line is of the form

$$r\cos(\theta - \alpha) = p,$$

where p and α are constants. The equation $r = 2a\cos\theta$ represents a circle passing through the origin; and the general equation of a circle is of the form

$$r^2 + c^2 - 2rc\cos(\theta - \alpha) = A^2,$$

where A, c, and α are constants.

* Polar coordinates are sometimes defined so that r may be positive or negative. In this case two pairs of coordinates—*e.g.* $(1, 0)$ and $(-1, \pi)$—correspond to the same point. The distinction between the two systems may be illustrated by means of the equation $l/r = 1 - e\cos\theta$, where $l > 0$, $e > 1$. According to our definitions r must be positive and therefore $\cos\theta < 1/e$: the equation represents one branch only of a hyperbola, the other having the equation $-l/r = 1 - e\cos\theta$. With the system of coordinates which admits negative values of r, the equation represents the whole hyperbola.

23. Further examples of functions and their graphical representation. The examples which follow will give the reader a better notion of the infinite variety of possible types of functions.

A. **Polynomials.** A *polynomial in* x is a function of the form

$$a_0x^m + a_1x^{m-1} + \ldots + a_m,$$

where $a_0, a_1, \ldots, a_m$ are constants. The simplest polynomials are the simple powers $y = x, x^2, x^3, \ldots, x^m, \ldots$. The graph of the function x^m is of two distinct types, according as m is even or odd.

First let $m = 2$. Then three points on the graph are $(0, 0)$, $(1, 1)$, $(-1, 1)$. Any number of additional points on the graph may be found by assigning other special values to x: thus the values

$$x = \tfrac{1}{2},\ 2,\ 3,\ -\tfrac{1}{2},\ -2,\ 3$$

give

$$y = \tfrac{1}{4},\ 4,\ 9,\ \ \tfrac{1}{4},\ \ 4,\ 9.$$

If the reader will plot off a fair number of points on the graph, he will be led to conjecture that the form of the graph is something like that shown in Fig. 8. If he draws a curve through the special points which he has proved to lie on the graph and then tests its accuracy by giving x new values, and calculating the corresponding values of y, he will find that they lie as near to the curve as it is reasonable to expect, when the inevitable inaccuracies of drawing are considered. The curve is of course a parabola.

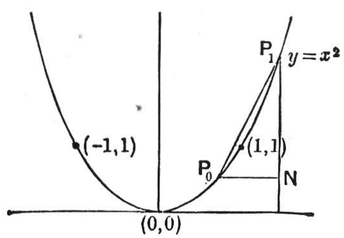

Fig. 8.

There is, however, one fundamental question which we cannot answer adequately at present. The reader has no doubt some notion as to what is meant by a *continuous* curve, a curve without breaks or jumps; such a curve, in fact, as is roughly represented in Fig. 8. The question is whether the graph of the function $y = x^2$ is in fact such a curve. This cannot be *proved* by merely

constructing any number of isolated points on the curve, although the more such points we construct the more probable it will appear.

This question cannot be discussed properly until Ch. V. In that chapter we shall consider in detail what our common sense idea of continuity really means, and how we can prove that such graphs as the one now considered, and others which we shall consider later on in this chapter, are really continuous curves. For the present the reader may be content to draw his curves as common sense dictates.

It is easy to see that the curve $y = x^2$ is everywhere convex to the axis of x. Let P_0, P_1 (Fig. 8) be the points (x_0, x_0^2), (x_1, x_1^2). Then the coordinates of a point on the chord P_0P_1 are $x = \lambda x_0 + \mu x_1$, $y = \lambda x_0^2 + \mu x_1^2$, where λ and μ are positive numbers whose sum is 1. And

$$y - x^2 = (\lambda + \mu)(\lambda x_0^2 + \mu x_1^2) - (\lambda x_0 + \mu x_1)^2 = \lambda\mu(x_1 - x_0)^2 \geqq 0,$$

so that the chord lies entirely above the curve.

The curve $y = x^4$ is similar to $y = x^2$ in general appearance, but flatter near O, and steeper beyond the points A, A' (Fig. 9), and $y = x^m$, where m is even and greater than 4, is still more so. As m gets larger and larger the flatness and steepness grow more and more pronounced, until the curve is practically indistinguishable from the thick line in the figure.

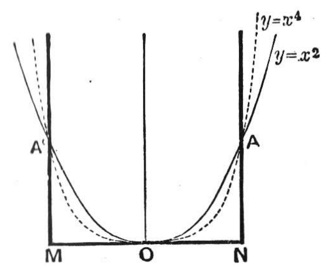

Fig. 9.

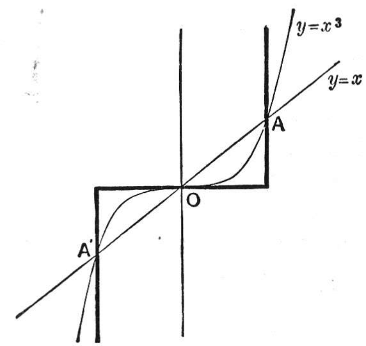

Fig. 10.

The reader should next consider the curves given by $y = x^m$, when m is odd. The fundamental difference between the two cases is that whereas when m is even $(-x)^m = x^m$, so that the curve is symmetrical about OY, when m is odd $(-x)^m = -x^m$, so that y is negative when x is negative. Fig. 10 shows the curves $y=x$, $y=x^3$, and the form to which $y=x^m$ approximates for larger odd values of m

It is now easy to see how (theoretically at any rate) the graph of any polynomial may be constructed. In the first place, from the graph of $y=x^m$ we can at once derive that of Cx^m, where C is a constant, by multiplying the ordinate of every point of the curve by C. And if we know the graphs of $f(x)$ and $F(x)$, we can find that of $f(x)+F(x)$ by taking the ordinate of every point to be the sum of the ordinates of the corresponding points on the two original curves.

The drawing of graphs of polynomials is however so much facilitated by the use of more advanced methods, which will be explained later on, that we shall not pursue the subject further here.

Examples XI. 1. Trace the curves $y=7x^4$, $y=3x^5$, $y=x^{10}$.

[The reader should draw the curves carefully, and all three should be drawn in one figure*. He will then realise how rapidly the higher powers of x increase, as x gets larger and larger, and will see that, in such a polynomial as

$$x^{10}+3x^5+7x^4$$

(or even $x^{10}+30x^5+700x^4$), it is the *first* term which is of really preponderant importance when x is fairly large. Thus even when $x=4$, $x^{10}>1{,}000{,}000$, while $30x^5<35{,}000$ and $700x^4<180{,}000$; while if $x=10$ the preponderance of the first term is still more marked.]

