INTRODUCTION TO SEMANTICS and FORMALIZATION OF LOGIC

By RUDOLF CARNAP

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While the first chapter contains explanations that are easily comprehensible, the remainder of the book is on a more technical level. Some devices are used to facilitate reading. Material not absolutely necessary for an understanding of the main text is printed in small type, e.g. digressions into more technical problems, examples, proofs, references to other authors, etc. Among the numbered definitions and theorems, the more important are marked by '+'. Each chapter and each section is preceded by a brief summary. This will enable the reader to look back over what has been covered and to anticipate the path immediately ahead, so that he will not feel lost in the jungle.

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R. C.

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INTRODUCTION TO SEMANTICS

A. SEMIOTIC AND ITS PARTS

Semiotic, the theory of signs and languages, is divided into three parts: pragmatics, semantics, and syntax. Semantics is divided into descriptive and pure semantics; syntax is divided analogously into descriptive and pure syntax. The present book deals with pure semantics, pure syntax, and their relations.

§ 1. Object Language and Metalanguage

The language spoken about in some context is called the *object language*; the language in which we speak about the first is called the *metalanguage.*

A **language**, as it is usually understood, is a system of sounds, or rather of the habits of producing them by the speaking organs, for the purpose of communicating with other persons, i.e. of influencing their actions, decisions, thoughts, etc. Instead of speech sounds other movements or things are sometimes produced for the same purpose, e.g. gestures, written marks, signals by drums, flags, trumpets, rockets, etc. It seems convenient to take the term 'language' in such a wide sense as to cover all these kinds of systems of means of communication, no matter what material they use. Thus we will distinguish between speech language (or spoken language), language of writing (or written language), gesture language, etc. Of course, speech language is the most important practically, and is, moreover, in most cases the basis of any other language, in the sense that this other language is learned with the help of the speech language. But this fact is accidental; any of the other kinds of language could be learned and used in a way independent of the speech language.

If we investigate, analyze, and describe a language L_1, we need a language L_2 for formulating the results of our investigation of L_1 or the rules for the use of L_1. In this case we call L_1 the **object language**, L_2 the **metalanguage**. The sum total of what can be known about L_1 and said in L_2 may be called the *metatheory* of L_1 (in L_2). If we describe in English the grammatical structure of modern German and French or describe the historical development of speech forms or analyze literary works in these languages, then German and French are our object languages and English is our metalanguage. Any language whatever can be taken as an object language; any language containing expressions suitable for describing the features of languages may be taken as a metalanguage. Object language and metalanguage may also be identical, e.g. when we are speaking in English about English grammar, literature, etc.

§ 2. Signs and Expressions

The smallest units of a language are called *signs*; sequences of signs are called *expressions*.

A continuous utterance in a language, e.g. a speech, a book, or a flag message, may be analyzed into smaller and smaller parts. Thus a speech may be divided into sentences, each sentence into words, each word into phonemes. A book or letter may be divided into (written) sentences, each sentence into (written) words, each word into letters of the alphabet, each letter into the simple strokes of which it consists. Where we stop the analysis is to some extent arbitrary, depending upon the purpose of our investigation. When interested in grammar, we may take (spoken or written) words or certain parts of words as ultimate units; when interested in spelling, letters; when interested in the historical development of letter forms, the single form elements of the letters. When we speak *in abstracto* about analysis of language, we use the term '**sign**' to designate the ultimate units of the expressions of the languages. Thereby it remains undecided whether words or letters or whatever else are taken as signs; this may be specified as soon as we go over from the general discussion to a special investigation of some one language.

By an **expression** in a language we mean any finite sequence of signs in that language, no matter whether meaningful or not. Thus we treat all utterances in language as being of linear form. This is convenient because it enables us to specify the positions of signs in an expression by enumeration. A spoken utterance in one of the ordinary languages is a temporal series of sounds; a written utterance consists of marks ordered in lines; either of them can therefore easily be taken as linear, i.e. as one sequence. Where in practice a second dimension is used — as e.g. in written accents or similar discriminating marks, in a statistical table of figures, or in a diagram of a configuration in chess — it is always possible by some device to regard the whole expression as linear (e.g. by counting the accent in 'très' as the fourth sign, the 's' as the fifth).

§ 3. Sign-Events and Sign-Designs

The word 'sign' is ambiguous. It means sometimes a single object or event, sometimes a kind to which many objects belong. Whenever necessary, we shall use 'sign-*event*' in the first case, 'sign-*design*' in the second.

In the ordinary way of speaking about signs and expressions, e.g. letters of the alphabet, words, phrases, and sentences in English, certain ambiguities often occur. Thus, for instance, the word 'letter' — and analogously the words 'word', 'sentence', etc. — is used in two different ways, as exhibited by the following two sets of examples: 1. "There are two letters 's' in the eighth word of this paragraph"; "The second letter 's' in that word is a plural ending". 2. "The letter 's' occurs twice in the word 'signs' "; "The letter 's' is in many cases used as a plural ending in English". In (1) we say "many letters 's' ", in (2) "the letter 's' ", thus indicating that there is only one; hence the phrase "letter 's' " has two different meanings. In (1), a letter is a single thing or event, e.g. a body consisting of printer's ink or a sound event; therefore, it is at a certain time-moment or during a certain time-interval, and at each time-moment within its duration it occupies a certain place. In (2), on the other hand, a letter is not a single thing but a class of things to which many things may belong, e.g. the letter 's' is that class of written or printed marks to which all lower case S's belong. Although, in most cases, the context leaves no doubt as to which of the two meanings is intended, it will sometimes be advisable to distinguish them explicitly. In cases of this kind we shall use the term '**event**' — or 'letter-event', and analogously 'word-event', 'expression-event', 'sentence-event', etc. — for meaning (1), and the term '**design**' — or 'letter-design', and analogously 'word-design', 'expression-design', 'sentence-design', etc. — for meaning (2).

In historical descriptions of particular acts of speaking or writing, expression-events are often dealt with. But they are usually characterized by the designs to which they belong. When we say "Caesar wrote 'vici' ", then we are speaking about a certain word-event produced by Caesar's hand; but we describe it by its design; the sentence is meant to say: "Caesar wrote a word-event of the design 'vici' ". When we are not concerned with the history of single acts but with the linguistic description of a certain language or the logical (syntactical or semantical) analysis of a certain language system, then the features which we study are common to all events of a design. Therefore, in this kind of investigation, it is convenient to drop reference to expression-events entirely and to speak only about designs. Instead of saying, "Every event of the word-design 'Hund' is a noun-event (in German)", we may simply say, "The word-design 'Hund' is a noun-design". Since in these fields we are dealing with designs only, we may establish the convention that, in texts belonging to these fields, e.g. in this treatise, 'word' is to be understood as 'word-design', 'noun' as 'noun-design', and analogously with 'sign', 'expression', 'sentence', etc. Thus we come to the ordinary formulation, "The word 'Hund' is a noun". In the same way, if we say in syntax that a certain sentence is provable in a certain calculus, or in semantics that a certain sentence is true, then we mean to attribute these properties to sentence-designs, because they are shared by all sentence-events of a design; the same holds for all other concepts of syntax and semantics.

An expression-event consists of (one or more) sign-events, and an expression-design consists of sign-designs. However, the relation is not the same in the two cases. In an expression-event all elements are different (i.e. non-identical); there is no repetition of sign-events, because an event (e.g. a physical object) can only be at one place at a time. On the other hand, in an expression-design a certain sign-design may occupy several positions; in this case we speak of the several *occurrences* of the sign (-design) within the expression (-design).

Examples. The first and the last letter-event in the eighth word-event of § 3 in your copy (-event) of this book (-design) are two bodies of ink. They are different (i.e. non-identical), although similar (i.e. of similar geometrical shape); their similarity enables you to recognize them as belonging to the same design. Thus that word-event contains two letter-events 's'. On the other hand, the word-design 'signs' cannot contain two letter-designs 's' because there is only one letter-design 's'; but this design 's' occurs at two positions in the design 'signs' just as one and the same color or kind of substance or disease or architectural style may occur at different places, i.e. be exhibited by different things.

In an exact exhibition, an expression-event may be represented either as a (discrete, finite) series of sign-events or as a sequence without repetitions. But an expression-design has to be represented as a (finite) sequence of sign-designs because the same sign-design may occur in it several times. (Concerning the difference between series and sequences, see § 6.)

Whether in the metalanguage names of sign-events or names of sign-designs are assigned to the zero-level, i.e. taken as individual constants, depends upon the purpose of the investigation. If sign-events are dealt with at all (as in descriptive semiotic), they will in general be taken as individuals and hence be designated by individual constants. In this case, a sign-design is a property or class of sign-events and hence to be designated by a predicate (level 1, degree 1; see § 6). If, however, only designs and not events are referred to — as is mostly the case in pure semiotic, especially in pure syntax and pure semantics — then sign-designs may be taken as individuals.

Another ambiguity of the word 'word' may be mentioned, although it is of less importance for our subsequent discussions. 'Speak', 'speaks', 'speaking', 'spoken' are sometimes, e.g. in grammar books, called four forms of the same word, but at other times four different words (of the same word group). We prefer the second use of the phrase 'the same word(-design)', hence applying it only in cases of literal similarity, i.e. where the word-events consist of letter-events of the same designs.

§ 4. The Parts of Semiotic: Pragmatics, Semantics, and Syntax

In an application of language, we may distinguish three chief factors: the speaker, the expression uttered, and the designatum of the expression, i.e. that to which the speaker intends to refer by the expression. In *semiotic*†, the general theory of signs and languages, three fields are distinguished. An investigation of a language belongs to *pragmatics* if explicit reference to a speaker is made; it belongs to *semantics*† if designata but not speakers are referred to; it belongs to *syntax*† if neither speakers nor designata but only expressions are dealt with.

† For *terminological remarks* concerning the terms marked by an obelisk, see § 37.

When we observe an application of language, we observe an organism, usually a human being, producing a sound, mark, gesture, or the like as an expression in order to refer by it to something, e.g. an object. Thus we may distinguish three factors involved: the speaker, the expression, and what is referred to, which we shall call the **designatum** of the expression. (We say e.g. that in German 'Rhein' designates the Rhine, and that the Rhine is the designatum of 'Rhein'; likewise, the designatum of 'rot' is a certain property, namely the color red; the designatum of 'kleiner' is a certain relation, that of 'Temperatur' a certain physical function, etc.)

