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A COURSE OF PURE MATHEMATICS

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EXTRACT FROM THE PREFACE TO THE FIRST EDITION

This book has been designed primarily for the use of first year students at the Universities whose abilities reach or approach something like what is usually described as 'scholarship standard'. I hope that it may be useful to other classes of readers, but it is this class whose wants I have considered first. It is in any case a book for mathematicians: I have nowhere made any attempt to meet the needs of students of engineering or indeed any class of students whose interests are not primarily mathematical.

I regard the book as being really elementary. There are plenty of hard examples (mainly at the ends of the chapters): to these I have added, wherever space permitted, an outline of the solution. But I have done my best to avoid the inclusion of anything that involves really difficult ideas. For instance, I make no use of the 'principle of convergence': uniform convergence, double series, infinite products, are never alluded to: and I prove no general theorems whatever concerning the inversion of limit-operations—I never even define $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$. In the last two chapters I have occasion once or twice to integrate a power-series, but I have confined myself to the very simplest cases and given a special discussion in each instance. Anyone who has read this book will be in a position to read with profit Dr Bromwich's *Infinite Series*, where a full and adequate discussion of all these points will be found.

G. H. H.

September 1908

CHAPTER X

THE GENERAL THEORY OF THE LOGARITHMIC, EXPONENTIAL, AND CIRCULAR FUNCTIONS

CHAPTER I

REAL VARIABLES

1. Rational numbers. A fraction $r = p/q$, where p and q are positive or negative integers, is called a *rational number*. We can suppose (i) that p and q have no common factor, as if they have a common factor we can divide each of them by it, and (ii) that q is positive, since

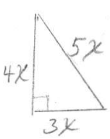
$$p/(-q) = (-p)/q, \quad (-p)/(-q) = p/q.$$

To the rational numbers thus defined we may add the 'rational number 0' obtained by taking $p = 0$.

We assume that the reader is familiar with the ordinary arithmetical rules for the manipulation of rational numbers. The examples which follow demand no knowledge beyond this.

Examples I. 1. If r and s are rational numbers, then $r+s$, $r-s$, rs, and r/s are rational numbers, unless in the last case $s=0$ (when r/s is of course meaningless).

2. If λ, m, and n are positive rational numbers, and $m > n$, then $\lambda(m^2-n^2)$, $2\lambda mn$, and $\lambda(m^2+n^2)$ are positive rational numbers. Hence show how to determine any number of right-angled triangles the lengths of all of whose sides are rational.



3. Any terminated decimal represents a rational number whose denominator contains no factors other than 2 or 5. Conversely, any such rational number can be expressed, and in one way only, as a terminated decimal.

[The general theory of decimals will be considered in Ch. IV.]

4. The positive rational numbers may be arranged in the form of a simple series as follows:

$$\tfrac{1}{1}, \tfrac{2}{1}, \tfrac{1}{2}, \tfrac{3}{1}, \tfrac{2}{2}, \tfrac{1}{3}, \tfrac{4}{1}, \tfrac{3}{2}, \tfrac{2}{3}, \tfrac{1}{4}, \ldots.$$

Show that p/q is the $[\tfrac{1}{2}(p+q-1)(p+q-2)+q]$th term of the series.

[In this series every rational number is repeated indefinitely. Thus 1 occurs as $\tfrac{1}{1}, \tfrac{2}{2}, \tfrac{3}{3}, \ldots$. We can of course avoid this by omitting every number which has already occurred in a simpler form, but then the problem of determining the precise position of p/q becomes more complicated.]

2. The representation of rational numbers by points on a line. It is convenient, in many branches of mathematical analysis, to make a good deal of use of geometrical illustrations.

The use of geometrical illustrations in this way does not, of course, imply that analysis has any sort of dependence upon geometry: they are illustrations and nothing more, and are employed merely for the sake of clearness of exposition. This being so, it is not necessary that we should attempt any logical analysis of the ordinary notions of elementary geometry; we may be content to suppose, however far it may be from the truth, that we know what they mean.

Assuming, then, that we know what is meant by a *straight line*, a *segment* of a line, and the *length* of a segment, let us take a straight line Λ, produced indefinitely in both directions, and a segment A_0A_1 of any length. We call A_0 the *origin*, or *the point* 0, and A_1 *the point* 1, and we regard these points as representing the numbers 0 and 1.

In order to obtain a point which shall represent a positive rational number $r = p/q$, we choose the point A_r such that

$$A_0A_r/A_0A_1 = r,$$

A_0A_r being a stretch of the line extending in the same direction along the line as A_0A_1, a direction which we shall suppose to be from left to right when, as in Fig. 1, the line is drawn horizontally across the paper. In order to obtain a point to represent a

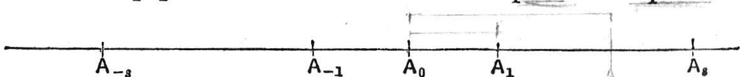


Fig. 1.

negative rational number $r = -s$, it is natural to regard length as a magnitude capable of sign, positive if the length is measured in one direction (that of A_0A_1), and negative if measured in the other, so that $AB = -BA$; and to take as the point representing r the point A_{-s} such that

$$A_0A_{-s} = -A_{-s}A_0 = -A_0A_s.$$

We thus obtain a point A_r on the line corresponding to every rational value of r, positive or negative, and such that

$$A_0A_r = r \,.\, A_0A_1;$$

and if, as is natural, we take A_0A_1 as our unit of length, and write $A_0A_1 = 1$, then we have

$$A_0A_r = r.$$

We shall call the points A_r the *rational points* of the line.

3. Irrational numbers. If the reader will mark off on the line all the points corresponding to the rational numbers whose denominators are 1, 2, 3, ... in succession, he will readily convince himself that he can cover the line with rational points as closely as he likes. We can state this more precisely as follows: *if we take any segment BC on Λ, we can find as many rational points as we please on BC.*

Suppose, for example, that BC falls within the segment A_1A_2. It is evident that if we choose a positive integer k so that

$$k \,.\, BC > 1 \quad \text{.........................(1)},^*$$

and divide A_1A_2 into k equal parts, then at least one of the points of division (say P) must fall inside BC, without coinciding with either B or C. For if this were not so, BC would be entirely included in one of the k parts into which A_1A_2 has been divided, which contradicts the supposition (1). But P obviously corresponds to a rational number whose denominator is k. Thus at least one rational point P lies between B and C. But then we can find another such point Q between B and P, another between B and Q, and so on indefinitely; *i.e.*, as we asserted above, we can find as many as we please. We may express this by saying that BC includes *infinitely many* rational points.

The meaning of such phrases as '*infinitely many*' or '*an infinity of*', in such sentences as 'BC includes infinitely many rational points' or 'there are an infinity of rational points on BC' or 'there are an infinity of positive integers', will be considered more closely in Ch. IV. The assertion 'there are an infinity of positive integers' means 'given any positive integer n, however large, we can find more than n positive integers'. This is plainly true

* The assumption that this is possible is equivalent to the assumption of what is known as the Axiom of Archimedes.

whatever n may be, *e.g.* for $n = 100{,}000$ or $100{,}000{,}000$. The assertion means exactly the same as 'we can find *as many positive integers as we please*'.

The reader will easily convince himself of the truth of the following assertion, which is substantially equivalent to what was proved in the second paragraph of this section: given any rational number r, and any positive integer n, we can find another rational number lying on either side of r and differing from r by less than $1/n$. It is merely to express this differently to say that we can find a rational number lying on either side of r and differing from r *by as little as we please*. Again, given any two rational numbers r and s, we can interpolate between them a chain of rational numbers in which any two consecutive terms differ by as little as we please, that is to say by less than $1/n$, where n is any positive integer assigned beforehand.

From these considerations the reader might be tempted to infer that an adequate view of the nature of the line could be obtained by imagining it to be formed simply by the rational points which lie on it. And it is certainly the case that if we imagine the line to be made up solely of the rational points, and all other points (if there are any such) to be eliminated, the figure which remained would possess most of the properties which common sense attributes to the straight line, and would, to put the matter roughly, look and behave very much like a line.

A little further consideration, however, shows that this view would involve us in serious difficulties.

Let us look at the matter for a moment with the eye of common sense, and consider some of the properties which we may reasonably expect a straight line to possess if it is to satisfy the idea which we have formed of it in elementary geometry.

The straight line must be composed of points, and any segment of it by all the points which lie between its end points. With any such segment must be associated a certain entity called its *length*, which must be a *quantity* capable of *numerical measurement* in terms of any standard or unit length, and these lengths must be capable of combination with one another, according to the ordinary rules of algebra, by means of addition or multiplication. Again, it must be possible to construct a line whose length is the sum or product of any two given lengths. If the length PQ, along a given line, is a, and the length QR, along the same straight line, is b, the length PR must be $a + b$.