2. Compare the relative magnitudes of x^{12}, $1{,}000{,}000x^6$, $1{,}000{,}000{,}000{,}000x$ when $x=1$, 10, 100, etc.

[The reader should make up a number of examples of this type for himself. This idea of the *relative rate of growth* of different functions of x is one with which we shall often be concerned in the following chapters.]

3. Draw the graph of $ax^2+2bx+c$.

[Here $y-\{(ac-b^2)/a\}=a\{x+(b/a)\}^2$. If we take new axes parallel to the old and passing through the point $x=-b/a$, $y=(ac-b^2)/a$, the new equation is $y'=ax'^2$. The curve is a parabola.]

4. Trace the curves $y=x^3-3x+1$, $y=x^2(x-1)$, $y=x(x-1)^2$.

* It will be found convenient to take the scale of measurement along the axis of y a good deal smaller than that along the axis of x, in order to prevent the figure becoming of an awkward size.

24. B. Rational Functions. The class of functions which ranks next to that of polynomials in simplicity and importance is that of *rational functions*. A rational function is the quotient of one polynomial by another: thus if $P(x)$, $Q(x)$ are polynomials, we may denote the general rational function by

$$R(x)=\frac{P(x)}{Q(x)}.$$

In the particular case when $Q(x)$ reduces to unity or any other constant (*i.e.* does not involve x), $R(x)$ reduces to a polynomial: thus the class of rational functions includes that of polynomials as a sub-class. The following points concerning the definition should be noticed.

(1) We usually suppose that $P(x)$ and $Q(x)$ have no common factor $x+a$ or $x^p+ax^{p-1}+bx^{p-2}+\dots+k$, all such factors being removed by division.

(2) It should however be observed that this removal of common factors *does as a rule change the function*. Consider for example the function x/x, which is a rational function. On removing the common factor x we obtain $1/1=1$. But the original function is not *always* equal to 1: it is equal to 1 only so long as $x\neq 0$. If $x=0$ it takes the form $0/0$, which is meaningless. Thus the function x/x is equal to 1 if $x\neq 0$ and is undefined when $x=0$. It therefore differs from the function 1, which is *always* equal to 1.

(3) Such a function as

$$\left(\frac{1}{x+1}+\frac{1}{x-1}\right)\bigg/\left(\frac{1}{x}+\frac{1}{x-2}\right)$$

may be reduced, by the ordinary rules of algebra, to the form

$$\frac{x^2(x-2)}{(x-1)^2(x+1)},$$

which is a rational function of the standard form. But here again it must be noticed that the reduction is not *always* legitimate. In order to calculate the value of a function for a given value of x we must substitute the value for x in the function *in the form in which it is given*. In the case of this function the values $x=-1, 1, 0, 2$ all lead to a meaningless expression, and so the function is not defined for these values. The same is true of the reduced form, so far as the values -1 and 1 are concerned. But $x=0$ and $x=2$ give the value 0. Thus once more the two functions are not the same.

(4) But, as appears from the particular example considered under (3), there will generally be a certain number of values of x for which the function is not defined even when it has been reduced to a rational function of the standard form. These are the values of x (if any) for which the denominator vanishes. Thus $(x^2-7)/(x^2-3x+2)$ is not defined when $x=1$ or 2.

(5) Generally we agree, in dealing with expressions such as those considered in (2) and (3), to disregard the exceptional values of x for which such processes of simplification as were used there are illegitimate, and to reduce our function to the standard form of rational function. The reader will easily verify that (on this understanding) the sum, product, or quotient of two rational functions may themselves be reduced to rational functions of the standard type. And generally *a rational function of a rational function is itself a rational function*: *i.e.* if in $z=P(y)/Q(y)$, where P and Q are polynomials, we substitute $y=P_1(x)/Q_1(x)$, we obtain on simplification an equation of the form $z=P_2(x)/Q_2(x)$.

(6) It is in no way presupposed in the definition of a rational function that the constants which occur as coefficients should be rational *numbers*. The word rational has reference solely to the way in which the variable x appears in the function. Thus

$$\frac{x^2+x+\sqrt{3}}{x\sqrt[3]{2}-\pi}$$

is a rational function

The use of the word rational arises as follows. The rational function $P(x)/Q(x)$ may be generated from x by a finite number of operations upon x, including only multiplication of x by itself or a constant, addition of terms thus obtained, and division of one function, obtained by such multiplications and additions, by another. In so far as the variable x is concerned, this procedure is very much like that by which all rational numbers can be obtained from unity, a procedure exemplified in the equation

$$\frac{5}{3}=\frac{1+1+1+1+1}{1+1+1}.$$

Again, *any* function which can be deduced from x by the elementary operations mentioned above, using at each stage of the process functions which have already been obtained from x in the same way, can be reduced to the standard type of rational function. The most general kind of function which can be obtained in this way is sufficiently illustrated by the example

$$\left(\frac{x}{x^2+1}+\frac{2x+7}{x^2+\dfrac{11x-3\sqrt{2}}{9x+1}}\right)\Big/\left(17+\frac{2}{x^3}\right),$$

which can obviously be reduced to the standard type of rational function.

25. The drawing of graphs of rational functions, even more than that of polynomials, is immensely facilitated by the use of methods depending upon the differential calculus. We shall therefore content ourselves at present with a very few examples.

Examples XII. 1. Draw the graphs of $y=1/x$, $y=1/x^2$, $y=1/x^3$, ...

[The figures show the graphs of the first two curves. It should be observed that, since $1/0$, $1/0^2$, ... are meaningless expressions, these functions are not defined for $x=0$.]

2. Trace $y=x+(1/x)$, $x-(1/x)$, $x^2+(1/x^2)$, $x^2-(1/x^2)$ and $ax+(b/x)$ taking various values, positive and negative, for a and b.

3. Trace

$$y=\frac{x+1}{x-1},\quad \left(\frac{x+1}{x-1}\right)^2,\quad \frac{1}{(x-1)^2},\quad \frac{x^2+1}{x^2-1}.$$

4. Trace $y=1/(x-a)(x-b)$, $1/(x-a)(x-b)(x-c)$, where $a<b<c$.

5. Sketch the general form assumed by the curves $y=1/x^m$ as m becomes larger and larger, considering separately the cases in which m is odd or even.

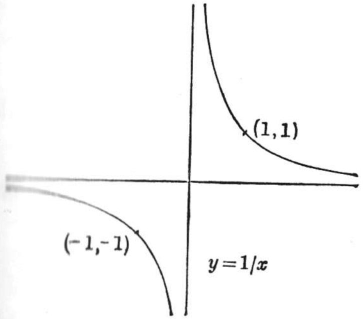

Fig. 11.