If we are analyzing a language, then we are concerned, of course, with expressions. But we need not necessarily also deal with speakers and designata. Although these factors are present whenever language is used, we may abstract from one or both of them in what we intend to say about the language in question. Accordingly, we distinguish three fields of investigation of languages. If in an investigation explicit reference is made to the speaker, or, to put it in more general terms, to the user of a language, then we assign it to the field of **pragmatics**. (Whether in this case reference to designata is made or not makes no difference for this classification.) If we abstract from the user of the language and analyze only the expressions and their designata, we are in the field of **semantics**†. And if, finally, we abstract from the designata also and analyze only the relations between the expressions, we are in (logical) **syntax**†. The whole science of language, consisting of the three parts mentioned, is called **semiotic**†.

The distinction between the three parts of semiotic has been made by C. W. Morris [Foundations] (see bibliography at the end of this book) on the basis of earlier distinctions of the three factors mentioned. There is a slight difference in the use of the term 'pragmatics', which is defined by Morris as the field dealing with the relations between speakers (or certain processes in them) and expressions. In practice, however, there does not seem to be a sharp line between investigations of this kind and those which refer also to designata.

Examples of *pragmatical* investigations are: a physiological analysis of the processes in the speaking organs and in the nervous system connected with speaking activities; a psychological analysis of the relations between speaking behavior and other behavior; a psychological study of the different connotations of one and the same word for different individuals; ethnological and sociological studies of the speaking habits and their differences in different tribes, different age groups, social strata; a study of the procedures applied by scientists in recording the results of experiments, etc. *Semantics* contains the theory of what is usually called the meaning of expressions, and hence the studies leading to the construction of a dictionary translating the object language into the metalanguage. But we shall see that theories of an apparently quite different subject-matter also belong to semantics, e.g. the theory of truth and the theory of logical deduction. It turns out that truth and logical consequence are concepts based on the relation of designation, and hence semantical concepts.

An investigation, a method, a concept concerning expressions of a language are called **formal**† if in their application reference is made not to the designata of the expressions but only to their form, i.e. to the kinds of signs occurring in an expression and the order in which they occur. Hence anything represented in a formal way belongs to *syntax.* It can easily be seen that it is possible to formulate rules for the construction of sentences, so-called *rules of formation*, in a strictly formal way (see e.g. the rules for S_3 in § 8). One might perhaps think at first that syntax would be restricted to a formulation and investigation of rules of this kind and hence would be a rather poor field. But it turns out that, in addition, *rules of deduction* can be formulated in a formal way and hence within syntax. This can, among other possibilities, be done in such a way that these rules lead to the same results as the semantical rules of logical deduction. In this way it is possible to represent logic in syntax.

The representation of certain concepts or procedures in a formal way and hence within syntax is sometimes called *formalization.* The formalization of semantical systems, i.e. the construction of corresponding syntactical systems, will be explained in § 36.

The result that logical deduction can be represented in a formal way — in other words, the possibility of a *formalization of logic* — is one of the most important results of the development of modern logic. The trend in this direction is as old as logic itself; but in different periods of its development the formal side has been emphasized sometimes more and sometimes less (comp. Scholz, *Geschichte der Logik*, 1931). The problem of the possibility of a full formalization of logic will be the chief subject-matter of Volume II.

For terminological remarks concerning the terms 'syntax' and 'formal', see § 37.

§ 5. Descriptive and Pure Semantics

Descriptive semantics is the empirical investigation of the semantical features of historically given languages. *Pure semantics* is the analysis of semantical systems, i.e. systems of semantical rules. Syntax is divided analogously. The present book is concerned with semantical and syntactical systems and their relations, hence only with pure semantics and syntax.

Semantical investigations are of two different kinds; we shall distinguish them as descriptive and pure semantics. By **descriptive semantics** we mean the description and analysis of the semantical features either of some particular historically given language, e.g. French, or of all historically given languages in general. The first would be *special* descriptive semantics; the second, *general* descriptive semantics. Thus, descriptive semantics describes facts; it is an empirical science. On the other hand, we may set up a system of semantical rules, whether in close connection with a historically given language or freely invented; we call this a *semantical system.* The construction and analysis of semantical systems is called **pure semantics**. The rules of a semantical system S constitute, as we shall see, nothing else than a definition of certain semantical concepts with respect to S, e.g. 'designation in S' or 'true in S'. Pure semantics consists of definitions of this kind and their consequences; therefore, in contradistinction to descriptive semantics, it is entirely analytic and without factual content.

We make an analogous distinction between **descriptive** and **pure syntax** (compare [Syntax] §§ 2 and 24), and divide these fields into two parts, *special* and *general syntax* (compare [Syntax] § 46). Descriptive syntax is an empirical investigation of the syntactical features of given languages. Pure syntax deals with syntactical systems. A syntactical system (or calculus) K consists of rules which define syntactical concepts, e.g. 'sentence in K', 'provable in K', 'derivable in K'. Pure syntax contains the analytic sentences of the metalanguage which follow from these definitions. Both in semantics and in syntax the relation between the pure and the descriptive field is perfectly analogous to the relation between pure or mathematical geometry, which is a part of mathematics and hence analytic, and physical geometry, which is a part of physics and hence empirical (compare [Syntax] § 25; [Foundations] § 22).

Sometimes the question is discussed whether semantics and syntax are dependent upon pragmatics or not. The answer is that in one sense they are but in another they are not. Descriptive semantics and syntax are indeed based on pragmatics. Suppose we wish to study the semantical and syntactical properties of a certain Eskimo language not previously investigated. Obviously, there is no other way than first to observe the speaking habits of the people who use it. Only after finding by observation the pragmatical

fact that those people have the habit of using the word 'igloo' when they intend to refer to a house are we in a position to make the semantical statement "'igloo' means (designates) house" and the syntactical statement "'igloo' is a predicate". In this way all knowledge in the field of descriptive semantics and descriptive syntax is based upon previous knowledge in pragmatics. *Linguistics*, in the widest sense, is that branch of science which contains all empirical investigation concerning languages. It is the descriptive, empirical part of semiotic (of spoken or written languages); hence it consists of pragmatics, descriptive semantics, and descriptive syntax. But these three parts are not on the same level; *pragmatics is the basis for all of linguistics.* However, this does not mean that, within linguistics, we must always explicitly refer to the users of the language in question. Once the semantical and syntactical features of a language have been found by way of pragmatics, we may turn our attention away from the users and restrict it to those semantical and syntactical features. Thus e.g. the two statements mentioned before no longer contain explicit pragmatical references. In this way, descriptive semantics and syntax are, strictly speaking, parts of pragmatics.

With respect to pure semantics and syntax the situation is different. These fields are independent of pragmatics. Here we lay down definitions for certain concepts, usually in the form of rules, and study the analytic consequences of these definitions. In choosing the rules we are entirely free. Sometimes we may be guided in our choice by the consideration of a given language, that is, by pragmatical facts. But this concerns only the motivation of our choice and has no bearing upon the correctness of the results of our analysis of the rules. (Analogy: the fact that somebody's garden has the shape of a pentagon may induce him to direct his studies in mathematical geometry to pentagons, or rather to certain abstract structures which correspond in a certain way to bodies of pentagonal shape; the shape of his garden guides his interests but does not constitute a basis for the results of his study.)

This treatise is devoted to *pure semantics* and *pure syntax*, or rather to the field in which semantical systems and syntactical systems, and in addition their relations, are analyzed. (There is so far no suitable name for this field; see terminological remarks, § 37, 'Theory of Systems'.) There will occasionally also occur examples referring to semantical or syntactical features of historical languages, say English or French, apparently belonging to descriptive semantics or syntax. But these examples are in fact meant as referring to semantical or syntactical systems which either are actually constructed or could be constructed in close connection with those languages.

Examples. Suppose that we make the statement, "The sentences 'Napoleon was born in Corsica' and 'Napoleon was not born in Corsica' are logically exclusive (incompatible) in English". This is meant as based upon a system E of semantical rules, especially a rule for 'not', constructed in consideration of the English language. The system E is tacitly or explicitly presupposed in this statement; it might be that a rule for 'not' has really been given previously, or it might be that it has not but easily could be given. In any case, concepts of logical analysis like 'logically exclusive', 'logically equivalent', etc., can only be applied on the basis of a system of rules.

The subject-matter of this treatise is restricted in still another direction, as compared with that of semiotic in general. Our discussions apply *only to declarative sentences*, leaving aside all sentences of other kinds, e.g. questions, imperatives, etc.; and hence only to language systems (semantical systems) consisting of declarative sentences. Our terminology is to be understood in this restricted sense; 'sentence' is short for 'declarative sentence', 'language' for 'language

(system) consisting of declarative sentences', 'English' for 'that part of English which consists of declarative sentences', 'interpretation of a sentence of a calculus' for 'interpretation of the sentence as a declarative sentence', etc.

Not much has been done so far in the logical analysis of other than declarative sentences. Concerning *imperatives* and ought-sentences see: E. Mally, *Grundgesetze des Sollens; Elemente der Logik des Willens*, 1926; W. Dubislav, "Zur Unbegründbarkeit der Forderungssätze", *Theoria* 3, 1937; J. Jørgensen, "Imperatives and Logic", *Erkenntnis* 7, 1938; K. Menger, "A Logic of the Doubtful: On Optative and Imperative Logic", *Reports of a Math. Colloquium*, 2nd ser., no. 1, pp. 53–64; R. Rand, "Logik der Forderungssätze", *Zeitschr. f. Theorie d. Rechtes*, 1939; A. Hofstadter and J. C. C. McKinsey, "On the Logic of Imperatives", *Phil. of Sc.* 6, pp. 446–457, 1939. Concerning *questions* see short remarks in [Syntax] § 76, and in Hofstadter and McKinsey, *loc. cit.*, p. 454.

§ 6. Survey of Some Symbols and Terms of Symbolic Logic

Symbols and technical terms are listed here for later use in this book. Features deviating from other authors are chiefly found in the following paragraphs: use of letters; terminology of designata; (series and sequences); German letters; metalanguage.

† For terminological remarks concerning the terms marked by an obelisk, see § 37.

In the subsequent discussions we shall often make use of symbolic logic, especially its elementary parts. Therefore a brief survey of the symbols, letters, and terms used will be given here. We shall later apply these symbols chiefly in examples of sentences in object languages, but occasionally also in a metalanguage. While we usually take the ordinary English word-language as metalanguage, it will sometimes be convenient, for greater clarity and precision, to use a few symbols in the metalanguage, either in combination with English words or alone.