Moreover, if the lengths OP, OQ, along one straight line, are 1 and a, and the length OR along another straight line is b, and if we determine the length OS by Euclid's construction (Euc. VI. 12) for a fourth proportional to the lines OP, OQ, OR, this length must be ab, the algebraical fourth proportional to 1, a, b. And it is hardly necessary to remark that the sums and products thus defined must obey the ordinary 'laws of algebra'; viz.

$$a + b = b + a, \quad a + (b + c) = (a + b) + c,$$
$$ab = ba, \quad a(bc) = (ab)c, \quad a(b + c) = ab + ac.$$

The lengths of our lines must also obey a number of obvious laws concerning inequalities as well as equalities: thus if A, B, C are three points lying along Λ from left to right, we must have $AB < AC$, and so on. Moreover it must be possible, on our fundamental line Λ, to find a point P such that A_0P is equal to any segment whatever taken along Λ or along any other straight line. All these properties of a line, and more, are involved in the presuppositions of our elementary geometry.

Now it is very easy to see that the idea of a straight line as composed of a series of points, each corresponding to a rational number, cannot possibly satisfy all these requirements. There are various elementary geometrical constructions, for example, which purport to construct a length x such that $x^2 = 2$. For instance, we

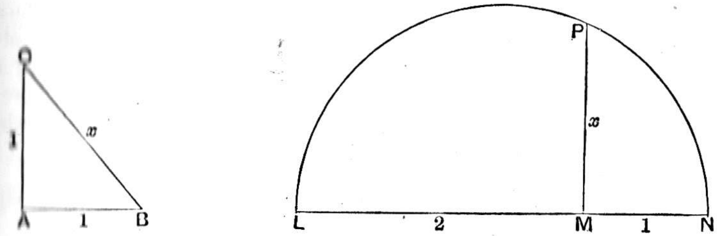


Fig. 2.

may construct an isosceles right-angled triangle ABC such that $AB = AC = 1$. Then if $BC = x$, $x^2 = 2$. Or we may determine the length x by means of Euclid's construction (Euc. VI. 13) for a mean proportional to 1 and 2, as indicated in the figure. Our requirements therefore involve the existence of a length measured by a number x, and a point P on Λ such that

$$A_0P = x, \quad x^2 = 2.$$

But it is easy to see that *there is no rational number such that its square is* 2. In fact we may go further and say that there is no rational number whose square is m/n, where m/n is any positive fraction in its lowest terms, unless m and n are both perfect squares.

For suppose, if possible, that

$$p^2/q^2 = m/n.$$

p having no factor in common with q, and m no factor in common with n. Then $np^2 = mq^2$. Every factor of q^2 must divide np^2, and as p and q have no common factor, every factor of q^2 must divide n. Hence $n = \lambda q^2$, where λ is an integer. But this involves $m = \lambda p^2$: and as m and n have no common factor, λ must be unity. Thus $m = p^2$, $n = q^2$, as was to be proved. In particular it follows, by taking $n = 1$, that an integer cannot be the square of a rational number, unless that rational number is itself integral.

It appears then that our requirements involve the existence of a number x and a point P, not one of the rational points already constructed, such that $A_0P = x$, $x^2 = 2$; and (as the reader will remember from elementary algebra) we write $x = \sqrt{2}$.

The following alternative proof that no rational number can have its square equal to 2 is interesting.

Suppose, if possible, that p/q is a positive fraction, in its lowest terms, such that $(p/q)^2 = 2$ or $p^2 = 2q^2$. It is easy to see that this involves $(2q-p)^2 = 2(p-q)^2$; and so $(2q-p)/(p-q)$ is another fraction having the same property. But clearly $q < p < 2q$, and so $p - q < q$. Hence there is another fraction equal to p/q and having a smaller denominator, which contradicts the assumption that p/q is in its lowest terms.

Examples II. 1. Show that no rational number can have its cube equal to 2.

2. Prove generally that a rational fraction p/q in its lowest terms cannot be the cube of a rational number unless p and q are both perfect cubes.

3. A more general proposition, which is due to Gauss and includes those which precede as particular cases, is the following: *an algebraical equation*

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \dots + p_n = 0,$$

with integral coefficients, cannot have a rational but non-integral root.

[For suppose that the equation has a root a/b, where a and b are integers

without a common factor, and b is positive. Writing a/b for x, and multiplying by b^{n-1}, we obtain

$$-\frac{a^n}{b} = p_1 a^{n-1} + p_2 a^{n-2} b + \dots + p_n b^{n-1},$$

a fraction in its lowest terms equal to an integer, which is absurd. Thus $b = 1$, and the root is a. It is evident that a must be a divisor of p_n.]

4. Show that if $p_n = 1$ and neither of

$$1 + p_1 + p_2 + p_3 + \dots, \quad 1 - p_1 + p_2 - p_3 + \dots$$

is zero, then the equation cannot have a rational root.

5. Find the rational roots (if any) of

$$x^4 - 4x^3 - 8x^2 + 13x + 10 = 0.$$

[The roots can only be integral, and so ± 1, ± 2, ± 5, ± 10 are the only possibilities: whether these are roots can be determined by trial. It is clear that we can in this way determine the rational roots of any such equation.]

4. Irrational numbers (*continued*). The result of our geometrical representation of the rational numbers is therefore to suggest the desirability of enlarging our conception of 'number' by the introduction of further numbers of a new kind.

The same conclusion might have been reached without the use of geometrical language. One of the central problems of algebra is that of the solution of equations, such as

$$x^2 = 1, \quad x^2 = 2.$$

The first equation has the two rational roots 1 and -1. But, if our conception of number is to be limited to the rational numbers, we can only say that the second equation has no roots; and the same is the case with such equations as $x^3 = 2$, $x^4 = 7$. These facts are plainly sufficient to make some generalisation of our idea of number desirable, if it should prove to be possible.

Let us consider more closely the equation $x^2 = 2$.

We have already seen that there is no rational number x which satisfies this equation. The square of any rational number is either less than or greater than 2. We can therefore divide the positive rational numbers (to which for the present we confine our attention) into two classes, one containing the numbers whose squares are less than 2, and the other those whose squares are greater than 2. We shall call these two classes *the class L*, or *the lower class*, or *the left-hand class*, and *the class R*, or *the upper* *class*, or *the right-hand class*. It is obvious that every member of R is greater than all the members of L. Moreover it is easy to convince ourselves that we can find a member of the class L whose square, though less than 2, differs from 2 by as little as we please, and a member of R whose square, though greater than 2, also differs from 2 by as little as we please. In fact, if we carry out the ordinary arithmetical process for the extraction of the square root of 2, we obtain a series of rational numbers, viz.

$$1,\ 1{\cdot}4,\ 1{\cdot}41,\ 1{\cdot}414,\ 1{\cdot}4142, \ldots$$

whose squares

$$1,\ 1{\cdot}96,\ 1{\cdot}9881,\ 1{\cdot}999396,\ 1{\cdot}99996164, \ldots$$

are all less than 2, but approach nearer and nearer to it; and by taking a sufficient number of the figures given by the process we can obtain as close an approximation as we want. And if we increase the last figure, in each of the approximations given above, by unity, we obtain a series of rational numbers

$$2,\ 1{\cdot}5,\ 1{\cdot}42,\ 1{\cdot}415,\ 1{\cdot}4143, \ldots$$

whose squares

$$4,\ 2{\cdot}25,\ 2{\cdot}0164,\ 2{\cdot}002225,\ 2{\cdot}00024449, \ldots$$

are all greater than 2 but approximate to 2 as closely as we please.

The reasoning which precedes, although it will probably convince the reader, is hardly of the precise character required by modern mathematics. We can supply a formal proof as follows. In the first place, we can find a member of L and a member of R, differing by as little as we please. For we saw in § 3 that, given any two rational numbers a and b, we can construct a chain of rational numbers, of which a and b are the first and last, and in which any two consecutive numbers differ by as little as we please. Let us then take a member x of L and a member y of R, and interpolate between them a chain of rational numbers of which x is the first and y the last, and in which any two consecutive numbers differ by less than δ, δ being any positive rational number as small as we please, such as ·01 or ·0001 or ·000001. In this chain there must be a last which belongs to L and a first which belongs to R, and these two numbers differ by less than δ.