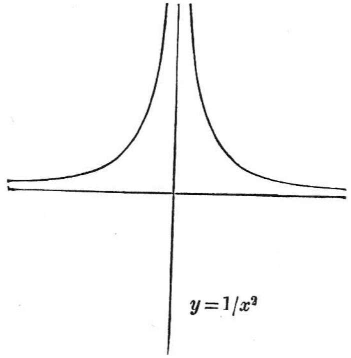

Fig. 12.

26. C. Explicit Algebraical Functions. The next important class of functions is that of *explicit algebraical functions*. These are functions which can be generated from x by a finite number of operations such as those used in generating rational functions, together with a finite number of operations of root extraction. Thus

$$\frac{\sqrt{(1+x)}-\sqrt[3]{(1-x)}}{\sqrt{(1+x)}+\sqrt[3]{(1-x)}},\qquad \sqrt{x}+\sqrt{(x+\sqrt{x})},$$

$$\left(\frac{x^2+x+\sqrt{3}}{x\sqrt[3]{2}-\pi}\right)^{\frac{2}{3}}$$

are explicit algebraical functions, and so is $x^{m/n}$ (*i.e.* $\sqrt[n]{x^m}$), where m and n are any integers.

It should be noticed that there is an ambiguity of notation involved in such an equation as $y=\sqrt{x}$. We have, up to the present, regarded (*e.g.*) $\sqrt{2}$ as denoting the *positive* square root of 2, and it would be natural to denote by $\sqrt{x}$, where x is any positive number, the positive square root of x, in which case $y=\sqrt{x}$ would be a one-valued function of x. It is however often more convenient to regard $\sqrt{x}$ as standing for the two-valued function whose two values are the positive and negative square roots of x.

The reader will observe that, when this course is adopted, the function $\sqrt{x}$ differs fundamentally from rational functions in two respects. In the first place a rational function is always defined for all values of x with a certain number of isolated exceptions. But $\sqrt{x}$ is undefined for *a whole range* of values of x (*i.e.* all negative values). Secondly the function, when x has a value for which it is defined, has generally two values of opposite signs.

The function $\sqrt[3]{x}$, on the other hand, is one-valued and defined for all values of x.

Examples XIII. 1. $\sqrt{\{(x-a)(b-x)\}}$, where $a<b$, is defined only for $a \leqq x \leqq b$. If $a<x<b$ it has two values: if $x=a$ or b only one, viz. 0.

2. Consider similarly

$$\sqrt{\{(x-a)(x-b)(x-c)\}} \quad (a<b<c),$$

$$\sqrt{\{x(x^2-a^2)\}}, \quad \sqrt[3]{\{(x-a)^2(b-x)\}} \quad (a<b),$$

$$\frac{\sqrt{(1+x)}-\sqrt{(1-x)}}{\sqrt{(1+x)}+\sqrt{(1-x)}}, \quad \sqrt{\{x+\sqrt{x}\}}.$$

3. Trace the curves $y^2=x$, $y^3=x$, $y^2=x^3$.

4. Draw the graphs of the functions $y=\sqrt{(a^2-x^2)}$, $y=b\sqrt{\{1-(x^2/a^2)\}}$.

27. D. Implicit Algebraical Functions. It is easy to verify that if

$$y=\frac{\sqrt{(1+x)}-\sqrt[3]{(1-x)}}{\sqrt{(1+x)}+\sqrt[3]{(1-x)}},$$

then

$$\left(\frac{1+y}{1-y}\right)^6=\frac{(1+x)^3}{(1-x)^2};$$

or if

$$y=\sqrt{x}+\sqrt{(x+\sqrt{x})},$$

then

$$y^4-(4y^2+4y+1)x=0.$$

Each of these equations may be expressed in the form

$$y^m+R_1y^{m-1}+\dots+R_m=0 \quad\dots\dots\dots(1),$$

where $R_1, R_2, \dots, R_m$ are rational functions of x: and the reader will easily verify that, if y is any one of the functions considered in the last set of examples, y satisfies an equation of this form.

It is naturally suggested that the same is true of any explicit algebraic function. And this is in fact true, and indeed not difficult to prove, though we shall not delay to write out a formal proof here. An example should make clear to the reader the lines on which such a proof would proceed. Let

$$y=\frac{x+\sqrt{x}+\sqrt{\{x+\sqrt{x}\}}+\sqrt[3]{(1+x)}}{x-\sqrt{x}+\sqrt{\{x+\sqrt{x}\}}-\sqrt[3]{(1+x)}}.$$

Then we have the equations

$$y=\frac{x+u+v+w}{x-u+v-w},$$

$$u^2=x, \quad v^2=x+u, \quad w^3=1+x,$$

and we have only to eliminate u, v, w between these equations in order to obtain an equation of the form desired.

We are therefore led to give the following definition: *a function $y=f(x)$ will be said to be an algebraical function of x if it is the root of an equation such as* (1), *i.e. the root of an equation of the m^{th} degree in y, whose coefficients are rational functions of x.* There is plainly no loss of generality in supposing the first coefficient to be unity.

This class of functions includes all the explicit algebraical functions considered in § 26. But it also includes other functions which cannot be expressed as explicit algebraical functions. For it is known that in general such an equation as (1) cannot be solved explicitly for y in terms of x, when m is greater than 4, though such a solution is always possible if $m=1$, 2, 3, or 4 and in special cases for higher values of m.

The definition of an algebraical function should be compared with that of an algebraical number given in the last chapter (Misc. Exs. 32).

Examples XIV. 1. If $m=1$, y is a rational function.

2. If $m=2$, the equation is $y^2+R_1y+R_2=0$, so that

$$y=\tfrac{1}{2}\{-R_1\pm\sqrt{(R_1^2-4R_2)}\}.$$

This function is defined for all values of x for which $R_1^2 \geqq 4R_2$. It has two values if $R_1^2>4R_2$ and one if $R_1^2=4R_2$.

If $m=3$ or 4, we can use the methods explained in treatises on Algebra for the solution of cubic and biquadratic equations. But as a rule the process is complicated and the results inconvenient in form, and we can generally study the properties of the function better by means of the original equation.

3. Consider the functions defined by the equations

$$y^2-2y-x^2=0, \quad y^2-2y+x^2=0, \quad y^4-2y^2+x^2=0,$$

in each case obtaining y as an explicit function of x, and stating for what values of x it is defined.

4. Find algebraical equations, with coefficients rational in x, satisfied by each of the functions

$$\sqrt{x}+\sqrt{(1/x)}, \quad \sqrt[3]{x}+\sqrt[3]{(1/x)}, \quad \sqrt{(x+\sqrt{x})}, \quad \sqrt{\{x+\sqrt{(x+\sqrt{x})}\}}.$$

5. Consider the equation $\qquad y^4=x^2.$

[Here $y^2=\pm x$. If x is positive, $y=\sqrt{x}$: if negative, $y=\sqrt{(-x)}$. Thus the function has two values for all values of x save $x=0$.]