	Symbol	Translation
1. *Propositional† calculus†*	*connectives*	
one-place { negation†	'∼ . . .'	'not . . .'
two-place connections: disjunction†	'. . . ∨ - - -'	'. . . or - - -'
conjunction†	'. . . • - - -'	'. . . and - - -'
implication†	'. . . ⊃ - - -'	'not . . . or - - -' (or: 'if . . . then - - -')
equivalence†	'. . . ≡ - - -'	'. . . if and only if - - -'
2. *Functional† calculus†*		
universality	{'. . *x* . .' / '(*x*) (. . *x* . .)'}	'for every *x*, . . *x* . .'
existence	'(∃*x*) (. . *x* . .)'	'for some (i.e. at least one) *x*, . . *x* . .' (or: 'there is an *x* such that . . *x* . .')
abstraction	'(λ*x*) (. . *x* . .)'	'the class of all *x* such that . . *x* . .'
	'(λ*x*, *y*) (. . *x* . . *y* . .)'	'the relation between *x* and *y* such that . . *x* . . *y* . .'
identity	'*x* = *y*'	'*x* is identical with (i.e. the same object as) *y*'

Use of *letters* for the different types.

	Constants	Variables
individual signs	'a', 'b', etc.	'*x*', '*y*', '*z*', etc.
predicates (level 1), degree 1	'P', 'Q'	'*F*', '*G*'
predicates (level 1), degree 2	'R', 'S'	'*H*', '*L*'
functors	'k', 'l'	'*f*', '*g*'
propositional signs†	'A', 'B', etc.	'*p*', '*q*', etc.
signs without types		'*u*', '*v*', etc.

Examples of sentences. 'P(a)' means "a is P (i.e. has the property P)"; 'R(a,b)' "a has the relation R to b"; 'M(P)' "P is M (i.e. the property P has the property of second level M)".

Individual signs designate the individuals of the realm in

question (objects); they belong to the zero **level**. Their properties and relations, and the **predicates** by which these are designated, belong to the first level. An **attribute** (i.e. a property or a relation) attributed to something of the level *n*, and the predicate designating it, belong to the level $n + 1$. A predicate of **degree** 1 (also called one-place predicate) designates a property; a predicate of degree *n* (*n*-place predicate) designates an *n*-adic relation, i.e. a relation holding between *n* members.

Examples of **functors**: 'prod', 'temp'; 'prod(*m*,*n*)' designates the product of the numbers *m* and *n*, 'temp(*x*)' the temperature of the body *x*.

A **definition** has the form '. . . =Df - - -'; this means: "'. . .' is to be interchangeable with '- - -'" (see § 24). Sometimes, instead of '=Df', '≡' (between sentences) or '=' (between other expressions) is used. '. . .' is called the **definiendum**, '- - -' the **definiens**.

Classification of forms of sentences. **Atomic sentences** are those which contain neither connectives nor variables (e.g. 'R(a,b)', 'b = c'); a **molecular** sentence is one not containing variables but consisting of atomic sentences (called its **components**) and connectives (e.g. '∼P(a)', 'A ∨ B'); a **general** sentence is one containing a variable (e.g. '(∃*x*)P(*x*)').

In a sentence of the form '(*x*) (. . .)' or '(∃*x*) (. . .)' or an expression of the form '(λ*x*) (. . *x* . .)', '(*x*)', '(∃*x*)', and '(λ*x*)' are called **operators** (universal, existential, and lambda- operator, respectively); '. . .' is called the **operand** belonging to the operator. A variable at a certain place in an expression is called **bound** if it stands at that place in an operator or in an operand whose operator contains the same variable; otherwise it is called **free**. An expression is called **open**, if it contains a free variable; otherwise **closed**. (A class of sentences is called closed if all its sentences are closed; this concept must be distinguished from that of a class closed with respect to a certain relation.) An open expression will also be called an **expressional function**†; and, moreover, an expressional function of degree n, if the number of (different) variables occurring in it as free variables is n. An expressional function such that it or the closed expressions constructed out of it by substitution are sentences is called a **sentential function**†.

Terminology of designata. In this treatise, the following terms for designata will be used. (Some of them do not seem to me quite satisfactory; they will be changed as soon as better ones have been proposed.)

SIGNS OR EXPRESSIONS	DESIGNATA			
individual constants	**individuals**			**entities**
predicates of degree 1	**properties** (classes)	**attributes**	**functions** (†II) or **concepts**†	**entities**
predicates of degree 2 and higher	**relations**	**attributes**	**functions** (†II) or **concepts**†	**entities**
functors	**functions** (†IIB)		**functions** (†II) or **concepts**†	**entities**
sentences	**propositions**†			**entities**

Series and sequences. There are two different ways of ordering objects in a linear order; it can be done by a series or by a sequence. A **series** of n objects is a transitive, irreflexive, and connected relation ('x precedes y'). A **sequence** with n members is, so to speak, an enumeration of the objects (at most n); it can be represented in two different ways: (1) by a predicate of degree 2 which designates a one-many relation between the objects and the ordinal numbers up to n, (2) by an argument expression containing n terms (in this case, the argument expression and the sequence designated are said to be of degree n). [*Example:* Suppose we want to order the objects b, c, d in such a way that we take first b, then c, then d, then c again. Thus we have a sequence with $n = 4$ but only three objects. This sequence may be represented in either of the following ways: (1) by '{b;1, c;2, d;3, c;4}' i.e. as the relation which correlates the object b to the number 1, c to 2 and also to 4, and d to 3; (2) by 'b;c;d;c'. If the objects

are individuals, the expression in (1) is of the first level, that in (2) of the zero level. Thus method (2) leads to simpler formulations; we shall apply it in this book. The sentence 'T(b,c,d,c)' is usually paraphrased in about this way: "The relation T (of degree 4) holds for the objects b, c, d and c in this order"; on the basis of method (2), we shall permit, in addition, the following formulation: "The relation T (of degree 4) holds for the sequence (of degree 4) b;c;d;c."] In a sequence, repetitions are possible, i.e. the same member may occur at several places (e.g., c in the example given). In a series, this is impossible because of its irreflexivity. Therefore, in many cases we cannot use series but have to use sequences (e.g. in the representation of expression-designs, § 3 at the end).

German letters are used as signs of the metalanguage designating kinds of signs or expressions of the object language. '$\mathfrak{i}$' designates (the class of) individual variables, '$\mathfrak{in}$' individual signs (including variables), '$\mathfrak{p}$' predicate variables, '$\mathfrak{pr}$' predicates (including variables), '$\mathfrak{f}$' functor variables, '$\mathfrak{fu}$' functors (including variables), '$\mathfrak{s}$' propositional variables†, '$\mathfrak{se}$' propositional signs (including variables), '$\mathfrak{S}$' sentences (including propositional signs); '$\mathfrak{v}$' variables (of any kind), '$\mathfrak{c}$' constants, '$\mathfrak{a}$' signs, '$\mathfrak{A}$' expressions; '$\mathfrak{K}$' classes of expressions (in most cases classes of sentences); '$\mathfrak{T}$' sentences and classes of sentences (see § 9). '$\mathfrak{pr}^n$' designates predicates of degree n, '${}^m\mathfrak{pr}$' predicates of level m, e.g. '${}^2\mathfrak{pr}^1$' predicates of first degree and second level; analogously with '$\mathfrak{p}$', '$\mathfrak{fu}$', and '$\mathfrak{f}$'. A constant of the metalanguage designating a particular sign (-design) or expression (-design) of one of the kinds mentioned is formed with the help of a figure as subscript; a corresponding variable of the metalanguage with the help of a letter 'i', 'j', etc., as subscript. Thus '$\mathfrak{in}_1$' is the name (in the metalanguage) of a particular individual constant (of the object language), e.g. 'a'; '$\mathfrak{in}_2$' of another one, e.g. 'b', etc.; '${}^1\mathfrak{pr}_1^2$' of a predicate of first level and second degree, e.g. 'R'; '$\mathfrak{S}_3$' of a particular sentence, e.g. 'Q(b)'. "If $\mathfrak{pr}_i$ occurs in $\mathfrak{S}_j$, then . . ." is short for "if a predicate $\mathfrak{pr}_i$ occurs in a sentence $\mathfrak{S}_j$, then . . .". '$\mathfrak{A}_j\binom{\mathfrak{v}_k}{\mathfrak{A}_i}$' designates that expression which is constructed out of $\mathfrak{A}_j$ by substituting $\mathfrak{A}_i$ for $\mathfrak{v}_k$ (i.e. by replacing $\mathfrak{v}_k$ at every place where it occurs as a free variable in $\mathfrak{A}_j$ by $\mathfrak{A}_i$). The designation of a compound expression is formed by putting the designations of its parts one after the other in the order in which the parts occur in the expression; signs which are not letters (e.g. brackets, comma, connectives, etc.) are in this procedure designated by themselves. Thus e.g. '$\mathfrak{pr}_1$ ($\mathfrak{in}_2$, $\mathfrak{in}_1$)' (with the above examples) designates the expression 'R(b,a)'; $\mathfrak{S}_3 \vee \mathfrak{S}_2$ is the sentence which consists of $\mathfrak{S}_3$ (this may be 'Q(b)') followed by '$\vee$' followed by $\mathfrak{S}_2$.

As *metalanguage* we shall usually employ the English word-language, but supplemented by symbols, for the sake of brevity and precision. In this way, we shall use the German letter symbols just explained, and occasionally also certain symbols of symbolic logic, among them variables (e.g. 'x', 'F', etc.), operators (e.g. '(x)', '$(\exists F)$', '(λx)', etc.), the signs of identity ('$=$') and of definition ('$=_{Df}$'). '$=_{Df}$' is to mean 'is (hereby defined to be) the same as' or 'if and only if'. Further, with respect to classes, especially $\mathfrak{K}$, we use the customary symbols of the theory of sets: '$x \epsilon \mathfrak{K}_j$' means "$x$ is an element of $\mathfrak{K}_j$"; '$\mathfrak{K}_i \subset \mathfrak{K}_j$' means "$\mathfrak{K}_i$ is a sub-class of $\mathfrak{K}_j$"; $-\mathfrak{K}_i$ is the complement of $\mathfrak{K}_i$, i.e. the class of all elements (of the type in question) not belonging to $\mathfrak{K}_i$; $\mathfrak{K}_i + \mathfrak{K}_j$ is the sum of $\mathfrak{K}_i$ and $\mathfrak{K}_j$, i.e. the class containing all elements of $\mathfrak{K}_i$ and all elements of $\mathfrak{K}_j$; $\mathfrak{K}_i \times \mathfrak{K}_j$ is the product of $\mathfrak{K}_i$ and $\mathfrak{K}_j$, i.e. the class of all elements belonging to both classes. (If in $\mathfrak{T}_i + \mathfrak{T}_j$, $\mathfrak{T}_i$ or $\mathfrak{T}_j$ is not a class but a sentence $\mathfrak{S}_k$, then its unit class $\{\mathfrak{S}_k\}$ is meant as component of the sum.) $\{x\}$ is the class whose only element is x; $\{x_1, x_2, .. x_n\}$ is the class whose elements are $x_1, x_2, .. x_n$. If $\mathfrak{M}_i$ is a class of

classes, pr($\mathfrak{M}_i$) is the product of the classes of $\mathfrak{M}_i$ (if $\mathfrak{M}_i$ is null, pr($\mathfrak{M}_i$) is the universal class).