We can now prove that *an x can be found in L and a y in R such that $2-x^2$ and y^2-2 are as small as we please*, say less than δ. Substituting $\frac{1}{4}\delta$ for δ in the argument which precedes, we see that we can choose x and y so that $y-x<\frac{1}{4}\delta$; and we may plainly suppose that both x and y are less than 2. Thus

$$y+x<4, \quad y^2-x^2=(y-x)(y+x)<4(y-x)<\delta;$$

and since $x^2<2$ and $y^2>2$ it follows *a fortiori* that $2-x^2$ and y^2-2 are each less than δ.

It follows also that *there can be no largest member of L or smallest member of R*. For if x is any member of L, then $x^2<2$. Suppose that $x^2=2-\delta$. Then we can find a member x_1 of L such that x_1^2 differs from 2 by less than δ, and so $x_1^2>x^2$ or $x_1>x$. Thus there are larger members of L than x; and as x is *any* member of L, it follows that no member of L can be larger than all the rest. Hence L has no largest member, and similarly R has no smallest.

5. Irrational numbers (*continued*). We have thus divided the positive rational numbers into two classes, L and R, such that (i) every member of R is greater than every member of L, (ii) we can find a member of L and a member of R whose difference is as small as we please, (iii) L has no greatest and R no least member. Our common-sense notion of the attributes of a straight line, the requirements of our elementary geometry and our elementary algebra, alike demand *the existence of a number x greater than all the members of L and less than all the members of R, and of a corresponding point P on Λ such that P divides the points which correspond to members of L from those which correspond to members of R.*

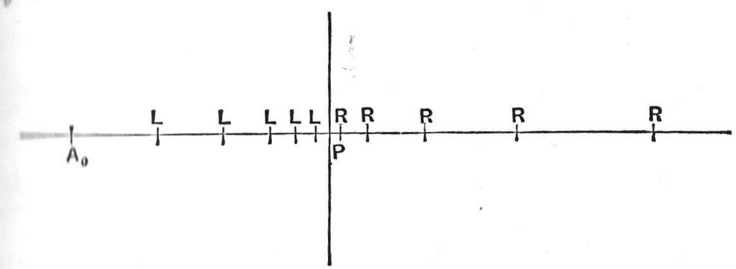


Fig. 3.

Let us suppose for a moment that there is such a number x, and that it may be operated upon in accordance with the laws of algebra, so that, for example, x^2 has a definite meaning. Then x^2 cannot be either less than or greater than 2. For suppose, for example, that x^2 is less than 2. Then it follows from what precedes that we can find a positive rational number ξ such that ξ^2 lies between x^2 and 2. That is to say, we can find a member of L greater than x; and this contradicts the supposition that x divides the members of L from those of R. Thus x^2 cannot be less than 2, and similarly it cannot be greater than 2. We are therefore driven to the conclusion that $x^2 = 2$, and that x is the number which in algebra we denote by $\sqrt{2}$. And of course this number $\sqrt{2}$ is not rational, for no rational number has its square equal to 2. It is the simplest example of what is called an **irrational** number.

But the preceding argument may be applied to equations other than $x^2 = 2$, almost word for word; for example to $x^2 = N$, where N is any integer which is not a perfect square, or to

$$x^3 = 3, \quad x^3 = 7, \quad x^4 = 23,$$

or, as we shall see later on, to $x^3 = 3x + 8$. We are thus led to believe in the existence of irrational numbers x and points P on Λ such that x satisfies equations such as these, even when these lengths cannot (as $\sqrt{2}$ can) be constructed by means of elementary geometrical methods.

The reader will no doubt remember that in treatises on elementary algebra the root of such an equation as $x^q = n$ is denoted by $\sqrt[q]{n}$ or $n^{1/q}$, and that a meaning is attached to such symbols as

$$n^{p/q}, \quad n^{-p/q}$$

by means of the equations

$$n^{p/q} = (n^{1/q})^p, \quad n^{p/q} n^{-p/q} = 1.$$

And he will remember how, in virtue of these definitions, the 'laws of indices' such as

$$n^r \times n^s = n^{r+s}, \quad (n^r)^s = n^{rs}$$

are extended so as to cover the case in which r and s are any rational numbers whatever.

The reader may now follow one or other of two alternative courses. He may, if he pleases, be content to assume that 'irrational numbers' such as $\sqrt{2}$, $\sqrt[3]{3}$, ... exist and are amenable to the algebraical laws with which he is familiar*. If he does this he will be able to avoid the more abstract discussions of the next few sections, and may pass on at once to §§ 13 *et seq.*

If, on the other hand, he is not disposed to adopt so *naive* an

* This is the point of view which was adopted in the first edition of this book.

attitude, he will be well advised to pay careful attention to the sections which follow, in which these questions receive fuller consideration*.

Examples III. 1. Find the difference between 2 and the squares of the decimals given in § 4 as approximations to $\sqrt{2}$.

2. Find the differences between 2 and the squares of

$$\tfrac{1}{1},\ \tfrac{3}{2},\ \tfrac{7}{5},\ \tfrac{17}{12},\ \tfrac{41}{29},\ \tfrac{99}{70}.$$

3. Show that if m/n is a good approximation to $\sqrt{2}$, then $(m+2n)/(m+n)$ is a better one, and that the errors in the two cases are in opposite directions. Apply this result to continue the series of approximations in the last example.

4. If x and y are approximations to $\sqrt{2}$, by defect and by excess respectively, and $2 - x^2 < \delta$, $y^2 - 2 < \delta$, then $y - x < \delta$.

5. The equation $x^2 = 4$ is satisfied by $x = 2$. Examine how far the argument of the preceding sections applies to this equation (writing 4 for 2 throughout). [If we define the classes L, R as before, they do not include *all* rational numbers. The rational number 2 is an exception, since 2^2 is neither less than or greater than 4.]

6. Irrational numbers (*continued*). In § 4 we discussed a special mode of division of the positive rational numbers x into two classes, such that $x^2 < 2$ for the members of one class and $x^2 > 2$ for those of the others. Such a mode of division is called a **section** of the numbers in question. It is plain that we could equally well construct a section in which the numbers of the two classes were characterised by the inequalities $x^3 < 2$ and $x^3 > 2$, or $x^4 < 7$ and $x^4 > 7$. Let us now attempt to state the principles of the construction of such 'sections' of the positive rational numbers in quite general terms.

Suppose that P and Q stand for two properties which are mutually exclusive and one of which must be possessed by every positive rational number. Further, suppose that every such number which possesses P is less than any such number which possesses Q. Thus P might be the property '$x^2 < 2$' and Q the property '$x^2 > 2$.' Then we call the numbers which possess P the lower or left-hand class L and those which possess Q the upper or

* In these sections I have borrowed freely from Appendix I of Bromwich's *Infinite Series*.

right-hand class R. In general both classes will exist; but it may happen in special cases that one is non-existent and that every number belongs to the other. This would obviously happen, for example, if P (or Q) were the property of being rational, or of being positive. For the present, however, we shall confine ourselves to cases in which both classes do exist; and then it follows, as in § 4, that we can find a member of L and a member of R whose difference is as small as we please.

In the particular case which we considered in § 4, L had no greatest member and R no least. This question of the existence of greatest or least members of the classes is of the utmost importance. We observe first that it is impossible in any case that L should have a greatest member *and* R a least. For if l were the greatest member of L, and r the least of R, so that $l < r$, then $\frac{1}{2}(l+r)$ would be a positive rational number lying between l and r, and so could belong neither to L nor to R; and this contradicts our assumption that every such number belongs to one class or to the other. This being so, there are but three possibilities, which are mutually exclusive. Either (i) L has a greatest member l, or (ii) R has a least member r, or (iii) L has no greatest member and R no least.

The section of § 4 gives an example of the last possibility. An example of the first is obtained by taking P to be '$x^2 \leqq 1$' and Q to be '$x^2 > 1$'; here $l = 1$. If P is '$x^2 < 1$' and Q is '$x^2 \geqq 1$,' we have an example of the second possibility, with $r = 1$. It should be observed that we do not obtain a section at all by taking P to be '$x^2 < 1$' and Q to be '$x^2 > 1$'; for the special number 1 escapes classification (cf. Ex. III. 5).

7. Irrational numbers (*continued*). In the first two cases we say that the section *corresponds to* a positive rational number a, which is l in the one case and r in the other. Conversely, it is clear that to any such number a corresponds a section which we shall denote by α*. For we might take P and Q to be the properties expressed by

$$x \leqq a, \quad x > a$$

respectively, or by $x < a$ and $x \geqq a$. In the first case a would be the greatest member of L, and in the second case the least member

* It will be convenient to denote a section, corresponding to a rational number denoted by an English letter, by the corresponding Greek letter.

of R. There are in fact just two sections corresponding to any positive rational number. In order to avoid ambiguity we select one of them; let us select that in which the number itself belongs to the *upper* class. In other words, let us agree that we will consider only sections in which the lower class L has no greatest number.