6. An algebraical function of an algebraical function of x is itself an algebraical function of x.

[For we have

$$y^m+R_1(z)\,y^{m-1}+\ldots+R_m(z)=0,$$

where

$$z^n+S_1(x)\,z^{n-1}+\ldots+S_n(x)=0.$$

Eliminating z we find an equation of the form

$$y^p+T_1(x)\,y^{p-1}+\ldots+T_p(x)=0.$$

Here all the capital letters denote rational functions.]

7. An example should perhaps be given of an algebraical function which cannot be expressed in an explicit algebraical form. Such an example is the function y defined by the equation

$$y^5-y-x=0.$$

But the proof that we cannot find an explicit algebraical expression for y in terms of x is difficult, and cannot be attempted here.

28. Transcendental functions. All functions of x which are not rational or even algebraical are called *transcendental* functions. This class of functions, being defined in so purely negative a manner, naturally includes an infinite variety of whole kinds of functions of varying degrees of simplicity and importance. Among these we can at present distinguish two kinds which are particularly interesting.

E. **The direct and inverse trigonometrical or circular functions.** These are the sine and cosine functions of elementary trigonometry, and their inverses, and the functions derived from them. We may assume provisionally that the reader is familiar with their most important properties*.

* The definitions of the circular functions given in elementary trigonometry presuppose that any sector of a circle has associated with it a definite number called its *area*. How this assumption is justified will appear in Ch. VII.

Examples XV. 1. Draw the graphs of $\cos x$, $\sin x$, and $a\cos x+b\sin x$.

[Since $a\cos x+b\sin x=\beta\cos(x-a)$, where $\beta=\sqrt{(a^2+b^2)}$, and a is an angle whose cosine and sine are $a/\sqrt{(a^2+b^2)}$ and $b/\sqrt{(a^2+b^2)}$, the graphs of these three functions are similar in character.]

2. Draw the graphs of $\cos^2 x$, $\sin^2 x$, $a\cos^2 x+b\sin^2 x$.

3. Suppose the graphs of $f(x)$ and $F(x)$ drawn. Then the graph of

$$f(x)\cos^2 x+F(x)\sin^2 x$$

is a wavy curve which oscillates between the curves $y=f(x)$, $y=F(x)$. Draw the graph when $f(x)=x$, $F(x)=x^2$.

4. Show that the graph of $\cos px+\cos qx$ lies between those of $2\cos\frac{1}{2}(p-q)x$ and $-2\cos\frac{1}{2}(p+q)x$, touching each in turn. Sketch the graph when $(p-q)/(p+q)$ is small. (*Math. Trip.* 1908.)

5. Draw the graphs of $x+\sin x$, $(1/x)+\sin x$, $x\sin x$, $(\sin x)/x$.

6. Draw the graph of $\sin(1/x)$.

[If $y=\sin(1/x)$, then $y=0$ when $x=1/m\pi$, where m is any integer. Similarly $y=1$ when $x=1/(2m+\frac{1}{2})\pi$ and $y=-1$ when $x=1/(2m-\frac{1}{2})\pi$. The curve is entirely comprised between the lines $y=-1$ and $y=1$ (Fig. 13). It oscillates up and down, the rapidity of the oscillations becoming greater and greater as x approaches 0. For $x=0$ the function is undefined. When x is large y is small*. The negative half of the curve is similar in character to the positive half.]

7. Draw the graph of $x\sin(1/x)$.

[This curve is comprised between the lines $y=-x$ and $y=x$ just as the last curve is comprised between the lines $y=-1$ and $y=1$ (Fig. 14).]

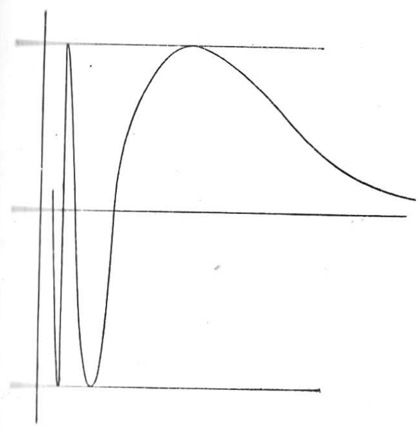

Fig. 13.

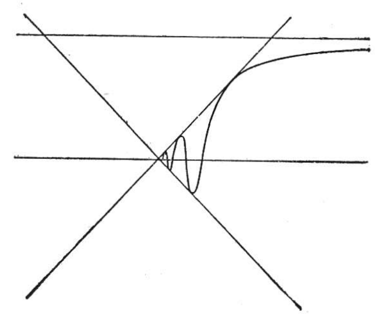

Fig. 14.

* See Chs. IV and V for explanations as to the precise meaning of this phrase.

8. Draw the graphs of $x^2 \sin(1/x)$, $(1/x)\sin(1/x)$, $\sin^2(1/x)$, $\{x\sin(1/x)\}^2$, $a\cos^2(1/x)+b\sin^2(1/x)$, $\sin x+\sin(1/x)$, $\sin x \sin(1/x)$.

9. Draw the graphs of $\cos x^2$, $\sin x^2$, $a\cos x^2+b\sin x^2$.

10. Draw the graphs of $\arccos x$ and $\arcsin x$.

[If $y=\arccos x$, $x=\cos y$. This enables us to draw the graph of x, considered as a function of y, and the same curve shows y as a function of x. It is clear that y is only defined for $-1 \leqq x \leqq 1$, and is infinitely many-valued for these values of x. As the reader no doubt remembers, there is, when $-1<x<1$, a value of y between 0 and π, say α, and the other values of y are given by the formula $2n\pi \pm \alpha$, where n is any integer, positive or negative.]

11. Draw the graphs of

$$\tan x,\quad \cot x,\quad \sec x,\quad \operatorname{cosec} x,\quad \tan^2 x,\quad \cot^2 x,\quad \sec^2 x,\quad \operatorname{cosec}^2 x.$$

12. Draw the graphs of $\operatorname{arc\,tan} x$, $\operatorname{arc\,cot} x$, $\operatorname{arc\,sec} x$, $\operatorname{arc\,cosec} x$. Give formulae (as in Ex. 10) expressing all the values of each of these functions in terms of any particular value.