As first introductions into *symbolic logic* for beginners see Cooley [Logic] and Tarski [Logic]. On a higher technical level see Whitehead and Russell [Princ. Math.], Quine [Math. Logic], Church [Logic], Carnap [Logic].

B. SEMANTICS

The construction of semantical systems is explained. Semantical concepts are introduced, especially truth, designation, and other concepts defined with their help.

§ 7. Semantical Systems

A *semantical system* is a system of rules which state *truth-conditions* for the sentences of an object language and thereby determine the meaning of these sentences. A semantical system *S* may consist of *rules of formation*, defining 'sentence in *S*', *rules of designation*, defining 'designation in *S*', and *rules of truth*, defining 'true in *S*'. The sentence in the metalanguage '$\mathfrak{S}_i$ is true in *S*' means the same as the sentence $\mathfrak{S}_i$ itself. This characteristic constitutes a condition for the *adequacy* of definitions of truth.

By a **semantical system** (or interpreted system) we understand a system of rules, formulated in a metalanguage and referring to an object language, of such a kind that the rules determine a **truth-condition** for every sentence of the object language, i.e. a sufficient and necessary condition for its truth. In this way the sentences are *interpreted* by the rules, i.e. made understandable, because to understand a sentence, to know what is asserted by it, is the same as to know under what conditions it would be true. To formulate it in still another way: the rules determine the *meaning* or *sense* of the sentences. Truth and falsity are called the **truth-values** of sentences. To know the truth-condition of a sentence is (in most cases) much less than to know its truth-value, but it is the necessary starting point for finding out its truth-value.

Example. Suppose that Pierre says: "Mon crayon est noir" ($\mathfrak{S}_1$). Then, if we know French, we understand the sentence $\mathfrak{S}_1$ although we may not know its truth-value. Our understanding of $\mathfrak{S}_1$ consists

in our knowledge of its truth-condition; we know that $\mathfrak{S}_1$ is true if and only if a certain object, Pierre's pencil, has a certain color, black. This knowledge of the truth-condition for $\mathfrak{S}_1$ tells us what we must do in order to determine the truth-value of $\mathfrak{S}_1$, i.e. to find out whether $\mathfrak{S}_1$ is true or false; what we must do in this case is to observe the color of Pierre's pencil.

In what way can the truth-conditions for the sentences of a system be stated? If the system contains only a finite number of sentences, then we may give a full list of the truth-conditions, one for each sentence. This is done, for instance, in the ordinary cable codes. A code translates each sentence separately and thereby interprets it. Hence a code is a semantical system, but one of a primitive kind. We may thus distinguish two chief kinds of semantical systems, *code systems* and *language systems*. A code system lists the truth-conditions separately for each sentence, while a language system gives general rules for partial expressions of sentences in such a way that the truth-condition for every sentence is determined by the rules for the expressions of which it consists. In the case of the ordinary cable codes, flag codes, and the like, only the first form, that of particular rules, is possible. In the case of a language system containing an infinite number of sentences, only the second form, that of general rules, is possible, because we cannot formulate an infinite number of rules. There are cases of languages with a finite number of sentences where either form is applicable.

Examples. 1. We construct a semantical system S_1 in the following way. S_1 (that is to say, the object language of S_1) contains seven signs: three individual constants, $\mathfrak{in}_1$, $\mathfrak{in}_2$, $\mathfrak{in}_3$, two predicates, $\mathfrak{pr}_1$ and $\mathfrak{pr}_2$, and the two parentheses '(' and ')'. [In order to be able to write down actual examples of sentences of S_1, we may choose some letters as the first five signs, e.g. 'a', 'b', 'c', 'P', 'Q'. But this choice is obviously irrelevant for the semantical properties of S_1 and is therefore, strictly speaking, outside of pure semantics. Its role is the same as that of diagrams in geometry; they facilitate the operations practically but have no theoretical bearing on the proofs.] Sentences of S_1 are the expressions of the form 𝔭𝔯 (𝔦𝔫). The truth-conditions are given separately for each sentence by the following rules:

1. $\mathfrak{pr}_1$ ($\mathfrak{in}_1$) is true if and only if Chicago is large.
2. $\mathfrak{pr}_1$ ($\mathfrak{in}_2$) is true if and only if New York is large.
3. $\mathfrak{pr}_1$ ($\mathfrak{in}_3$) is true if and only if Carmel is large.
4. $\mathfrak{pr}_2$ ($\mathfrak{in}_1$) is true if and only if Chicago has a harbor.
5. $\mathfrak{pr}_2$ ($\mathfrak{in}_2$) is true if and only if New York has a harbor.
6. $\mathfrak{pr}_2$ ($\mathfrak{in}_3$) is true if and only if Carmel has a harbor.

2. We construct the semantical system S_2 in the following way. S_2 contains the same signs and sentences as S_1. We give five particular rules each specifying the designatum of one of the five chief signs, and one general rule for the truth-conditions of the sentences:

1. $\mathfrak{in}_1$ designates Chicago.
2. $\mathfrak{in}_2$ designates New York.
3. $\mathfrak{in}_3$ designates Carmel.
4. $\mathfrak{pr}_1$ designates the property of being large.
5. $\mathfrak{pr}_2$ designates the property of having a harbor.
6. A sentence $\mathfrak{pr}_i$ ($\mathfrak{in}_j$) is true if and only if the designatum of $\mathfrak{in}_j$ has the designatum of $\mathfrak{pr}_i$ (i.e. the object designated by $\mathfrak{in}_j$ has the property designated by $\mathfrak{pr}_i$). The systems S_1 and S_2 contain the same sentences, and every sentence has the same truth-condition (interpretation, meaning) in both systems. Hence they are essentially alike, but differ with respect to the kinds of rules applied; S_1 is a code system, S_2 a language system.

As the previous and the following examples show, a *semantical system* may be constructed in this way: first a *classification of the signs* is given, then **rules of formation** are laid down, then **rules of designation**, and finally **rules of truth**. By the rules of formation of a system *S* the term 'sentence of *S*' is defined; by the rules of designation 'designation in *S*'; by the rules of truth 'true in *S*'. The definition of 'true in *S*' is the real aim of the whole system *S*; the other definitions serve as preparatory steps for this one, making its formulation simpler. On the basis of 'true in *S*', other semantical concepts with respect to *S* can be defined, as we

shall see later. (The simplest one is the definition of falsity: a sentence $\mathfrak{S}_i$ of *S* is *false* in *S* $=_{\text{Df}}$ $\mathfrak{S}_i$ is not true in *S*.) It is especially important to be aware of the fact that the rules of designation do not make factual assertions as to what are the designata of certain signs. There are no factual assertions in pure semantics. The rules merely lay down conventions in the form of a definition of 'designation in *S*'; this is done by an enumeration of the cases in which the relation of designation is to hold. Sometimes the term 'designation' is also used for compound expressions and even for sentences; this will be discussed later (§ 12). In this case, the rules of designation define by enumeration the preliminary term 'direct designation'; and with its help the more general term 'designation' is defined recursively.

In the case of the very simple system S_2 it can easily be shown that the rules of designation define 'designation' by enumeration. We can transform those rules into an explicit definition:

> *x designates t* in S_2 $=_{\text{Df}}$ ($x = \mathfrak{in}_1$ and t = Chicago) or ($x = \mathfrak{in}_2$ and t = New York) or ($x = \mathfrak{in}_3$ and t = Carmel) or ($x = \mathfrak{pr}_1$ and t = the property of being large) or ($x = \mathfrak{pr}_2$ and t = the property of having a harbor).

('t' is here a variable not satisfying the ordinary rule of types; its range of values comprehends both individuals and properties. The problem involved here will be discussed later; see § 12.)

It will now be shown that the whole set of rules of formation, rules of designation and rules of truth for S_2 can be brought into the form of a definition for 'true in S_2', based upon a classification of the signs of S_2. (The classes $\mathfrak{K}_1$ to $\mathfrak{K}_4$ are meant as 𝔦𝔫, 𝔭𝔯, {'('}, and {')'} respectively; but this need not be mentioned in the formulation of the system.)

1. Classification. S_2 contains four (mutually exclusive) classes of signs, $\mathfrak{K}_1$, $\mathfrak{K}_2$, $\mathfrak{K}_3$, and $\mathfrak{K}_4$; $\mathfrak{K}_1$ contains (only) the signs $\mathfrak{a}_1$, $\mathfrak{a}_2$, $\mathfrak{a}_3$; $\mathfrak{K}_2$, $\mathfrak{a}_4$ and $\mathfrak{a}_5$; $\mathfrak{K}_3$, $\mathfrak{a}_6$; $\mathfrak{K}_4$, $\mathfrak{a}_7$.

2. $\mathfrak{A}_i$ is true in S_2 $=_{\text{Df}}$ $(\exists x)$ $(\exists y)$ $(\exists z)$ $(\exists F)$ [$\mathfrak{A}_i$ consists of x, $\mathfrak{a}_6$, y, $\mathfrak{a}_7$ in this order and $x \in \mathfrak{K}_2$ and $y \in \mathfrak{K}_1$ and [($y = \mathfrak{a}_1$ and z = Chicago) or ($y = \mathfrak{a}_2$ and z = New York) or ($y = \mathfrak{a}_3$ and z = Carmel)] and [($x = \mathfrak{a}_4$

and F = the property of being large) or ($x = \mathfrak{a}_5$ and F = the property of having a harbor)] and $F(z)$].

By this definition, the system S_2 is established.

A remark may be added as to the way in which the term '*true*' is used in these discussions. We apply this term chiefly to sentences (and later to classes of sentences also). [The term may also be applied in an analogous way to propositions as designata of sentences (see D17-1); but this use will not occur often in the following discussions; compare the terminological remarks in § 37.] We use the term here in such a sense that *to assert that a sentence is true means the same as to assert the sentence itself*; e.g. the two statements "The sentence 'The moon is round' is true" and "The moon is round" are merely two different formulations of the same assertion. (The two statements mean the same in a logical or semantical sense; from the point of view of pragmatics, in this as in nearly every case, two different formulations have different features and different conditions of application; from this point of view we may e.g. point to the difference between these two statements in emphasis and emotional function.)