There being this correspondence between the positive rational numbers and the sections defined by means of them, it would be perfectly legitimate, for mathematical purposes, to replace the numbers by the sections, and to regard the symbols which occur in our formulae as standing for the sections instead of for the numbers. Thus, for example, $\alpha > \alpha'$ would mean the same as $a > a'$, if α and α' are the sections which correspond to a and a'.

But when we have in this way substituted sections of rational numbers for the rational numbers themselves, we are almost forced to a generalisation of our number system. For there are sections (such as that of § 4) which do *not* correspond to any rational number. The aggregate of sections is a larger aggregate than that of the positive rational numbers; it includes sections corresponding to all these numbers, and more besides. It is this fact which we make the basis of our generalisation of the idea of number. We accordingly frame the following definitions, which will however be modified in the next section, and must therefore be regarded as temporary and provisional.

A section of the positive rational numbers, in which both classes exist and the lower class has no greatest member, is called a **positive real number.**

A positive real number which does not correspond to a positive rational number is called a positive **irrational** *number.*

8. Real numbers. We have confined ourselves so far to certain sections of the positive rational numbers, which we have agreed provisionally to call 'positive real numbers.' Before we frame our final definitions, we must alter our point of view a little. We shall consider sections, or divisions into two classes, not merely of the positive rational numbers, but of all rational numbers, including zero. We may then repeat all that we have said about sections of the positive rational numbers in §§ 6, 7, merely omitting the word positive occasionally.

DEFINITIONS. *A section of the rational numbers, in which both classes exist and the lower class has no greatest member, is called a* **real number**, *or simply a* **number.**

A real number which does not correspond to a rational number is called an **irrational** *number.*

If the real number does correspond to a rational number, we shall use the term 'rational' as applying to the real number also.

The term 'rational number' will, as a result of our definitions, be ambiguous; it may mean the rational number of § 1, or the corresponding real number. If we say that $\frac{1}{2} > \frac{1}{3}$, we may be asserting either of two different propositions, one a proposition of elementary arithmetic, the other a proposition concerning sections of the rational numbers. Ambiguities of this kind are common in mathematics, and are perfectly harmless, since the relations between different propositions are exactly the same whichever interpretation is attached to the propositions themselves. From $\frac{1}{2} > \frac{1}{3}$ and $\frac{1}{3} > \frac{1}{4}$ we can infer $\frac{1}{2} > \frac{1}{4}$; the inference is in no way affected by any doubt as to whether $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ are arithmetical fractions or real numbers. Sometimes, of course, the context in which (*e.g.*) '$\frac{1}{2}$' occurs is sufficient to fix its interpretation. When we say (see § 9) that $\frac{1}{2} < \sqrt{(\frac{1}{3})}$, we *must* mean by '$\frac{1}{2}$' the real number $\frac{1}{2}$.

The reader should observe, moreover, that no particular logical importance is to be attached to the precise form of definition of a 'real number' that we have adopted. We defined a 'real number' as being a section, *i.e.* a pair of classes. We might equally well have defined it as being the lower, or the upper, class; indeed it would be easy to define an infinity of classes of entities each of which would possess the properties of the class of real numbers. What is essential in mathematics is that its symbols should be capable of *some* interpretation; generally they are capable of *many*, and then, so far as mathematics is concerned, it does not matter which we adopt. Mr Bertrand Russell has said that 'mathematics is the science in which we do not know what we are talking about, and do not care whether what we say about it is true', a remark which is expressed in the form of a paradox but which in reality embodies a number of important truths. It would take too long to analyse the meaning of Mr Russell's epigram in detail, but one at any rate of its implications is this, that the symbols of mathematics are capable of varying interpretations, and that we are in general at liberty to adopt whichever we prefer.

There are now three cases to distinguish. It may happen that all negative rational numbers belong to the lower class and zero and all positive rational numbers to the upper. We describe this section as the **real number zero.** Or again it may happen that the lower class includes some positive numbers. Such a section

we describe as a **positive real number.** Finally it may happen that some negative numbers belong to the upper class. Such a section we describe as a **negative real number***.

The difference between our present definition of a positive real number a and that of § 7 amounts to the addition to the lower class of zero and all the negative rational numbers. An example of a negative real number is given by taking the property P of § 6 to be $x+1<0$ and Q to be $x+1 \geqslant 0$. This section plainly corresponds to the negative rational number -1. If we took P to be $x^3 < -2$ and Q to be $x^3 > -2$, we should obtain a negative real number which is not rational.

9. Relations of magnitude between real numbers. It is plain that, now that we have extended our conception of number, we are bound to make corresponding extensions of our conceptions of equality, inequality, addition, multiplication, and so on. We have to show that these ideas can be applied to the new numbers, and that, when this extension of them is made, all the ordinary laws of algebra retain their validity, so that we can operate with real numbers in general in exactly the same way as with the rational numbers of § 1. To do all this systematically would occupy a considerable space, and we shall be content to indicate summarily how a more systematic discussion would proceed.

We denote a real number by a Greek letter such as $\alpha, \beta, \gamma, \ldots$; the rational numbers of its lower and upper classes by the corresponding English letters a, A; b, B; c, C; .... The classes themselves we denote by $(a), (A), \ldots$.

If α and β are two real numbers, there are three possibilities:

(i) every a is a b and every A a B; in this case (a) is identical with (b) and (A) with (B);

* There are also sections in which every number belongs to the lower or to the upper class. The reader may be tempted to ask why we do not regard these sections also as defining numbers, which we might call the *real numbers positive and negative infinity*.

There is no logical objection to such a procedure, but it proves to be inconvenient in practice. The most natural definitions of addition and multiplication do not work in a satisfactory way. Moreover, for a beginner, the chief difficulty in the elements of analysis is that of learning to attach precise senses to phrases containing the word 'infinity'; and experience seems to show that he is likely to be confused by any addition to their number.

(ii) every a is a b, but not all A's are B's; in this case (a) is a proper part of (b)*, and (B) a proper part of (A);

(iii) every A is a B, but not all a's are b's.

These three cases may be indicated graphically as in Fig. 4.

In case (i) we write $\alpha=\beta$, in case (ii) $\alpha<\beta$, and in case (iii) $\alpha>\beta$. It is clear that, when α and β are both rational, these definitions agree with the ideas of equality and inequality between rational numbers which we began by taking for granted; and that any positive number is greater than any negative number.

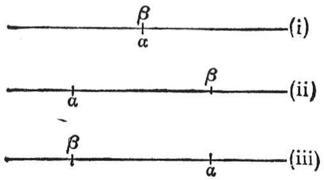


Fig. 4.

It will be convenient to define at this stage the negative $-\alpha$ of a positive number α. We suppose first that α is irrational. If (a), (A) are the classes which constitute α, we can define another section of the rational numbers by putting all numbers $-A$ in the lower class and all numbers $-a$ in the upper. The real number thus defined, which is clearly negative, we denote by $-\alpha$. Similarly we can define $-\alpha$ when α is negative; if α is negative, $-\alpha$ is positive. It is plain also that $-(-\alpha)=\alpha$. Of the two numbers α and $-\alpha$ one is always positive. The one which is positive we denote by $|\alpha|$ and call the *modulus* of α.

There is a complication if α is rational. In this case α belongs to (A), and the classes $(-A)$, $(-a)$ do not define a real number in the sense of § 8, since $-\alpha$ belongs to the lower class instead of to the upper. We must therefore modify our definition of $-\alpha$ by agreeing that, when α is rational, the rational $-\alpha$ is to be included in the upper class.

Examples IV. 1. Prove that $0=-0$.

2. Prove that $\beta=\alpha$, $\beta<\alpha$, or $\beta>\alpha$ according as $\alpha=\beta$, $\alpha>\beta$, or $\alpha<\beta$.

3. If $\alpha=\beta$ and $\beta=\gamma$, then $\alpha=\gamma$.

4. If $\alpha\leqq\beta$, $\beta<\gamma$, then $\alpha<\gamma$.

5. Prove that $-\beta<-\alpha$ if $\alpha<\beta$.

6. Prove that $\alpha>0$ if α is positive, and $\alpha<0$ if α is negative.

7. Prove that $\alpha\leqq|\alpha|$.

8. Prove that $1<\sqrt{2}<\sqrt{3}<2$.

[All these results are immediate consequences of our definitions.]