13. Draw the graphs of $\tan(1/x)$, $\cot(1/x)$, $\sec(1/x)$, $\operatorname{cosec}(1/x)$.

14. Show that $\cos x$ and $\sin x$ are not rational functions of x.

[A function is said to be *periodic*, with period a, if $f(x)=f(x+a)$ for all values of x for which $f(x)$ is defined. Thus $\cos x$ and $\sin x$ have the period 2π. It is easy to see that no periodic function can be a rational function, unless it is a constant. For suppose that

$$f(x)=P(x)/Q(x),$$

where P and Q are polynomials, and that $f(x)=f(x+a)$, each of these equations holding for all values of x. Let $f(0)=k$. Then the equation $P(x)-kQ(x)=0$ is satisfied by an infinite number of values of x, viz. $x=0$, a, $2a$, etc., and therefore for all values of x. Thus $f(x)=k$ for all values of x, *i.e.* $f(x)$ is a constant.]

15. Show, more generally, that no function with a period can be an algebraical function of x.

[Let the equation which defines the algebraical function be

$$y^m+R_1y^{m-1}+\dots+R_m=0 \dots\dots\dots\dots\dots\dots\dots\dots (1)$$

where R_1, ... are rational functions of x. This may be put in the form

$$P_0y^m+P_1y^{m-1}+\dots+P_m=0,$$

where P_0, P_1, ... are polynomials in x. Arguing as above, we see that

$$P_0k^m+P_1k^{m-1}+\dots+P_m=0$$

for all values of x. Hence $y=k$ satisfies the equation (1) for all values of x, and one set of values of our algebraical function reduces to a constant.

Now divide (1) by $y-k$ and repeat the argument. Our final conclusion is that our algebraical function has, for any value of x, the same set of values k, k', ...; *i.e.* it is composed of a certain number of constants.]

16. The inverse sine and inverse cosine are not rational or algebraical functions. [This follows from the fact that, for any value of x between -1 and $+1$, $\arcsin x$ and $\arccos x$ have infinitely many values.]

29. F. Other classes of transcendental functions. Next in importance to the trigonometrical functions come the exponential and logarithmic functions, which will be discussed in Chs. IX and X. But these functions are beyond our range at present. And most of the other classes of transcendental functions whose properties have been studied, such as the elliptic functions, Bessel's and Legendre's functions, Gamma-functions, and so forth, lie altogether beyond the scope of this book. There are however some elementary types of functions which, though of much less importance theoretically than the rational, algebraical, or trigonometrical functions, are particularly instructive as illustrations of the possible varieties of the functional relation.

Examples XVI. 1. Let $y=[x]$, where $[x]$ denotes the greatest integer not greater than x. The graph is shown in Fig. 15 *a*. The left-hand end points of the thick lines, but not the right-hand ones, belong to the graph.

2. $y=x-[x]$. (Fig. 15 *b*.)

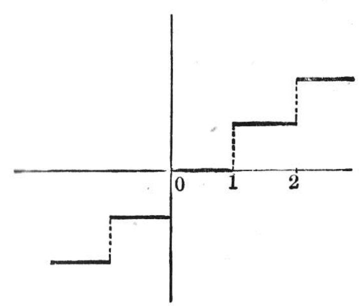

Fig. 15 *a*.

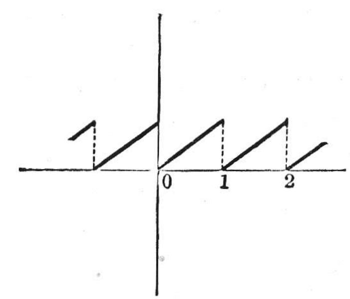

Fig. 15 *b*.

3. $y=\sqrt{\{x-[x]\}}$. (Fig. 15 *c*.) 4. $y=[x]+\sqrt{\{x-[x]\}}$. (Fig. 15 *d*.)

5. $y=(x-[x])^2$, $[x]+(x-[x])^2$.

6. $y=[\sqrt{x}]$, $[x^2]$, $\sqrt{x}-[\sqrt{x}]$, $x^2-[x^2]$, $[1-x^2]$.

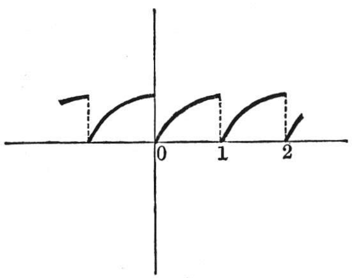

Fig. 15*c*.

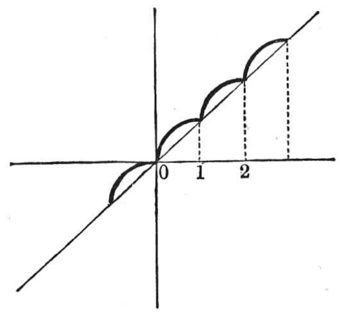

Fig. 15*d*.

7. Let y be defined as *the largest prime factor of* x (cf. Exs. x. 6). Then y is defined only for integral values of x. If

$$x=1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, \ldots,$$

then $$y=1, 2, 3, 2, 5, 3, 7, 2, 3, 5, 11, 3, 13, \ldots.$$

The graph consists of a number of isolated points.

8. Let y be *the denominator of* x (Exs. x. 7). In this case y is defined only for rational values of x. We can mark off as many points on the graph as we please, but the result is not in any ordinary sense of the word a curve, and there are no points corresponding to any irrational values of x.

Draw the straight line joining the points $(N-1, N)$, (N, N), where N is a positive integer. Show that the number of points of the locus which lie on this line is equal to the number of positive integers less than and prime to N.

9. Let $y=0$ when x is an integer, $y=x$ when x is not an integer. The graph is derived from the straight line $y=x$ by taking out the points

$$\ldots(-1, -1),\quad (0, 0),\quad (1, 1),\quad (2, 2), \ldots$$

and adding the points $(-1, 0)$, $(0, 0)$, $(1, 0)$, ... on the axis of x.

The reader may possibly regard this as an unreasonable function. *Why*, he may ask, if y is equal to x for all values of x save integral values, should it not be equal to x for integral values too? The answer is simply, *why should it?* The function y does in point of fact answer to the definition of a function: there is a relation between x and y such that when x is known y is known. We are perfectly at liberty to take this relation to be what we please, however arbitrary and apparently futile. This function y is, of course, a quite different function from that one which is *always* equal to x, whatever value, integral or otherwise, x may have.

10. Let $y=1$ when x is rational, but $y=0$ when x is irrational. The graph consists of two series of points arranged upon the lines $y=1$ and $y=0$. To the eye it is not distinguishable from two continuous straight lines, but in reality an infinite number of points are missing from each line.