The decision just mentioned concerning the use of the term 'true' is itself not a definition for 'true'. It is rather a standard by which we judge whether a definition for truth is adequate, i.e. in accordance with our intention. If a definition of a predicate $\mathfrak{pr}_i$ — e.g. the word 'true' or 'valid' or any sign arbitrarily chosen — is proposed as a definition of truth, then we shall accept it as an adequate definition of truth if and only if, on the basis of this definition, $\mathfrak{pr}_i$ fulfills the condition mentioned above, namely that it yields sentences like " 'The moon is round' is . . . if and only if the moon is round", where $\mathfrak{pr}_i$ (e.g. 'true') is to be put at the place of '. . .'. This leads to the following definition D7-A.

D7-A. A predicate $\mathfrak{pr}_i$ is an *adequate* predicate (and its

definition an adequate definition) for the concept of *truth* within a certain class of sentences $\mathfrak{K}_j$ $=_{\mathrm{Df}}$ every sentence which is constructed out of the sentential function 'x is F if and only if p' by substituting $\mathfrak{pr}_i$ for 'F', any sentence $\mathfrak{S}_k$ of $\mathfrak{K}_j$ for 'p', and any name (syntactical description) of $\mathfrak{S}_k$ for 'x', follows from the definition of $\mathfrak{pr}_i$.

Example. Let $\mathfrak{K}_j$ contain the sentence 'Chicago is a city'. Let '$\mathfrak{S}_1$' be a name of this sentence. Suppose that somebody introduces the word 'verum' into English by a certain definition D. In order to apply D7-A, we have to examine all sentences constructed in the way described in D7-A. By putting 'verum' for 'F', 'Chicago is a city' for 'p', and '$\mathfrak{S}_1$' for 'x', we obtain '$\mathfrak{S}_1$ is verum if and only if Chicago is a city'. If our examination comes to the result that D is of such a kind that this and all analogous sentences follow from D, then, according to D7-A, we shall call 'verum' an adequate predicate for truth and the proposed definition D an adequate definition for truth. This is practically justified by the fact that the result mentioned shows that the new word 'verum' as introduced by D is used in the same way as the ordinary word 'true' according to the decision mentioned above.

D7-A is the simplest form of the definition of adequacy; it refers only to the special case where the sentences to which the predicate for the concept of truth is applied belong to the same language as this predicate — in other words, where the object language is the same as (or part of) the metalanguage. In general, object language S and metalanguage M are different. In this case, the following more general definition of adequacy applies. (This definition is due to Tarski; see below.)

D7-B. A predicate $\mathfrak{pr}_i$ in M is an *adequate* predicate (and its definition an adequate definition) for the concept of *truth* with respect to an object language S $=_{\mathrm{Df}}$ from the definition of $\mathfrak{pr}_i$ every sentence in M follows which is constructed out of the sentential function 'x is F if and only if p' by substituting $\mathfrak{pr}_i$ for 'F', a translation of any sentence $\mathfrak{S}_k$ of S

into M for 'p', and any name (syntactical description) of $\mathfrak{S}_k$ for 'x'.

Example. Let S be a certain part of the German language, containing among others the sentence 'Der Mond ist rund'. Let '$\mathfrak{S}_2$' be the name of this sentence. We take English as metalanguage M. The translation of $\mathfrak{S}_2$ in M is 'The moon is round'. Suppose that a definition D_2 for the sign 'T' is proposed and that we wish to find out whether D_2 is an adequate definition for truth with respect to the part S of the German language. According to D7–B, one of the sentences to be examined is constructed by substituting 'T' for 'F', the translation 'The moon is round' for 'p', and '$\mathfrak{S}_2$' for 'x'. Thus we obtain the sentence '$\mathfrak{S}_2$ is T if and only if the moon is round'. If this and all analogous sentences are found to follow from the definition D_2 of 'T', then D_2 is an adequate definition and 'T' an adequate predicate for truth in S.

It can easily be shown that two predicates each of which is an adequate predicate for truth with respect to the same object language S have the same extension (they are equivalent, D10–11b, and even L-equivalent, T22–13).

It is especially to be noticed that the concept of truth in the sense just explained — we may call it the *semantical concept of truth* — is fundamentally different from concepts like 'believed', 'verified', 'highly confirmed', etc. The latter concepts belong to pragmatics and require a reference to a person.

In order to make clearer the distinction just mentioned, let us consider the following example. 'The moon has no atmosphere' ($\mathfrak{S}_1$); '$\mathfrak{S}_1$ is true' ($\mathfrak{S}_2$); '$\mathfrak{S}_1$ is confirmed to a very high degree by scientists at the present time' ($\mathfrak{S}_3$). $\mathfrak{S}_2$ says the same as $\mathfrak{S}_1$; $\mathfrak{S}_2$ is, like $\mathfrak{S}_1$, an astronomical statement and is, like $\mathfrak{S}_1$, to be tested by astronomical observations of the moon. On the other hand, $\mathfrak{S}_3$ is a historical statement; it is to be tested by historical, psychological observations of the behavior of astronomers.

Wittgenstein ([Tractatus] 4.024, 4.46) has emphasized the point of view that the truth-conditions of a sentence constitute its meaning, and that understanding consists in knowing these conditions. This

view is also connected with his conception of logical truth (compare quotations given at the end of § 18A).

According to Tarski ([Wahrheitsbegriff] p. 267), S. Lesniewski was the first to formulate an exact requirement of adequacy for the definition of truth, in the simple form of D7–A above (in unpublished lectures since 1919); and similar formulations are found in a Polish book on the theory of knowledge by T. Kotarbinski (1926). F. P. Ramsey, in his review (1923) of Wittgenstein's book, gives a related formulation: "If a thought or proposition token 'p' says p, then it is called true if p, and false if $\sim p$" ("Foundations of Mathematics", p. 275). Tarski himself gave the more general form (like D7–B above) of the definition of adequacy (his "Konvention $\mathfrak{W}$", [Wahrheitsbegriff] p. 305). Further, he gave the first exact definition for truth with respect to certain formalized languages; his definition fulfills the requirement of adequacy and simultaneously avoids the antinomies connected with an unrestricted use of the concept of truth as e.g. in everyday language. In the same work [Wahrheitsbegriff], Tarski comes to very valuable results by his analysis of the concept of truth and related semantical concepts. These results are of a highly technical nature and therefore cannot be explained in this introductory Volume I.

The requirement mentioned is not meant as a new theory or conception of truth. Kotarbinski has already remarked that it is the old classical conception which dates back to Aristotle. The new feature is only the more precise formulation of the requirement. Tarski says further that the characterization given is also in agreement with the ordinary use of the word 'true'. It seems to me that he is right in this assertion, at least as far as the use in science, in judicial proceedings, in discussions of everyday life on theoretical questions is concerned. But I will not stress this point; it may be remarked that Arne Ness has expressed some doubts about the assertion, based on systematic questioning of people. At any rate, this question is of a pragmatical (historical, psychological) nature and has not much bearing on the questions of the method and results of semantics.

§ 8. Truth-Tables as Semantical Rules

The customary truth-tables are semantical truth-rules in the form of diagrams. The rules of formation, and likewise the rules of truth, for molecular sentences may be stated in the form of a recursive definition, specifying the condition first for atomic sentences and then for molecular sentences with reference to their components.

The semantical systems considered so far contain only atomic sentences. Now we come to systems possessing connectives and molecular sentences constructed with their help. The number of sentences in a system of this kind is infinite. This is the case with nearly all symbolic systems usually dealt with, and also with the natural languages. [In English, for instance, for any given sentence, however long, we can construct a longer sentence by adding 'and the moon is round'; therefore the number of sentences is infinite.]

The connectives are often introduced with the help of **truth-tables**. It is easily seen that a truth-table is nothing but a semantical rule in the form of a diagram. Take e.g. the table of disjunction (usually written in a less correct way with variables '*p*', etc., of the object language instead of signs '$\mathfrak{S}_i$', etc., of the metalanguage):

	$\mathfrak{S}_i$	$\mathfrak{S}_j$	$\mathfrak{S}_i \vee \mathfrak{S}_j$
1.	T	T	T
2.	T	F	T
3.	F	T	T
4.	F	F	F

The four lines of the table are meant to say this: 1. If $\mathfrak{S}_i$ is true and $\mathfrak{S}_j$ is true, $\mathfrak{S}_i \vee \mathfrak{S}_j$ is true; 2. if $\mathfrak{S}_i$ is true and $\mathfrak{S}_j$ is false, $\mathfrak{S}_i \vee \mathfrak{S}_j$ is true; 3. if $\mathfrak{S}_i$ is false and $\mathfrak{S}_j$ is true, $\mathfrak{S}_i \vee \mathfrak{S}_j$ is true; 4. if $\mathfrak{S}_i$ is false and $\mathfrak{S}_j$ is false, $\mathfrak{S}_i \vee \mathfrak{S}_j$ is false. Hence the whole table says: $\mathfrak{S}_i \vee \mathfrak{S}_j$ is true if and

only if $\mathfrak{S}_i$ is true or $\mathfrak{S}_j$ is true or both. Thus the table states a truth-condition for the sentences of the form $\mathfrak{S}_i \vee \mathfrak{S}_j$; it says the same as rule (4c) in the example S_3 below.

The customary truth-table for negation is this:

	$\mathfrak{S}_i$	$\sim\mathfrak{S}_i$
1.	T	F
2.	F	T

It says: 1. If $\mathfrak{S}_i$ is true, $\sim\mathfrak{S}_i$ is false; 2. if $\mathfrak{S}_i$ is false, $\sim\mathfrak{S}_i$ is true. In other words, $\sim\mathfrak{S}_i$ is true if and only if $\mathfrak{S}_i$ is false, i.e. not true. Hence it says the same as rule (4b) in the example S_3 below.

In the same way, the customary truth-tables for the other connectives are truth-rules in the form of diagrams. Some of them are reformulated in words in the rules of the example S_4 below.

The rules of formation for a system *S* in which the number of components in a sentence is not limited may be formulated in the following way. First, the form or forms of atomic sentences of *S* are stated, and, second, the operations are described by which compound sentences of *S* may be constructed out of sentences (and sometimes other expressions) of atomic form. Thus the definition of 'sentence in *S*' is not an explicit but a recursive definition. The term defined occurs also in the definiens (see e.g. rules (2) for S_3 below, where '$\mathfrak{S}$' occurs in the definiens). This fact, however, does not make the definition circular. If we wish to determine whether a given expression $\mathfrak{A}_k$ is a sentence, the definition refers us back to the question whether another expression $\mathfrak{A}_i$ is a sentence. But it does so in such a way that $\mathfrak{A}_i$ is a proper part of $\mathfrak{A}_k$. Therefore, after a finite number of applications of the second part of the recursive definition we come to an expression of atomic form and hence to a solution with the help of the first part of the definition. The situation with the rules of truth is similar. They give a recursive definition for 'true in S' in strict analogy to the definition for 'sentence in S'. Therefore, for any given sentence $\mathfrak{S}_i$ of S, the rules of truth determine a truth-condition, although in general they do not determine the truth-value of $\mathfrak{S}_i$.