* *I.e.* is included in but not identical with (b).

10. Algebraical operations with real numbers. We now proceed to define the meaning of the elementary algebraical operations such as addition, as applied to real numbers in general.

(i) *Addition.* In order to define the sum of two numbers α and β, we consider the following two classes: (i) the class (c) formed by all sums $c=a+b$, (ii) the class (C) formed by all sums $C=A+B$. Plainly $c<C$ in all cases.

Again, there cannot be more than one rational number which does not belong either to (c) or to (C). For suppose there were two, say r and s, and let s be the greater. Then both r and s must be greater than every c and less than every C; and so $C-c$ cannot be less than $s-r$. But

$$C-c=(A-a)+(B-b);$$

and we can choose a, b, A, B so that both $A-a$ and $B-b$ are as small as we like; and this plainly contradicts our hypothesis.

If every rational number belongs to (c) or to (C), the classes (c), (C) form a section of the rational numbers, that is to say, a number γ. If there is one which does not, we add it to (C). We have now a section or real number γ, which must clearly be rational, since it corresponds to the least member of (C). *In any case we call γ the sum of α and β, and write*

$$\gamma=\alpha+\beta.$$

If both α and β are rational, they are the least members of the upper classes (A) and (B). In this case it is clear that $\alpha+\beta$ is the least member of (C), so that our definition agrees with our previous ideas of addition.

(ii) *Subtraction.* We define $\alpha-\beta$ by the equation

$$\alpha-\beta=\alpha+(-\beta).$$

The idea of subtraction accordingly presents no fresh difficulties.

Examples V. 1. Prove that $\alpha+(-\alpha)=0$.

2. Prove that $\alpha+0=0+\alpha=\alpha$.

3. Prove that $\alpha+\beta=\beta+\alpha$. [This follows at once from the fact that the classes $(a+b)$ and $(b+a)$, or $(A+B)$ and $(B+A)$, are the same, since, *e.g.*, $a+b=b+a$ when a and b are rational.]

4. Prove that $\alpha+(\beta+\gamma)=(\alpha+\beta)+\gamma$.

H.

5. Prove that $a - a = 0$.

6. Prove that $\alpha - \beta = -(\beta - \alpha)$.

7. From the definition of subtraction, and Exs. 4, 1, and 2 above, it follows that

$$(\alpha-\beta)+\beta=\{\alpha+(-\beta)\}+\beta=\alpha+\{(-\beta)+\beta\}=\alpha+0=\alpha.$$

We might therefore define the difference $\alpha-\beta=\gamma$ by the equation $\gamma+\beta=\alpha$.

8. Prove that $\alpha-(\beta-\gamma)=\alpha-\beta+\gamma$.

9. Give a definition of subtraction which does not depend upon a previous definition of addition. [To define $\gamma=\alpha-\beta$, form the classes (c), (C) for which $c=a-B$, $C=A-b$. It is easy to show that this definition is equivalent to that which we adopted in the text.]

10. Prove that

$$||\alpha|-|\beta|| \leqq |\alpha\pm\beta| \leqq |\alpha|+|\beta|.$$

11. Algebraical operations with real numbers (*continued*). (iii) *Multiplication.* When we come to multiplication, it is most convenient to confine ourselves to *positive* numbers (among which we may include 0) in the first instance, and to go back for a moment to the sections of positive rational numbers only which we considered in §§ 4—7. We may then follow practically the same road as in the case of addition, taking (c) to be (ab) and (C) to be (AB). The argument is the same, except when we are proving that all rational numbers with at most one exception must belong to (c) or (C). This depends, as in the case of addition, on showing that we can choose a, A, b, and B so that $C-c$ is as small as we please. Here we use the identity

$$C-c=AB-ab=(A-a)B+a(B-b).$$

Finally we include negative numbers within the scope of our definition by agreeing that, if α and β are positive, then

$$(-\alpha)\beta=-\alpha\beta, \quad \alpha(-\beta)=-\alpha\beta, \quad (-\alpha)(-\beta)=\alpha\beta.$$

(iv) *Division.* In order to define division, we begin by defining the reciprocal $1/\alpha$ of a number α (other than zero). Confining ourselves in the first instance to positive numbers and sections of positive rational numbers, we define the reciprocal of a positive number α by means of the lower class $(1/A)$ and the upper class $(1/a)$. We then define the reciprocal of a negative number $-\alpha$ by the equation $1/(-\alpha)=-(1/\alpha)$. Finally we define α/β by the equation

$$\alpha/\beta=\alpha\times(1/\beta).$$

We are then in a position to apply to all real numbers, rational or irrational, the whole of the ideas and methods of elementary algebra. Naturally we do not propose to carry out this task in detail. It will be more profitable and more interesting to turn our attention to some special, but particularly important, classes of irrational numbers.

Examples VI. Prove the theorems expressed by the following formulae:

1. $\alpha\times 0=0\times\alpha=0$.
2. $\alpha\times 1=1\times\alpha=\alpha$.
3. $\alpha\times(1/\alpha)=1$.
4. $\alpha\beta=\beta\alpha$.
5. $\alpha(\beta\gamma)=(\alpha\beta)\gamma$.
6. $\alpha(\beta+\gamma)=\alpha\beta+\alpha\gamma$.
7. $(\alpha+\beta)\gamma=\alpha\gamma+\beta\gamma$.
8. $|\alpha\beta|=|\alpha||\beta|$.

12. The number $\sqrt{2}$. Let us now return for a moment to the particular irrational number which we discussed in §§ 4—5. We there constructed a section by means of the inequalities $a^2<2$, $A^2>2$. This was a section of the positive rational numbers only; but we replace it (as was explained in § 8) by a section of all the rational numbers. We denote the section or number thus defined by the symbol $\sqrt{2}$.

The classes by means of which the product of $\sqrt{2}$ by itself is defined are (i) (aa'), where a and a' are positive rational numbers whose squares are less than 2, (ii) (AA'), where A and A' are positive rational numbers whose squares are greater than 2. These classes exhaust all positive rational numbers save one, which can only be 2 itself. Thus

$$(\sqrt{2})^2=\sqrt{2}\,\sqrt{2}=2.$$

Again

$$(-\sqrt{2})^2=(-\sqrt{2})(-\sqrt{2})=\sqrt{2}\,\sqrt{2}=(\sqrt{2})^2=2.$$

Thus *the equation $x^2=2$ has the two roots $\sqrt{2}$ and $-\sqrt{2}$.* Similarly we could discuss the equations $x^2=3$, $x^3=7$, ... and the corresponding irrational numbers $\sqrt{3}$, $-\sqrt{3}$, $\sqrt[3]{7}$, ...

13. Quadratic surds. A number of the form $\pm\sqrt{a}$, where a is a positive rational number which is not the square of another rational number, is called a *pure quadratic surd.* A number of the form $a\pm\sqrt{b}$, where a is rational, and $\sqrt{b}$ is a pure quadratic surd, is sometimes called a mixed quadratic surd.

The two numbers $a \pm \sqrt{b}$ are the roots of the quadratic equation

$$x^2 - 2ax + a^2 - b = 0.$$

Conversely, the equation $x^2 + 2px + q = 0$, where p and q are rational, and $p^2 - q > 0$, has as its roots the two quadratic surds $-p \pm \sqrt{(p^2 - q)}$.

The only kind of irrational numbers whose existence was suggested by the geometrical considerations of § 3 are these quadratic surds, pure and mixed, and the more complicated irrationals which may be expressed in a form involving the repeated extraction of square roots, such as

$$\sqrt{2} + \sqrt{(2 + \sqrt{2})} + \sqrt{\{2 + \sqrt{(2 + \sqrt{2})}\}}.$$

It is easy to construct geometrically a line whose length is equal to any number of this form, as the reader will easily see for himself. That irrational numbers of these kinds *only* can be constructed by Euclidean methods (*i.e.* by geometrical constructions with ruler and compasses) is a point the proof of which must be deferred for the present*. This property of quadratic surds makes them especially interesting.

Examples VII. 1. Give geometrical constructions for

$$\sqrt{2}, \quad \sqrt{(2+\sqrt{2})}, \quad \sqrt{\{2+\sqrt{(2+\sqrt{2})}\}}.$$

2. The quadratic equation $ax^2 + 2bx + c = 0$ has two real roots† if $b^2 - ac > 0$. Suppose a, b c rational. Nothing is lost by taking all three to be integers, for we can multiply the equation by the least common multiple of their denominators.

The reader will remember that the roots are $\{-b \pm \sqrt{(b^2 - ac)}\}/a$. It is easy to construct these lengths geometrically, first constructing $\sqrt{(b^2 - ac)}$. A much more elegant, though less straightforward, construction is the following.