11. Let $y=x$ when x is irrational and $y=\sqrt{\{(1+p^2)/(1+q^2)\}}$ when x is a rational fraction p/q.

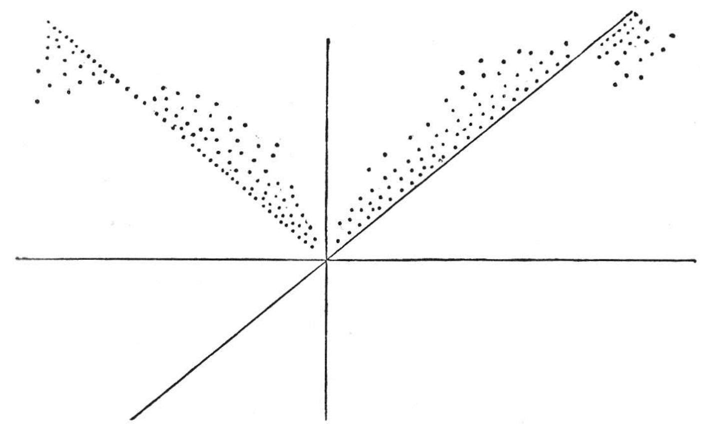

Fig. 16.

The irrational values of x contribute to the graph a curve in reality discontinuous, but apparently not to be distinguished from the straight line $y=x$.

Now consider the rational values of x. First let x be positive. Then $\sqrt{\{(1+p^2)/(1+q^2)\}}$ cannot be equal to p/q unless $p=q$, *i.e.* $x=1$. Thus all the points which correspond to rational values of x lie off the line, except the one point (1, 1). Again, if $p<q$, $\sqrt{\{(1+p^2)/(1+q^2)\}}>p/q$; if $p>q$, $\sqrt{\{(1+p^2)/(1+q^2)\}}<p/q$. Thus the points lie above the line $y=x$ if $0<x<1$, below if $x>1$. If p and q are large, $\sqrt{\{(1+p^2)/(1+q^2)\}}$ is nearly equal to p/q. Near any value of x we can find any number of rational fractions with large numerators and denominators. Hence the graph contains a large number of points which crowd round the line $y=x$. Its general appearance (for positive values of x) is that of a line surrounded by a swarm of isolated points which gets denser and denser as the points approach the line.

The part of the graph which corresponds to negative values of x consists of the rest of the discontinuous line together with the reflections of all these isolated points in the axis of y. Thus to the left of the axis of y the swarm of points is not round $y=x$ but round $y=-x$, which is not itself part of the graph. See Fig. 16.

30. Graphical solution of equations containing a single unknown number. Many equations can be expressed in the form

$$f(x) = \phi(x) \qquad\qquad (1),$$

where $f(x)$ and $\phi(x)$ are functions whose graphs are easy to draw. And if the curves

$$y = f(x), \quad y = \phi(x)$$

intersect in a point P whose abscissa is ξ, then ξ is a root of the equation (1).

Examples XVII. 1. The quadratic equation $ax^2+2bx+c=0$. This may be solved graphically in a variety of ways. For instance we may draw the graphs of

$$y = ax + 2b, \quad y = -c/x,$$

whose intersections, if any, give the roots. Or we may take

$$y = x^2, \quad y = -(2bx+c)/a.$$

But the most elementary method is probably to draw the circle

$$a(x^2+y^2)+2bx+c=0,$$

whose centre is $(-b/a, 0)$ and radius $\{\sqrt{(b^2-ac)}\}/a$. The abscissae of its intersections with the axis of x are the roots of the equation.

2. Solve by any of these methods

$$x^2+2x-3=0, \quad x^2-7x+4=0, \quad 3x^2+2x-2=0.$$

3. **The equation** $x^m+ax+b=0$. This may be solved by constructing the curves $y=x^m$, $y=-ax-b$. Verify the following table for the number of roots of

$$x^m+ax+b=0:$$

(*a*) *m even* $\begin{cases} b \text{ positive, } \textit{two or none,} \\ b \text{ negative, } \textit{two}. \end{cases}$

(*b*) *m odd* $\begin{cases} a \text{ positive, } \textit{one,} \\ a \text{ negative, } \textit{three or one.} \end{cases}$

Construct numerical examples to illustrate all possible cases.

4. Show that the equation $\tan x = ax + b$ has always an infinite number of roots.

5. Determine the number of roots of

$$\sin x = x, \quad \sin x = \tfrac{1}{3}x, \quad \sin x = \tfrac{1}{8}x, \quad \sin x = \tfrac{1}{120}x.$$

6. Show that if a is small and positive (*e.g.* $a = \cdot 01$), the equation

$$x - a = \tfrac{1}{2}\pi \sin^2 x$$

has three roots. Consider also the case in which a is small and negative. Explain how the number of roots varies as a varies.

31. Functions of two variables and their graphical representation. In § 20 we considered two variables connected by a relation. We may similarly consider *three* variables (x, y, and z) connected by a relation such that when the values of x and y are both given, the value or values of z are known. In this case we call z a *function of the two variables* x and y; x and y the *independent* variables, z the *dependent* variable; and we express this dependence of z upon x and y by writing

$$z = f(x, y).$$

The remarks of § 20 may all be applied, *mutatis mutandis*, to this more complicated case.

The method of representing such functions of two variables graphically is exactly the same in principle as in the case of functions of a single variable. We must take three axes, OX, OY, OZ in space of three dimensions, each axis being perpendicular to the other two. The point (a, b, c) is the point whose distances from the planes YOZ, ZOX, XOY, measured parallel to OX, OY, OZ, are a, b, and c. Regard must of course be paid to sign, lengths measured in the directions OX, OY, OZ being regarded as positive. The definitions of *coordinates, axes, origin* are the same as before.

Now let $$z = f(x, y).$$

As x and y vary, the point (x, y, z) will move in space. The aggregate of all the positions it assumes is called the *locus* of the point (x, y, z) or the *graph* of the function $z = f(x, y)$. When the relation between x, y, and z which defines z can be expressed in an analytical formula, this formula is called the *equation* of the locus. It is easy to show, for example, that the equation

$$Ax + By + Cz + D = 0$$

(*the general equation of the first degree*) represents a *plane*, and that the equation of any plane is of this form. The equation

$$(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 = \rho^2,$$

or

$$x^2 + y^2 + z^2 + 2Fx + 2Gy + 2Hz + C = 0,$$

where $F^2 + G^2 + H^2 - C > 0$, represents a *sphere*; and so on. For proofs of these propositions we must again refer to text-books of Analytical Geometry.