Examples of semantical systems. To facilitate understanding, we formulate the rules in the following systems by using signs and expressions of the object language in quotes. The exact method using names of the signs (German letters) has been shown in § 7.

Semantical System S_3

1. Classification of signs. Three $\mathfrak{in}$ ('a', 'b', 'c'), two $\mathfrak{pr}$ ('P', 'Q'); further single signs: '$\sim$', '$\vee$', '(', ')'.

2. Rules of formation. An expression $\mathfrak{A}_k$ in S_3 is a *sentence* ($\mathfrak{S}$) in $S_3 =_{Df}$ $\mathfrak{A}_k$ has one of the following forms:

   a. $\mathfrak{pr}(\mathfrak{in})$; b. $\sim(\mathfrak{S}_i)$; c. $(\mathfrak{S}_i) \vee (\mathfrak{S}_j)$.

3. Rules of designation. $\mathfrak{a}_i$ *designates* (an entity) u in $S_3 =_{Df}$ $\mathfrak{a}_i$ is the first and u the second member in one of the following pairs: a. 'a', Chicago; b. 'b', New York; c. 'c', Carmel; d. 'P', the property of being large; e. 'Q', the property of having a harbor.

4. Rules of truth. $\mathfrak{S}_k$ is *true* in $S_3 =_{Df}$ one of the following three conditions is fulfilled:

   a. $\mathfrak{S}_k$ has the form $\mathfrak{pr}_i(\mathfrak{in}_j)$, and the object designated by $\mathfrak{in}_j$ has the property designated by $\mathfrak{pr}_i$.
   b. $\mathfrak{S}_k$ has the form $\sim(\mathfrak{S}_i)$, and $\mathfrak{S}_i$ is not true.
   c. $\mathfrak{S}_k$ has the form $(\mathfrak{S}_i) \vee (\mathfrak{S}_j)$, and at least one of the sentences $\mathfrak{S}_i$ and $\mathfrak{S}_j$ is true.

Examples of application of the rules. (While the rules require every component of a connection to be included in parentheses, we shall omit the parentheses here and in later examples under the customary conditions.) Let us examine the expression '$P(c) \vee \sim Q(a)$' ($\mathfrak{A}_1$) on the basis of the rules of S_3. By applying rules (2c) and (2b), and rule (2a) twice, we find that $\mathfrak{A}_1$ is a sentence in S_3. Now we apply rules (4) in order to construct a truth-condition for $\mathfrak{A}_1$ in S_3. According to rule (4c), $\mathfrak{A}_1$ is true in S_3 if and only if '$P(c)$' is true or '$\sim Q(a)$' is true or both. According to (4b), '$\sim Q(a)$' is true if and only if '$Q(a)$' is not true. Hence, $\mathfrak{A}_1$ is true if and only if '$P(c)$' is true or '$Q(a)$' is not true or both. According to (4a) and (3), '$P(c)$' is true if and only if Carmel is large, and '$Q(a)$' is true if and only if Chicago has a harbor. Therefore, $\mathfrak{A}_1$ is true in S_3 if and only if either Carmel is large or Chicago does not have a harbor or both. Thus we have found a truth-condition for $\mathfrak{A}_1$ in S_3 as determined by the rules of S_3. But these rules do not suffice to determine the truth-value of $\mathfrak{A}_1$. In order to find this we must know certain facts in addition to the rules. This would lead us outside of semantics into empirical science, in this case into geography.

Semantical System S_4

1. Classification of signs. The same signs as in S_3, and in addition '$\bullet$', '$\supset$', '$\equiv$'.

2. Rules of formation. (a), (b), and (c) as in S_3; further: d. $(\mathfrak{S}_i) \bullet (\mathfrak{S}_j)$; e. $(\mathfrak{S}_i) \supset (\mathfrak{S}_j)$; f. $(\mathfrak{S}_i) \equiv (\mathfrak{S}_j)$.

3. Rules of designation. The same as in S_3.

4. Rules of truth. (a), (b), and (c) as in S_3; further:

   d. $\mathfrak{S}_k$ has the form $(\mathfrak{S}_i) \bullet (\mathfrak{S}_j)$, and both $\mathfrak{S}_i$ and $\mathfrak{S}_j$ are true.
   e. $\mathfrak{S}_k$ has the form $(\mathfrak{S}_i) \supset (\mathfrak{S}_j)$, and $\mathfrak{S}_i$ is not true or $\mathfrak{S}_j$ is true or both.
   f. $\mathfrak{S}_k$ has the form $(\mathfrak{S}_i) \equiv (\mathfrak{S}_j)$, and $\mathfrak{S}_i$ and $\mathfrak{S}_j$ are either both true or both not true.

§ 9. Radical Concepts

On the basis of the concept of truth, the following concepts, called radical semantical concepts, are defined: 'false', 'implicate', 'equivalent', 'disjunct', 'exclusive', 'comprehensive'. Theorems for these concepts are stated.

By the rules of a semantical system S the concept of truth in S (for sentences) is defined, as we have seen. We shall now define other semantical concepts on this basis. These concepts are called *radical concepts* and their terms *radical terms*, in distinction to terms formed with prefixes ('L-' and 'F-', §§ 14 and 21). We add some theorems; these are based merely on the definitions, not on any postulates; hence they are analytic. In the definitions and theorems we make no special assumptions concerning any particular features of S. Hence these definitions and theorems belong to general semantics. For the sake of brevity, we often omit the phrase 'in S' in connection with a semantical term; but it must be kept in mind that every semantical term has a meaning only with respect to a semantical system and therefore, in a complete formulation, must be accompanied by a reference to a semantical system.

Most of the theorems in this section are not of great importance in themselves but are lemmas to other theorems or serve for later reference. Here and later, the more important definitions, theorems, postulates, etc., are marked by a *plus symbol* '+'. In referring to a definition, a theorem, a postulate, etc., of the same section, we omit the section number (e.g. a reference 'D3' in this section refers to D9-3).

We shall apply the semantical concepts not only to sentences but also to *classes of sentences* (including the null class and transfinite classes). Thus we may e.g. regard a book or a paper as a (finite) class of sentences; and a theory may be regarded as the class (in general transfinite) of all those sentences which are deducible from a given finite set of sentences, e.g. physical laws. Now a book or a paper or a theory is meant as the joint assertion of all sentences belonging to it; hence it seems natural to call it true if and only if those sentences are true (D1).

+**D9-1.** $\mathfrak{K}_i$ is *true* (in S) $=_{\mathrm{Df}}$ every sentence of $\mathfrak{K}_i$ is true.

One possible way of defining the semantical terms for both sentences and sentential classes would be to define them for classes and then to add the general convention that a term may be applied to a sentence $\mathfrak{S}_i$ if and only if it applies to its unit class $\{\mathfrak{S}_i\}$. Instead, we formulate the definitions with the help of '$\mathfrak{T}$' (§ 6); '$\mathfrak{T}_i$' is a variable of the meta-

language whose range of values comprehends both sentences and sentential classes of the object language.

+**D9-2.** $\mathfrak{T}_i$ is **false** (in S) $=_{\mathrm{Df}}$ $\mathfrak{T}_i$ belongs to S and is not true in S.

+**T9-1.** $\mathfrak{K}_i$ is false if and only if at least one sentence of $\mathfrak{K}_i$ is false. (From D2 and 1.)

T9-2. $\mathfrak{T}_i$ is not both true and false. (From D2.)

T9-3. $\mathfrak{T}_i$ is either true or false. (From D2.)

T9-4. If $\mathfrak{K}_j \subset \mathfrak{K}_i$ and $\mathfrak{K}_i$ is true, then $\mathfrak{K}_j$ is true. (From D1.)

T9-5. If $\mathfrak{K}_j \subset \mathfrak{K}_i$ and $\mathfrak{K}_j$ is false, then $\mathfrak{K}_i$ is false. (From T1.)

T9-6. The class of all true sentences of S is true. (From D1.)

T9-7. There is a false sentential class in S if and only if there is a false sentence in S. (From T1; if $\mathfrak{S}_i$ is false, $\{\mathfrak{S}_i\}$ is false.)

T9-8. $\mathfrak{K}_i + \mathfrak{K}_j$ is false if and only if $\mathfrak{K}_i$ is false or $\mathfrak{K}_j$ is false. (From T1.)

The relation of *implication*, to be defined now (D3), must be clearly distinguished from logical implication, to be defined later ('L-implication', § 14). [In order to stress the difference, the first is sometimes called material implication; see terminological remarks, § 37, Connections (1).] Analogously, *equivalence* (D9-4) must be distinguished from logical equivalence ('L-equivalence', § 14). Implication and equivalence as defined here are not logical relations; they do not require any connection between the subject-matter of $\mathfrak{T}_i$ and that of $\mathfrak{T}_j$, but merely certain conditions with respect to the truth-values of $\mathfrak{T}_i$ and $\mathfrak{T}_j$. Therefore, these relations are much less important than the corresponding L-concepts and the corresponding concepts in syntax (C-concepts, § 28); they serve chiefly as a basis for these other concepts. The same holds for the terms 'disjunct', 'exclusive', and 'comprehensive' in relation to the corresponding L-terms and C-terms. For the sake of brevity, we shall often write '$\mathfrak{T}_i \rightarrow \mathfrak{T}_j$' instead of '$\mathfrak{T}_j$ is an implicate of $\mathfrak{T}_i$' (or '$\mathfrak{T}_i$ implies $\mathfrak{T}_j$', a formulation we usually avoid). (Thus the arrow '$\rightarrow$' is here not, as in Hilbert's notation, a connective (of implication) but a predicate of the metalanguage designating a certain relation between sentences, not between propositions.)

+**D9-3.** $\mathfrak{T}_j$ is an **implicate** of $\mathfrak{T}_i$ ($\mathfrak{T}_i$ implies $\mathfrak{T}_j$, $\mathfrak{T}_i \rightarrow \mathfrak{T}_j$) (in S) $=_{\mathrm{Df}}$ $\mathfrak{T}_i$ and $\mathfrak{T}_j$ belong to S, and either $\mathfrak{T}_i$ is false or $\mathfrak{T}_j$ is true (or both).