* See Ch. II, Misc. Exs. 22.

† *I.e.* there are two values of x for which $ax^2 + 2bx + c = 0$. If $b^2 - ac < 0$ there are no such values of x. The reader will remember that in books on elementary algebra the equation is said to have two 'complex' roots. The meaning to be attached to this statement will be explained in Ch. III.

When $b^2 = ac$ the equation has only one root. For the sake of uniformity it is generally said in this case to have 'two equal' roots, but this is a mere convention.

Draw a circle of unit radius, a diameter PQ, and the tangents at the ends of the diameters.

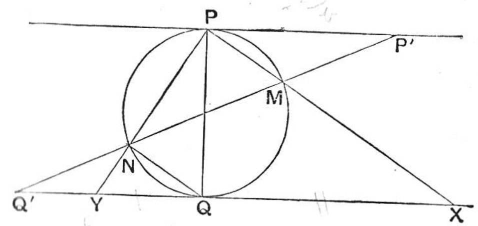


Fig. 5.

Take $PP' = -2a/b$ and $QQ' = -c/2b$, having regard to sign. Join $P'Q'$, cutting the circle in M and N. Draw PM and PN, cutting QQ' in X and Y. Then QX and QY are the roots of the equation with their proper signs†.*

The proof is simple and we leave it as an exercise to the reader. Another, perhaps even simpler, construction is the following. *Take a line AB of unit length. Draw $BC = -2b/a$ perpendicular to AB, and $CD = c/a$ perpendicular to BC and in the same direction as BA. On AD as diameter describe a circle cutting BC in X and Y. Then BX and BY are the roots.*

3. If ac is positive PP' and QQ' will be drawn in the same direction. Verify that $P'Q'$ will not meet the circle if $b^2 < ac$, while if $b^2 = ac$ it will be a tangent. Verify also that if $b^2 = ac$ the circle in the second construction will touch BC.

4. Prove that

$$\sqrt{(pq)} = \sqrt{p} \times \sqrt{q}, \quad \sqrt{(p^2 q)} = p\sqrt{q}.$$

14. Some theorems concerning quadratic surds. Two pure quadratic surds are said to be *similar* if they can be expressed as rational multiples of the same surd, and otherwise to be *dissimilar*. Thus

$$\sqrt{8} = 2\sqrt{2}, \quad \sqrt{\tfrac{25}{2}} = \tfrac{5}{2}\sqrt{2},$$

and so $\sqrt{8}$, $\sqrt{\frac{25}{2}}$ are similar surds. On the other hand, if M and N are integers which have no common factor, and neither of which is a perfect square, $\sqrt{M}$ and $\sqrt{N}$ are dissimilar surds. For suppose, if possible,

$$\sqrt{M} = \frac{p}{q}\sqrt{\frac{t}{u}}, \quad \sqrt{N} = \frac{r}{s}\sqrt{\frac{t}{u}},$$

where all the letters denote integers.

* The figure is drawn to suit the case in which b and c have the same and a the opposite sign. The reader should draw figures for other cases.

† I have taken this construction from Klein's *Leçons sur certaines questions de géométrie élémentaire* (French translation by J. Griess, Paris, 1896).

Then $\sqrt{MN}$ is evidently rational, and therefore (Ex. II. 3) integral. Thus $MN=P^2$, where P is an integer. Let $a, b, c, \ldots$ be the prime factors of P, so that

$$MN=a^{2\alpha}b^{2\beta}c^{2\gamma}\ldots,$$

where $\alpha, \beta, \gamma, \ldots$ are positive integers. Then MN is divisible by $a^{2\alpha}$, and therefore either (1) M is divisible by $a^{2\alpha}$, or (2) N is divisible by $a^{2\alpha}$, or (3) M and N are both divisible by a. The last case may be ruled out, since M and N have no common factor. This argument may be applied to each of the factors $a^{2\alpha}, b^{2\beta}, c^{2\gamma}, \ldots$, so that M must be divisible by some of these factors and N by the remainder. Thus

$$M=P_1^2, \quad N=P_2^2,$$

where P_1^2 denotes the product of some of the factors $a^{2\alpha}, b^{2\beta}, c^{2\gamma}, \ldots$ and P_2^2 the product of the rest. Hence M and N are both perfect squares, which is contrary to our hypothesis.

THEOREM. *If A, B, C, D are rational and*

$$A+\sqrt{B}=C+\sqrt{D},$$

then either (i) $A=C$, $B=D$ *or* (ii) B *and* D *are both squares of rational numbers.*

For $B-D$ is rational, and so is

$$\sqrt{B}-\sqrt{D}=C-A.$$

If B is not equal to D (in which case it is obvious that A is also equal to C), it follows that

$$\sqrt{B}+\sqrt{D}=(B-D)/(\sqrt{B}-\sqrt{D})$$

is also rational. Hence $\sqrt{B}$ and $\sqrt{D}$ are rational.

COROLLARY. *If* $A+\sqrt{B}=C+\sqrt{D}$, *then* $A-\sqrt{B}=C-\sqrt{D}$ *(unless $\sqrt{B}$ and $\sqrt{D}$ are both rational).*

Examples VIII. 1. Prove *ab initio* that $\sqrt{2}$ and $\sqrt{3}$ are not similar surds.

2. Prove that $\sqrt{a}$ and $\sqrt{(1/a)}$, where a is rational, are similar surds (unless both are rational).

3. If a and b are rational, then $\sqrt{a}+\sqrt{b}$ cannot be rational unless $\sqrt{a}$ and $\sqrt{b}$ are rational. The same is true of $\sqrt{a}-\sqrt{b}$, unless $a=b$.

4. If

$$\sqrt{A}+\sqrt{B}=\sqrt{C}+\sqrt{D},$$

then either (a) $A=C$ and $B=D$, or (b) $A=D$ and $B=C$, or (c) $\sqrt{A}$, $\sqrt{B}$, $\sqrt{C}$, $\sqrt{D}$ are all rational or all similar surds. [Square the given equation and apply the theorem above.]

5. Neither $(a+\sqrt{b})^3$ nor $(a-\sqrt{b})^3$ can be rational unless $\sqrt{b}$ is rational.

6. Prove that if $x=p+\sqrt{q}$, where p and q are rational, then x^m, where m is any integer, can be expressed in the form $P+Q\sqrt{q}$, where P and Q are rational. For example,

$$(p+\sqrt{q})^2=p^2+q+2p\sqrt{q}, \quad (p+\sqrt{q})^3=p^3+3pq+(3p^2+q)\sqrt{q}.$$

Deduce that any polynomial in x with rational coefficients (*i.e.* any expression of the form

$$a_0x^n+a_1x^{n-1}+\ldots+a_n,$$

where $a_0, \ldots a_n$ are rational numbers) can be expressed in the form $P+Q\sqrt{q}$.

7. If $a+\sqrt{b}$, where b is not a perfect square, is the root of an algebraical equation with rational coefficients, then $a-\sqrt{b}$ is another root of the same equation.

8. Express $1/(p+\sqrt{q})$ in the form prescribed in Ex. 6. [Multiply numerator and denominator by $p-\sqrt{q}$.]

9. Deduce from Exs. 6 and 8 that any expression of the form $G(x)/H(x)$, where $G(x)$ and $H(x)$ are polynomials in x with rational coefficients, can be expressed in the form $P+Q\sqrt{q}$, where P and Q are rational.

10. If p, q, and p^2-q are positive, we can express $\sqrt{(p+\sqrt{q})}$ in the form $\sqrt{x}+\sqrt{y}$, where

$$x=\tfrac{1}{2}\{p+\sqrt{(p^2-q)}\}, \quad y=\tfrac{1}{2}\{p-\sqrt{(p^2-q)}\}.$$

11. Determine the conditions that it may be possible to express $\sqrt{(p+\sqrt{q})}$, where p and q are rational, in the form $\sqrt{x}+\sqrt{y}$, where x and y are rational.

12. If a^2-b is positive, the necessary and sufficient conditions that

$$\sqrt{(a+\sqrt{b})}+\sqrt{(a-\sqrt{b})}$$

should be rational are that a^2-b and $\frac{1}{2}\{a+\sqrt{(a^2-b)}\}$ should both be squares of rational numbers.