32. Curves in a plane. We have hitherto used the notation

$$y=f(x) \qquad\qquad (1)$$

to express functional dependence of y upon x. It is evident that this notation is most appropriate in the case in which y is expressed explicitly in terms of x by means of a formula, as when for example

$$y=x^2, \quad \sin x, \quad a\cos^2 x+b\sin^2 x.$$

We have however very often to deal with functional relations which it is impossible or inconvenient to express in this form. If, for example, $y^5-y-x=0$ or $x^5+y^5-ay=0$, it is known to be impossible to express y explicitly as an algebraical function of x. If

$$x^2+y^2+2Gx+2Fy+C=0,$$

y can indeed be so expressed, viz. by the formula

$$y=-F+\sqrt{(F^2-x^2-2Gx-C)};$$

but the functional dependence of y upon x is better and more simply expressed by the original equation.

It will be observed that in these two cases the functional relation is fully expressed *by equating a function of the two variables x and y to zero*, *i.e.* by means of an equation

$$f(x, y)=0 \qquad\qquad (2).$$

We shall adopt this equation as the standard method of expressing the functional relation. It includes the equation (1) as a special case, since $y-f(x)$ is a special form of a function of x and y. We can then speak of the locus of the point (x, y) subject to $f(x, y)=0$, the graph of the function y defined by $f(x, y)=0$, the curve or locus $f(x, y)=0$, and the equation of this curve or locus.

There is another method of representing curves which is often useful. Suppose that x and y are both functions of a third variable t, which is to be regarded as essentially auxiliary and devoid of any particular geometrical significance. We may write

$$x=f(t), \quad y=F(t) \qquad\qquad (3).$$

If a particular value is assigned to t, the corresponding values of x and of y are known. Each pair of such values defines a point

(x, y). If we construct all the points which correspond in this way to different values of t, we obtain *the graph of the locus defined by the equations* (3). Suppose for example

$$x=a\cos t, \quad y=a\sin t.$$

Let t vary from 0 to 2π. Then it is easy to see that the point (x, y) describes the circle whose centre is the origin and whose radius is a. If t varies beyond these limits, (x, y) describes the circle over and over again. We can in this case at once obtain a direct relation between x and y by squaring and adding: we find that $x^2+y^2=a^2$, t being now eliminated.

Examples XVIII. 1. The points of intersection of the two curves whose equations are $f(x, y)=0$, $\phi(x, y)=0$, where f and ϕ are polynomials, can be determined if these equations can be solved as a pair of simultaneous equations in x and y. The solution generally consists of a finite number of pairs of values of x and y. The two equations therefore generally represent a finite number of isolated points.

2. Trace the curves $(x+y)^2=1$, $xy=1$, $x^2-y^2=1$.

3. The curve $f(x, y)+\lambda\phi(x, y)=0$ represents a curve passing through the points of intersection of $f=0$ and $\phi=0$.

4. What loci are represented by

(α) $x=at+b$, $y=ct+d$, (β) $x/a=2t/(1+t^2)$, $y/a=(1-t^2)/(1+t^2)$,

when t varies through all real values?

33. Loci in space. In space of three dimensions there are two fundamentally different kinds of loci, of which the simplest examples are the plane and the straight line.

A particle which moves along a straight line has only *one degree of freedom*. Its direction of motion is fixed; its position can be completely fixed by one measurement of position, *e.g.* by its distance from a fixed point on the line. If we take the line as our fundamental line Λ of Chap. I, the position of any of its points is determined by a single coordinate x. A particle which moves in a plane, on the other hand, has *two* degrees of freedom; its position can only be fixed by the determination of two coordinates.

A locus represented by a single equation

$$z=f(x, y)$$

plainly belongs to the second of these two classes of loci, and is called a *surface*. It may or may not (in the obvious simple cases it will) satisfy our common-sense notion of what a surface should be.

The considerations of § 31 may evidently be generalised so as to give definitions of a function $f(x, y, z)$ of *three* variables (or of functions of any number of variables). And as in § 32 we agreed to adopt $f(x, y) = 0$ as the standard form of the equation of a plane curve, so now we shall agree to adopt

$$f(x, y, z) = 0$$

as the standard form of equation of a surface.

The locus represented by *two* equations of the form $z = f(x, y)$ or $f(x, y, z) = 0$ belongs to the first class of loci, and is called a *curve*. Thus a *straight line* may be represented by two equations of the type $Ax + By + Cz + D = 0$. A *circle* in space may be regarded as the intersection of a sphere and a plane; it may therefore be represented by two equations of the forms

$$(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 = \rho^2, \quad Ax + By + Cz + D = 0.$$

Examples XIX. 1. What is represented by *three* equations of the type $f(x, y, z)=0$?

2. Three linear equations in general represent a single point. What are the exceptional cases?

3. What are the equations of a plane curve $f(x, y)=0$ in the plane XOY, when regarded as a curve in space? [$f(x, y)=0$, $z=0$.]

4. **Cylinders.** What is the meaning of a single equation $f(x, y)=0$, considered as a locus in space of three dimensions?

[All points on the surface satisfy $f(x, y)=0$, whatever be the value of z. The curve $f(x, y)=0$, $z=0$ is the curve in which the locus cuts the plane XOY. The locus is the surface formed by drawing lines parallel to OZ through all points of this curve. Such a surface is called a *cylinder*.]

5 **Graphical representation of a surface on a plane. Contour Maps.** It might seem to be impossible to represent a surface adequately by a drawing on a plane; and so indeed it is: but a very fair notion of the nature of the surface may often be obtained as follows. Let the equation of the surface be $z=f(x, y)$.

If we give z a particular value a, we have an equation $f(x, y)=a$, which we may regard as determining a plane curve on the paper. We trace this curve and mark it (a). Actually the curve (a) is the projection on the plane

XOY of the section of the surface by the plane $z=a$. We do this for all values of a (practically, of course, for a selection of values of a). We obtain some such figure as is shown in Fig. 17. It will at once suggest a contoured Ordnance Survey map: and in fact this is the principle on which such maps are constructed. The contour line 1000 is the projection, on the plane of the sea level, of the section of the surface of the land by the plane parallel to the plane of the sea level and 1000 ft. above it*.

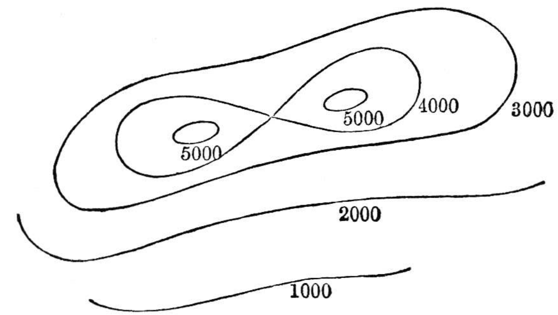

Fig. 17.

6. Draw a series of contour lines to illustrate the form of the surface $2z=3xy$.

7. **Right circular cones.** Take the origin of coordinates at the vertex of the cone and the axis of z along the axis of the cone; and let a be the semi-vertical angle of the cone. The equation of the cone (which must be regarded as extending both ways from its vertex) is $x^2+y^2-z^2\tan^2 a=0$.