+**T9-10.** If $\mathfrak{T}_i \rightarrow \mathfrak{T}_j$ and $\mathfrak{T}_i$ is true, $\mathfrak{T}_j$ is true. (From D3 and 2.)

+**T9-11.** If $\mathfrak{T}_i \rightarrow \mathfrak{T}_j$ and $\mathfrak{T}_j$ is false, $\mathfrak{T}_i$ is false. (From D3 and 2.)

T9-12. If $\mathfrak{T}_i$ is false, $\mathfrak{T}_i \rightarrow$ every $\mathfrak{T}_j$. (From D3.)

T9-13. If $\mathfrak{T}_j$ is true, every $\mathfrak{T}_i \rightarrow \mathfrak{T}_j$. (From D3.)

T9-14. The relation of implication is

a) reflexive (i.e. $\mathfrak{T}_i \rightarrow \mathfrak{T}_i$),

b) transitive (i.e., if $\mathfrak{T}_i \rightarrow \mathfrak{T}_j$ and $\mathfrak{T}_j \rightarrow \mathfrak{T}_k$, then $\mathfrak{T}_i \rightarrow \mathfrak{T}_k$). (From T3, D3; T10, T13, T12.)

T9-15. If $\mathfrak{S}_j \in \mathfrak{K}_i$, then $\mathfrak{K}_i \rightarrow \mathfrak{S}_j$. (From D3, D1.)

T9-16. If $\mathfrak{K}_j \subset \mathfrak{K}_i$, then $\mathfrak{K}_i \rightarrow \mathfrak{K}_j$. (From D3, D1.)

T9-17. $\mathfrak{T}_i \rightarrow \mathfrak{K}_j$ if and only if $\mathfrak{T}_i \rightarrow$ every sentence of $\mathfrak{K}_j$. (From D3, D1, T13, T12; T15, T14b.)

T9-18. $\mathfrak{T}_j$ is not an implicate of $\mathfrak{T}_i$ if and only if $\mathfrak{T}_i$ is true and $\mathfrak{T}_j$ is false. (From D3.)

+**D9-4.** $\mathfrak{T}_i$ is **equivalent** to $\mathfrak{T}_j$ (in S) $=_{\mathrm{Df}}$ $\mathfrak{T}_i$ and $\mathfrak{T}_j$ belong to S, and either both are true or neither of them is true.

T9-20. Each of the following conditions is a sufficient and necessary condition for $\mathfrak{T}_i$ and $\mathfrak{T}_j$ to be equivalent (to one another):

+**a.** $\mathfrak{T}_i$ and $\mathfrak{T}_j$ are both true or both false. (From D4, D2.)

+**b.** $\mathfrak{T}_i \rightarrow \mathfrak{T}_j$ and $\mathfrak{T}_j \rightarrow \mathfrak{T}_i$. (From D4, D3.)

T9-21. Each of the following conditions is a sufficient and necessary condition for $\mathfrak{T}_i$ and $\mathfrak{T}_j$ not to be equivalent:

a. Exactly one of them is true.

b. Exactly one of them is false. (From T20a.)

T9-22. $\mathfrak{S}_i$ and $\{\mathfrak{S}_i\}$ are equivalent. (From T20a, D1, T1.)

It is important to notice the difference (1) between a negation sentence, whether in a symbolic language (example $\mathfrak{S}_1$ below) or in English ($\mathfrak{S}_2$), and a sentence about falsity ($\mathfrak{S}_3$); and likewise (2) between an equivalence sentence ($\mathfrak{S}_4$ and $\mathfrak{S}_5$) and a sentence about equivalence ($\mathfrak{S}_6$), and (3) between an implication sentence ($\mathfrak{S}_7$ and $\mathfrak{S}_8$) and a sentence about implication ($\mathfrak{S}_9$).

Examples:

1. $\mathfrak{S}_1$: '$\sim$Q(c)'.
   $\mathfrak{S}_2$: 'Carmel does not have a harbor'.
   $\mathfrak{S}_3$: ' 'Q(c)' is false'.

2. $\mathfrak{S}_4$: 'P(a) $\equiv$ Q(b)'.
   $\mathfrak{S}_5$: 'Chicago is large if and only if New York has a harbor'.
   $\mathfrak{S}_6$: ' 'P(a)' is equivalent to 'Q(b)' '.

3. $\mathfrak{S}_7$: 'Q(c) $\supset$ P(b)'.
   $\mathfrak{S}_8$: 'If Carmel has a harbor, New York is large'.
   $\mathfrak{S}_9$: ' 'Q(c)' implies 'P(b)' ' (or ' 'P(b)' is an implicate of 'Q(c)' ' or ' 'Q(c)'$\rightarrow$'P(b)' ').

$\mathfrak{S}_2$, not $\mathfrak{S}_3$, is the direct translation of $\mathfrak{S}_1$ into English; likewise, $\mathfrak{S}_5$, not $\mathfrak{S}_6$, of $\mathfrak{S}_4$; and $\mathfrak{S}_8$, not $\mathfrak{S}_9$, of $\mathfrak{S}_7$. Here, for the sake of simplicity, we have translated '... $\equiv$ - - -' into '... if and only if - - -', and '... $\supset$ - - -' into 'if ... then - - -'. These translations are often appropriate; but in these examples they deviate somewhat from the customary use of the word 'if' and the phrase 'if and only if' in English, because these expressions are usually restricted to cases where there is a logical or causal or motivational connection between the two members. A more precise but somewhat lengthy translation of 'A ⊃ B' is 'not A, or B', and of 'A ≡ B' 'A and B, or, not A and not B'. The chief distinction is between $\mathfrak{S}_1$ and $\mathfrak{S}_2$ on the one hand and $\mathfrak{S}_3$ on the other. $\mathfrak{S}_1$ belongs to a symbolic object language. $\mathfrak{S}_2$ may be regarded as belonging either to English as an object language or, so to speak, to the object part of the Englih metalanguage, i.e. to that part which does not contain semiotical terms. On the other hand, $\mathfrak{S}_3$ belongs to the metalanguage and, moreover, to its semantical part. In the cases (2) and (3), the situation is analogous.

D9-5. $\mathfrak{T}_i$ is **disjunct** with $\mathfrak{T}_j$ (in S) $=_{\mathrm{Df}}$ at least one of them is true (and hence, not both of them false).

T9-25. If $\mathfrak{T}_i$ is disjunct with $\mathfrak{T}_j$, then $\mathfrak{T}_j$ is disjunct with $\mathfrak{T}_i$. (From D5.)

D9-6. $\mathfrak{T}_i$ is **exclusive** of $\mathfrak{T}_j$ (in S) $=_{\mathrm{Df}}$ not both of them are true (and hence, at least one is false).

T9-27. $\mathfrak{T}_i$ and $\mathfrak{T}_j$ are exclusive (of one another) if and only if $\mathfrak{T}_i + \mathfrak{T}_j$ is false. (From D6, T8, T1.)

We shall designate the **null class of sentences** in S, i.e. that class of the type of sentential classes which has no elements, by 'Λ_S' or simply 'Λ' (D7) and the **universal sentential class** in S, i.e. the class of all sentences of S by 'V_S' or simply 'V' (D8). Then Λ is true (T32); it fulfills the condition of D1 that every sentence of it is true, because there is no such sentence. There is no analogous theorem for V. Although in most semantical systems V is false, we cannot state it as a general theorem that V is false, but only that V is false if there is a false $\mathfrak{T}_i$ at all in S (T43b). There are systems in which every sentence and hence every $\mathfrak{K}_i$ and every $\mathfrak{T}_i$ is true, including V (e.g. in the system S_5, which is like S_2, § 7, except that in_3 designates San Francisco instead of Carmel). The fact that every system contains a true $\mathfrak{K}_i$, namely Λ, but not every system a false $\mathfrak{K}_i$, reveals an astonishing *lack of symmetry* in the edifice of semantics. We shall find in the discussion in [II] (see Bibliography) that this is due to a lack of symmetry in the customary way of dealing with sentential classes. By employing new concepts, which are not definable by the concepts ordinarily used, it will be possible to gain symmetry for semantics and simultaneously for syntax.

+**D9-7.** Λ (Λ_S) $=_{\mathrm{Df}}$ the null sentential class.

T9-30. For every $\mathfrak{K}_i$, $\Lambda \subset \mathfrak{K}_i$. (From D7.)

+**T9-32.** Λ is true. (From D7, D1; can also be seen with the help of T30, 6, and 4.)

T9-33. Every $\mathfrak{T}_i \rightarrow \Lambda$. (From T32 and 13.)

T9-34 (lemma). If $\Lambda \rightarrow \mathfrak{T}_j$, then $\mathfrak{T}_j$ is true. (From T32 and 10.)

T9-35. $\mathfrak{T}_i$ is true if and only if $\Lambda \rightarrow \mathfrak{T}_i$. (From T34; T13.)

+**D9-8.** V (V_S) $=_{\mathrm{Df}}$ the universal sentential class.

T9-37 (lemma). Every $\mathfrak{S}_i \in V$.

T9-38 (lemma). Every $\mathfrak{K}_i \subset V$.

T9-39 (lemma). $V \rightarrow$ every $\mathfrak{S}_i$. (From T37 and 15.)

T9-40 (lemma). $V \rightarrow$ every $\mathfrak{K}_i$. (From T38 and 16.)

+**T9-41.** $V \rightarrow$ every $\mathfrak{T}_i$. (From T39 and 40.)

T9-42. Each of the following conditions is a sufficient and necessary condition for V to be true in S:

a. Every $\mathfrak{S}_i$ in S is true.
b. Every $\mathfrak{K}_i$ in S is true.
c. Every $\mathfrak{T}_i$ in S is true. (From D8, D1.)

T9-43. Each of the following conditions is a sufficient and necessary condition for V to be false in S:

a. At least one sentence in S is false.
b. At least one sentential class in S is false
c. At least one $\mathfrak{T}_i$ in S is false.
(From T42.)

The term 'comprehensive' (D9) is introduced only for the sake of corresponding L- and C- terms (D14-5, D30-6).

D9-9. $\mathfrak{T}_i$ is *comprehensive* (in S) $=_{\mathrm{Df}}$ $\mathfrak{T}_i \rightarrow$ every sentence in S.

T9-50. Each of the following conditions is a sufficient and necessary condition for $\mathfrak{T}_i$ to be comprehensive:

a. $\mathfrak{T}_i \rightarrow$ V. (From D9, T17.)
b. $\mathfrak{T}_i$ is equivalent to V. (From (a), T41.)
c. $\mathfrak{T}_i \rightarrow$ every $\mathfrak{R}_j$. (From (a), T40, T14b.)
d. $\mathfrak{T}_i \rightarrow$ every $\mathfrak{T}_j$. (From D9, (c).)