15. The continuum. The aggregate of all real numbers, rational and irrational, is called the **arithmetical continuum.**

It is convenient to suppose that the straight line Λ of § 2 is composed of points corresponding to all the numbers of the arithmetical continuum, and of no others*. The points of the

* This supposition is merely a hypothesis adopted (i) because it suffices for the purposes of our geometry and (ii) because it provides us with convenient geometrical illustrations of analytical processes. As we use geometrical language only for purposes of illustration, it is not part of our business to study the foundations of geometry.

line, the aggregate of which may be said to constitute the **linear continuum**, then supply us with a convenient image of the arithmetical continuum.

We have considered in some detail the chief properties of a few classes of real numbers, such, for example, as rational numbers or quadratic surds. We add a few further examples to show how very special these particular classes of numbers are, and how, to put it roughly, they comprise only a minute fraction of the infinite variety of numbers which constitute the continuum.

(i) Let us consider a more complicated surd expression such as

$$z=\sqrt[3]{(4+\sqrt{15})}+\sqrt[3]{(4-\sqrt{15})}.$$

Our argument for supposing that the expression for z has a meaning might be as follows. We first show, as in § 12, that there is a number $y=\sqrt{15}$ such that $y^2=15$, and we can then, as in § 10, define the numbers $4+\sqrt{15}$, $4-\sqrt{15}$. Now consider the equation in z_1,

$$z_1^3=4+\sqrt{15}.$$

The right-hand side of this equation is not rational: but exactly the same reasoning which leads us to suppose that there is a real number x such that $x^3=2$ (or any other rational number) also leads us to the conclusion that there is a number z_1 such that $z_1^3=4+\sqrt{15}$. We thus define $z_1=\sqrt[3]{(4+\sqrt{15})}$, and similarly we can define $z_2=\sqrt[3]{(4-\sqrt{15})}$; and then, as in § 10, we define $z=z_1+z_2$.

Now it is easy to verify that

$$z^3=3z+8.$$

And we might have given a direct proof of the existence of a unique number z such that $z^3=3z+8$. It is easy to see that there cannot be two such numbers. For if $z_1^3=3z_1+8$ and $z_2^3=3z_2+8$, we find on subtracting and dividing by z_1-z_2 that $z_1^2+z_1z_2+z_2^2=3$. But if z_1 and z_2 are positive $z_1^3>8$, $z_2^3>8$ and therefore $z_1>2$, $z_2>2$, $z_1^2+z_1z_2+z_2^2>12$, and so the equation just found is impossible. And it is easy to see that neither z_1 nor z_2 can be negative. For if z_1 is negative and equal to $-\zeta$, ζ is positive and $\zeta^3-3\zeta+8=0$, or $3-\zeta^2=8/\zeta$. Hence $3-\zeta^2>0$, and so $\zeta<2$. But then $8/\zeta>4$, and so $8/\zeta$ cannot be equal to $3-\zeta^2$, which is less than 3.

Hence there is at most one z such that $z^3=3z+8$. And it cannot be rational. For any rational root of this equation must be integral and a factor of 8 (Ex. II. 3), and it is easy to verify that no one of 1, 2, 4, 8 is a root.

Thus $z^3=3z+8$ has at most one root and that root, if it exists, is positive and not rational. We can now divide the positive rational numbers x into two classes L, R according as $x^3<3x+8$ or $x^3>3x+8$. It is easy to see that if $x^3>3x+8$ and y is any number greater than x, then also $y^3>3y+8$. For suppose if possible $y^3\leqq 3y+8$. Then since $x^3>3x+8$ we obtain on subtracting $y^3-x^3<3(y-x)$, or $y^2+xy+x^2<3$, which is impossible; for y is

positive and $x>2$ (since $x^3>8$). Similarly we can show that if $x^3<3x+8$ and $y<x$ then also $y^3<3y+8$.

Finally, it is evident that the classes L and R both exist; and they form a section of the positive rational numbers or positive real number z which satisfies the equation $z^3=3z+8$. The reader who knows how to solve cubic equations by Cardan's method will be able to obtain the explicit expression of z directly from the equation.

(ii) The direct argument applied above to the equation $x^3=3x+8$ could be applied (though the application would be a little more difficult) to the equation

$$x^5=x+16.$$

and would lead us to the conclusion that a unique positive real number exists which satisfies this equation. In this case, however, it is not possible to obtain a simple explicit expression for x composed of any combination of surds. It can in fact be proved (though the proof is difficult) that it is *generally* impossible to find such an expression for the root of an equation of higher degree than 4. Thus, besides irrational numbers which can be expressed as pure or mixed quadratic or other surds, or combinations of such surds, there are others which are roots of algebraical equations but cannot be so expressed. It is only in very special cases that such expressions can be found.

(iii) But even when we have added to our list of irrational numbers roots of equations (such as $x^5=x+16$) which cannot be explicitly expressed as surds, we have not exhausted the different kinds of irrational numbers contained in the continuum. Let us draw a circle whose diameter is equal to A_0A_1, *i.e.* to unity. It is natural to suppose* that the circumference of such a circle has a length capable of numerical measurement. This length is usually denoted by π. And it has been shown† (though the proof is unfortunately long and difficult) that this number π is not the root of any algebraical equation with integral coefficients, such, for example, as

$$\pi^2=n, \quad \pi^3=n, \quad \pi^5=\pi+n,$$

* See Hobson's *Plane Trigonometry* (5th edition), pp. 7 *et seq.*

† See Hobson, *loc. cit.*, pp. 305 *et seq.*, or the same writer's *Squaring the Circle* (Cambridge, 1913).

where n is an integer. In this way it is possible to define a number which is not rational nor yet belongs to any of the classes of irrational numbers which we have so far considered. And this number π is no isolated or exceptional case. Any number of other examples can be constructed. In fact it is only special classes of irrational numbers which are roots of equations of this kind, just as it is only a still smaller class which can be expressed by means of surds.

16. The continuous real variable. The 'real numbers' may be regarded from two points of view. We may think of them *as an aggregate*, the 'arithmetical continuum' defined in the preceding section, or *individually*. And when we think of them individually, we may think either of a particular *specified* number (such as $1, -\frac{1}{2}, \sqrt{2}$, or π) or we may think of *any* number, *an unspecified* number, *the number* x. This last is our point of view when we make such assertions as 'x is a number', 'x is the measure of a length', 'x may be rational or irrational'. The x which occurs in propositions such as these is called *the continuous real variable*: and the individual numbers are called the *values* of the variable.

A 'variable', however, need not necessarily be continuous. Instead of considering the aggregate of *all* real numbers, we might consider some partial aggregate contained in the former aggregate, such as the aggregate of rational numbers, or the aggregate of positive integers. Let us take the last case. Then in statements about *any* positive integer, or *an unspecified* positive integer, such as 'n is either odd or even', n is called the variable, a *positive integral variable*, and the individual positive integers are its values.

Naturally 'x' and 'n' are only examples of variables, the variable whose 'field of variation' is formed by all the real numbers, and that whose field is formed by the positive integers. These are the most important examples, but we have often to consider other cases. In the theory of decimals, for instance, we may denote by x any figure in the expression of any number as a decimal. Then x is a variable, but a variable which has only ten different values, viz. 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. The reader should

think of other examples of variables with different fields of variation. He will find interesting examples in ordinary life: policeman x, the driver of cab x, the year x, the xth day of the week. The values of these variables are naturally not numbers.

17. Sections of the real numbers. In §§ 4—7 we considered 'sections' of the rational numbers, *i.e.* modes of division of the rational numbers (or of the positive rational numbers only) into two classes L and R possessing the following characteristic properties:

(i) that every number of the type considered belongs to one and only one of the two classes;

(ii) that both classes exist;

(iii) that any member of L is less than any member of R.

It is plainly possible to apply the same idea to the aggregate of all real numbers, and the process is, as the reader will find in later chapters, of very great importance.

Let us then suppose* that P and Q are two properties which are mutually exclusive, and one of which is possessed by every real number. Further let us suppose that any number which possesses P is less than any which possesses Q. We call the numbers which possess P the *lower* or *left-hand class* L, and those which possess Q the *upper* or *right-hand class* R.

Thus P might be $x \leqq \sqrt{2}$ and Q be $x > \sqrt{2}$. It is important to observe that a pair of properties which suffice to define a section of the rational numbers may not suffice to define one of the real numbers. This is so, for example, with the pair '$x < \sqrt{2}$' and '$x > \sqrt{2}$' or (if we confine ourselves to positive numbers) with '$x^2 < 2$' and '$x^2 > 2$'. Every rational number possesses one or other of the properties, but not every real number, since in either case $\sqrt{2}$ escapes classification.