8. **Surfaces of revolution in general.** The cone of Ex. 7 cuts ZOX in two lines whose equations may be combined in the equation $x^2=z^2\tan^2 a$. That is to say, the equation of the surface generated by the revolution of the curve $y=0$, $x^2=z^2\tan^2 a$ round the axis of z is derived from the second of these equations by changing x^2 into x^2+y^2. Show generally that the equation of the surface generated by the revolution of the curve $y=0$, $x=f(z)$, round the axis of z, is

$$\sqrt{(x^2+y^2)}=f(z).$$

9. **Cones in general.** A surface formed by straight lines passing through a fixed point is called a *cone*: the point is called the *vertex*. A particular case is given by the right circular cone of Ex. 7. Show that the equation of a cone whose vertex is O is of the form $f(z/x, z/y)=0$, and that any equation of this form represents a cone. [If (x, y, z) lies on the cone, so must $(\lambda x, \lambda y, \lambda z)$, for any value of λ.]

* We assume that the effects of the earth's curvature may be neglected.

10. **Ruled surfaces.** Cylinders and cones are special cases of *surfaces composed of straight lines.* Such surfaces are called *ruled surfaces.*

The two equations

$$x=az+b, \quad y=cz+d \quad \text{.............................(1)}$$

represent the intersection of two planes, *i.e.* a straight line. Now suppose that a, b, c, d instead of being fixed are *functions of an auxiliary variable* t. For any particular value of t the equations (1) give a line. As t varies, this line moves and generates a surface, whose equation may be found by eliminating t between the two equations (1). For instance, in Ex. 7 the equations of the line which generates the cone are

$$x=z\tan\alpha\cos t, \quad y=z\tan\alpha\sin t,$$

where t is the angle between the plane XOZ and a plane through the line and the axis of z.

Another simple example of a ruled surface may be constructed as follows. Take two sections of a right circular cylinder perpendicular to the axis and at a distance l apart (Fig. 18 *a*). We can imagine the surface of the cylinder to be made up of a number of thin parallel rigid rods of length l, such as PQ, the ends of the rods being fastened to two circular rods of radius a.

Now let us take a third circular rod of the same radius and place it round the surface of the cylinder at a distance h from one of the first two rods (see Fig. 18 *a*, where $Pq=h$). Unfasten the end Q of the rod PQ and turn PQ about P until Q can be fastened to the third circular rod in the position Q'. The angle $qOQ'=\alpha$ in the figure is evidently given by

$$l^2-h^2=qQ'^2=(2a\sin\tfrac{1}{2}\alpha)^2.$$

Let all the other rods of which the cylinder was composed be treated in the same way. We obtain a ruled surface whose form is indicated in Fig. 18 *b*. It is entirely built up of straight lines; but the surface is curved everywhere, and is in general shape not unlike certain forms of table-napkin rings (Fig. 18 *c*).

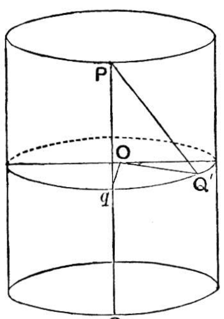

Fig. 18 *a*.

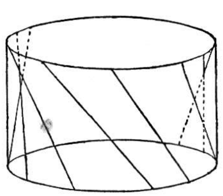

Fig. 18 *b*.

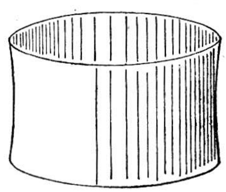

Fig. 18 *c*.

MISCELLANEOUS EXAMPLES ON CHAPTER II.

1. Show that if $y=f(x)=(ax+b)/(cx-a)$ then $x=f(y)$.

2. If $f(x)=f(-x)$ for all values of x, $f(x)$ is called an *even* function. If $f(x)=-f(-x)$, it is called an *odd* function. Show that any function of x, defined for all values of x, is the sum of an even and an odd function of x. [Use the identity $f(x)=\frac{1}{2}\{f(x)+f(-x)\}+\frac{1}{2}\{f(x)-f(-x)\}$.]

3. Draw the graphs of the functions

$$3\sin x+4\cos x, \quad \sin\left(\frac{\pi}{\sqrt{2}}\sin x\right). \qquad (\textit{Math. Trip. } 1896.)$$

4. Draw the graphs of the functions

$$\sin x\,(a\cos^2 x+b\sin^2 x), \quad \frac{\sin x}{x}(a\cos^2 x+b\sin^2 x), \quad \left(\frac{\sin x}{x}\right)^2.$$

5. Draw the graphs of the functions $x[1/x]$, $[x]/x$.

6. Draw the graphs of the functions

(i) $\operatorname{arc}\cos(2x^2-1)-2\operatorname{arc}\cos x$,

(ii) $\operatorname{arc}\tan\dfrac{a+x}{1-ax}-\operatorname{arc}\tan a-\operatorname{arc}\tan x$,

where the symbols $\operatorname{arc}\cos a$, $\operatorname{arc}\tan a$ denote, for any value of a, the least positive (or zero) angle, whose cosine or tangent is a.

7. Verify the following method of constructing the graph of $f\{\phi(x)\}$ by means of the line $y=x$ and the graphs of $f(x)$ and $\phi(x)$: take $OA=x$ along OX, draw AB parallel to OY to meet $y=\phi(x)$ in B, BC parallel to OX to meet $y=x$ in C, CD parallel to OY to meet $y=f(x)$ in D, and DP parallel to OX to meet AB in P; then P is a point on the graph required.

8. Show that the roots of $x^3+px+q=0$ are the abscissae of the points of intersection (other than the origin) of the parabola $y=x^2$ and the circle

$$x^2+y^2+(p-1)y+qx=0.$$

9. The roots of $x^4+nx^3+px^2+qx+r=0$ are the abscissae of the points of intersection of the parabola $x^2=y-\frac{1}{2}nx$ and the circle

$$x^2+y^2+(\tfrac{1}{8}n^2-\tfrac{1}{2}pn+\tfrac{1}{2}n+q)x+(p-1-\tfrac{1}{4}n^2)y+r=0.$$

10. Discuss the graphical solution of the equation

$$x^m+ax^2+bx+c=0$$

by means of the curves $y=x^m$, $y=-ax^2-bx-c$. Draw up a table of the various possible numbers of roots.

11. Solve the equation $\sec\theta+\operatorname{cosec}\theta=2\sqrt{2}$; and show that the equation $\sec\theta+\operatorname{cosec}\theta=c$ has two roots between 0 and 2π if $c^2<8$ and four if $c^2>8$.