There are now two possibilities†. Either L has a greatest member l, or R has a least member r. *Both* of these events

* The discussion which follows is in many ways similar to that of § 6. We have not attempted to avoid a certain amount of repetition. The idea of a 'section,' first brought into prominence in Dedekind's famous pamphlet *Stetigkeit und irrationale Zahlen*, is one which can, and indeed must, be grasped by every reader of this book, even if he be one of those who prefer to omit the discussion of the notion of an irrational number contained in §§ 6—12.

† There were three in § 6.

cannot occur. For if L had a greatest member l, and R a least member r, the number $\frac{1}{2}(l+r)$ would be greater than all members of L and less than all members of R, and so could not belong to either class. On the other hand *one* event must occur*.

For let L_1 and R_1 denote the classes formed from L and R by taking only the rational members of L and R. Then the classes L_1 and R_1 form a section of the rational numbers. There are now two cases to distinguish.

It may happen that L_1 has a greatest member α. In this case α must be also the greatest member of L. For if not, we could find a greater, say β. There are rational numbers lying between α and β, and these, being less than β, belong to L, and therefore to L_1; and this is plainly a contradiction. Hence α is the greatest member of L.

On the other hand it may happen that L_1 has no greatest member. In this case the section of the rational numbers formed by L_1 and R_1 is a real number α. This number α must belong to L or to R. If it belongs to L we can shew, precisely as before, that it is the greatest member of L, and similarly, if it belongs to R, it is the least member of R.

Thus in any case either L has a greatest member or R a least. Any section of the real numbers therefore 'corresponds' to a real number in the sense in which a section of the rational numbers sometimes, but not always, corresponds to a rational number. This conclusion is of very great importance; for it shows that the consideration of sections of all the real numbers does not lead to any further generalisation of our idea of number. Starting from the rational numbers, we found that the idea of a section of the rational numbers led us to a new conception of a number, that of a real number, more general than that of a rational number; and it might have been expected that the idea of a section of the real numbers would have led us to a conception more general still. The discussion which precedes shows that this is not the case, and that the aggregate of real numbers, or the continuum, has a kind of completeness which the aggregate of the rational numbers lacked, a completeness which is expressed in technical language by saying that the continuum is closed.

* This was not the case in § 6.

The result which we have just proved may be stated as follows:

Dedekind's Theorem. *If the real numbers are divided into two classes L and R in such a way that*

(i) *every number belongs to one or other of the two classes,*
(ii) *each class contains at least one number,*
(iii) *any member of L is less than any member of R,*

then there is a number α, which has the property that all the numbers less than it belong to L and all the numbers greater than it to R. The number α itself may belong to either class.

In applications we have often to consider sections not of *all* numbers but of all those contained in an *interval* (β, γ), that is to say of all numbers x such that $\beta \leqq x \leqq \gamma$. A 'section' of such numbers is of course a division of them into two classes possessing the properties (i), (ii), and (iii). Such a section may be converted into a section of *all* numbers by adding to L all numbers less than β and to R all numbers greater than γ. It is clear that the conclusion stated in Dedekind's Theorem still holds if we substitute 'the real numbers of the interval (β, γ)' for 'the real numbers', and that the number α in this case satisfies the inequalities $\beta \leqq \alpha \leqq \gamma$.

18. Points of accumulation. A system of real numbers, or of the points on a straight line corresponding to them, defined in any way whatever, is called an **aggregate** or **set** of numbers or points. The set might consist, for example, of all the positive integers, or of all the rational points.

It is most convenient here to use the language of geometry*. Suppose then that we are given a set of points, which we will denote by S. Take any point ξ, which may or may not belong to S. Then there are two possibilities. Either (i) it is possible to choose a positive number δ so that the interval $(\xi-\delta, \xi+\delta)$ does not contain any point of S, other than ξ itself†, or (ii) this is not possible.

Suppose, for example, that S consists of the points corresponding to all the positive integers. If ξ is itself a positive integer, we can take δ to be any number less than 1, and (i) will be true; or, if ξ is halfway between two positive integers, we can take δ to be any number less than $\frac{1}{2}$. On the other hand, if S consists of all the rational points, then, whatever the value of ξ, (ii) is true; for any interval whatever contains an infinity of rational points.

* The reader will hardly require to be reminded that this course is adopted solely for reasons of linguistic convenience.

† This clause is of course unnecessary if ξ does not itself belong to S.

Let us suppose that (ii) is true. Then any interval $(\xi-\delta, \xi+\delta)$, however small its length, contains at least one point ξ_1 which belongs to S and does not coincide with ξ; and this whether ξ itself be a member of S or not. In this case we shall say that ξ is a **point of accumulation** of S. It is easy to see that the interval $(\xi-\delta, \xi+\delta)$ must contain, not merely one, but infinitely many points of S. For, when we have determined ξ_1, we can take an interval $(\xi-\delta_1, \xi+\delta_1)$ surrounding ξ but not reaching as far as ξ_1. But this interval also must contain a point, say ξ_2, which is a member of S and does not coincide with ξ. Obviously we may repeat this argument, with ξ_2 in the place of ξ_1; and so on indefinitely. In this way we can determine as many points

$$\xi_1, \xi_2, \xi_3, \dots$$

as we please, all belonging to S, and all lying inside the interval $(\xi-\delta, \xi+\delta)$.

A point of accumulation of S may or may not be itself a point of S. The examples which follow illustrate the various possibilities.

Examples IX. 1. If S consists of the points corresponding to the positive integers, or all the integers, there are no points of accumulation.

2. If S consists of all the rational points, every point of the line is a point of accumulation.

3. If S consists of the points $1, \frac{1}{2}, \frac{1}{3}, \dots$, there is one point of accumulation, viz. the origin.

4. If S consists of all the positive rational points, the points of accumulation are the origin and all positive points of the line.

19. Weierstrass's Theorem. The general theory of sets of points is of the utmost interest and importance in the higher branches of analysis; but it is for the most part too difficult to be included in a book such as this. There is however one fundamental theorem which is easily deduced from Dedekind's Theorem and which we shall require later.

THEOREM. *If a set S contains infinitely many points, and is entirely situated in an interval (α, β), then at least one point of the interval is a point of accumulation of S.*

We divide the points of the line Λ into two classes in the following manner. The point P belongs to L if there are an

infinity of points of S to the right of P, and to R in the contrary case. Then it is evident that conditions (i) and (iii) of Dedekind's Theorem are satisfied; and since α belongs to L and β to R, condition (ii) is satisfied also.

Hence there is a point ξ such that, however small be δ, $\xi-\delta$ belongs to L and $\xi+\delta$ to R, so that the interval $(\xi-\delta, \xi+\delta)$ contains an infinity of points of S. Hence ξ is a point of accumulation of S.

This point may of course coincide with α or β, as for instance when $\alpha=0$, $\beta=1$, and S consists of the points $1, \frac{1}{2}, \frac{1}{3}, \dots$. In this case 0 is the sole point of accumulation.

MISCELLANEOUS EXAMPLES ON CHAPTER I.

1. What are the conditions that $ax+by+cz=0$, (1) for all values of x, y, z; (2) for all values of x, y, z subject to $\alpha x+\beta y+\gamma z=0$; (3) for all values of x, y, z subject to both $\alpha x+\beta y+\gamma z=0$ and $Ax+By+Cz=0$?

2. Any positive rational number can be expressed in one and only one way in the form

$$a_1+\frac{a_2}{1.2}+\frac{a_3}{1.2.3}+\dots+\frac{a_k}{1.2.3\dots k},$$

where $a_1, a_2, \dots, a_k$ are integers, and

$$0 \leqq a_1, \quad 0 \leqq a_2 < 2, \quad 0 \leqq a_3 < 3, \dots 0 < a_k < k.$$

3. Any positive rational number can be expressed in one and one way only as a simple continued fraction

$$a_1+\frac{1}{a_2+}\,\frac{1}{a_3+}\dots\frac{1}{+a_n},$$

where $a_1, a_2, \dots$ are positive integers, of which the first only may be zero.

[Accounts of the theory of such continued fractions will be found in text-books of algebra. For further information as to modes of representation of rational and irrational numbers, see Hobson, *Theory of Functions of a Real Variable*, 2nd edition, vol. I, pp. 45–49.]

4. Find the rational roots (if any) of $9x^3-6x^2+15x-10=0$.

5. A line AB is divided at C *in aurea sectione* (Euc. II. 11)—*i.e.* so that $AB \cdot AC = BC^2$. Show that the ratio AC/AB is irrational.

[A direct geometrical proof will be found in Bromwich's *Infinite Series*, § 143, p. 363.]

6. A is irrational. In what circumstances can $\dfrac{aA+b}{cA+d}$, where a, b, c, d are rational, be rational?