We shall now define the concept of equivalence of semantical systems; it must clearly be distinguished from the concept of equivalence of sentences or sentential classes (D9–4).

D9-11. The *semantical system* S_m is *equivalent* to the semantical system S_n $=_{\mathrm{Df}}$ the following two conditions are fulfilled:

a. S_m and S_n contain the same sentences.
b. For every $\mathfrak{S}_i$, $\mathfrak{S}_i$ is true in S_m if and only if $\mathfrak{S}_i$ is true in S_n.

T9-70. The systems S_m and S_n are equivalent if and only if the following three conditions are fulfilled:

a. S_m and S_n contain the same sentences.
b. For every $\mathfrak{S}_i$, if $\mathfrak{S}_i$ is true in S_m, it is true in S_n.
c. For every $\mathfrak{S}_i$, if $\mathfrak{S}_i$ is false in S_m, it is false in S_n.
(From D11.)

T9-71. If S_m and S_n are equivalent systems, then each of the following concepts (applied to sentences and sentential classes) has the same extension in S_m as in S_n: **a.** truth, **b.** falsity, **c.** implication, **d.** equivalence, **e.** disjunctness, **f.** exclusion, **g.** comprehensiveness. ((a), from D11, D1; (b) to (g), from D11 and the definitions of these concepts, which are all based on the concept of truth.)

§ 10. Further Radical Concepts

Some concepts applicable to attributes are defined, among them 'universal', 'empty', 'implicate', 'equivalent'. These concepts are absolute, i.e. not dependent upon language. With their help, corresponding semantical concepts ('universal in S', etc.), applicable to predicates, are defined. Further, the terms 'interchangeable', 'extensional sentence', and 'extensional system' are defined. Theorems for the concepts defined are stated.

There are some semantical properties and relations of predicates analogous to some of the properties and relations of sentences defined in § 9. As a preliminary step to the introduction of these semantical terms we shall first define some terms which may belong to any suitable object language rather than to the metalanguage. (They are, however, not descriptive but logical in the sense to be explained in § 13.) Therefore these terms are not accompanied by a reference to a language, but — as we shall say later (§ 17) — they are used in an absolute way. The concepts designated by these terms are thus not dependent upon language; we call them *absolute concepts* (§ 17).

In the following definitions, M and N are attributes of any degree, say n. '$M(u)$' means 'M holds for the argument u' or 'u possesses the attribute M', where u is a sequence of n members belonging to types suitable for M. H is a relation of degree two; '$H(x,y)$' means 'H holds between x and y'

D10-1. M is **universal** $=_{\mathrm{Df}}$ for every u, $M(u)$.

D10-2. M is **empty** $=_{\mathrm{Df}}$ for every u, not $M(u)$ (in other words, there is no u such that $M(u)$).

D10-3. M is **non-empty** $=_{\mathrm{Df}}$ M is not empty (in other words, there is at least one u such that $M(u)$).

D10-4. N is an **implicate** of M (or, M implies N) $=_{\mathrm{Df}}$ for every u, if $M(u)$ then $N(u)$ (in other words, the extension of M is contained in that of N).

D10-5. M is **equivalent** to N $=_{\text{Df}}$ for every u, $M(u)$ if and only if $N(u)$ (in other words, M and N imply one another, they coincide, they have the same extension).

D10-6. M is **exclusive** of N $=_{\text{Df}}$ there is no u such that $M(u)$ and $N(u)$.

Further, the familiar concepts of the theory of relations belong to this kind of absolute concept, e.g. 'symmetric', 'non-symmetric', 'asymmetric', 'reflexive', 'non-reflexive', 'irreflexive', 'transitive', 'non-transitive', 'intransitive', 'connected', 'one-many', 'many-one', 'one-one', etc. We shall give only one example here:

D10-7. H is *symmetric* $=_{\text{Df}}$ for every x and y, if $H(x,y)$ then $H(y,x)$.

Now we decide to use the same terms as semantical terms also, hence for different but closely corresponding concepts. While the terms in their absolute use defined above are applied to attributes, in their semantical use they will be applied to those predicates which designate attributes of the kind specified. For these concepts, the dependence upon a language system is essential. Thus e.g. (the property of being) large is non-empty independently of any language, just because of the fact that there are some large things. On the other hand, the predicate 'P' is non-empty *in* S_3 (§ 8) because of the same fact; the same predicate 'P' may be empty in some other system because there it may designate some other property which happens to be empty.

D10-10. A *predicate* $\mathfrak{pr}_i$ is **a. universal (b. empty, c. non-empty)** in S $=_{\text{Df}}$ the attribute designated by $\mathfrak{pr}_i$ in S is **a.** universal (**b.** empty, **c.** non-empty, respectively).

D10-11. $\mathfrak{pr}_i$ is **a.** an **implicate** of (**b. equivalent** to, **c. exclusive** of) $\mathfrak{pr}_j$ in S $=_{\text{Df}}$ the designatum of $\mathfrak{pr}_i$ in S is **a.** an implicate of (**b.** equivalent to, **c.** exclusive of, respectively) the designatum of $\mathfrak{pr}_j$.

D10-12. A *predicate* $\mathfrak{pr}_i$ of degree two is *symmetric* in S $=_{\text{Df}}$ the relation designated by $\mathfrak{pr}_i$ in S is symmetric.

Analogous definitions may be laid down for the other terms of the theory of relations.

The following concept is of interest chiefly because of the corresponding L- and C- concepts (D14-6, D31-6).

D10-15. $\mathfrak{A}_i$ is **interchangeable** with $\mathfrak{A}_j$ (in S) $=_{\text{Df}}$ any closed sentence $\mathfrak{S}_i$ is equivalent to every sentence $\mathfrak{S}_j$ constructed out of $\mathfrak{S}_i$ by either replacing $\mathfrak{A}_i$ at some place in $\mathfrak{S}_i$ by $\mathfrak{A}_j$ or $\mathfrak{A}_j$ by $\mathfrak{A}_i$, and there is at least one pair of sentences $\mathfrak{S}_i$ and $\mathfrak{S}_j$ of this kind. (The last condition is added in order to exclude trivial cases.)

If a sentence $\mathfrak{S}_i$ is constructed out of other sentences as components with the help of some of the ordinary sentential connectives (as e.g. in S_3 and S_4, § 8) then the truth-value of $\mathfrak{S}_i$ depends merely upon the truth-values of its components. (Therefore, a sentence of this kind is sometimes called a truth-function of its components; we shall call it *extensional* with respect to its partial sentences.) This concept is defined in a general way in D20.

D10-20. $\mathfrak{S}_i$ is **extensional** (in S) in relation to a partial sentence $\mathfrak{S}_j$ occurring at a certain place in $\mathfrak{S}_i$ $=_{\text{Df}}$ for every closed (§ 6) $\mathfrak{S}_k$, if $\mathfrak{S}_j$ is equivalent to $\mathfrak{S}_k$, then $\mathfrak{S}_i$ is equivalent to the sentence constructed out of $\mathfrak{S}_i$ by replacing $\mathfrak{S}_j$ at the place in question by $\mathfrak{S}_k$.

D10-21. The *system* S is **extensional** in relation to partial sentences $=_{\text{Df}}$ for every $\mathfrak{S}_i$ in S, if $\mathfrak{S}_i$ contains a closed sentence $\mathfrak{S}_j$ at some place, then $\mathfrak{S}_i$ is extensional in relation to $\mathfrak{S}_j$ at that place.

T10-20. If S is extensional in relation to partial sentences, then any two closed equivalent sentences in S are interchangeable. (From D21, D20, D15.)

§ 11. Variables

If a system S contains variables, then, on the basis of the rules of designation and as basis for the rules of truth, we lay down first *rules of values*, and then either *rules of determination* or *rules of fulfillment*. The rules of values specify which entities are the values of the variables of the kinds occurring in S; the rules of determination specify which attributes are determined by the sentential functions in S; the rules of fulfillment specify which entities fulfill the sentential functions in S.

The examples of semantical systems discussed so far (S_1 to S_4, §§ 7 and 8) are constructed in a very simple way. They lack one important feature, variables. The chief application of variables is in expressing universal and existential propositions.

If a system S is to contain variables, the classification of signs, which precedes the formulation of rules, has to specify the kinds of variables. The rules of formation refer to these kinds in describing the forms of sentences. Then, in a **rule of values** related to the rules of designation, it is stated for each kind of variable which entities are to be **values** of the variables of that kind. Their class is sometimes called the **range of values** of the variables in question. If an expression $\mathfrak{A}_i$ or a sign $\mathfrak{a}_i$ designates a value of a variable $\mathfrak{v}_j$, we call $\mathfrak{A}_i$ a **value expression** and $\mathfrak{a}_i$ a value sign of $\mathfrak{v}_j$. A rule of values might e.g. state that the range of values of the individual variables $\mathfrak{i}$ in the system S comprehends all space-time points, or all physical things, or all events, or all human beings in general, or all human beings living at a certain time, etc. The values of the $\mathfrak{i}$ are then called the individuals in S. A rule for another kind of variables, say $\mathfrak{p}$, might state that all properties of individuals are their values, or all second-degree relations of individuals, or all attributes of any degree of individuals, or all properties of any finite level, or all attributes of any finite level, etc. A rule for still an-

other kind of variables, say $\mathfrak{f}$, might state that the propositions (designata of sentences) are their values.

Further, for a system S containing variables, rules have to be given specifying which entities are **determined** *by the expressional functions* (i.e. expressions with free variables; see § 6) of various forms, and especially which attributes are determined by sentential functions. These rules which define 'determination in S' are called **rules of determination.**

Then, with the help of the concepts defined by the preceding rules, especially the range of values of a variable and the attribute determined by a sentential function, truth rules for general sentences have to be laid down.

Example of a semantical system containing variables. We construct the system S_6 out of S_3 (§ 8) by adding new signs and rules. (S_6 contains only individual variables; all sentences are closed; all operands have molecular form, i.e. they do not contain operators.) Here again, to facilitate understanding, we sometimes use expressions of the object language included in quotes.

Semantical system S_6

1. Classification of signs. In addition to the signs of S_3, S_6 contains '$\exists$' and an infinite number of $\mathfrak{i}$ ('x', 'y', etc.).

2. Rules of formation. An expression $\mathfrak{A}_k$ in S_6 is a *sentential function* in S_6 $=_{Df}$ $\mathfrak{A}_k$ has one of the following forms: α. $\mathfrak{pr}(\mathfrak{i})$; β. $\sim(\mathfrak{A}_i)$, where $\mathfrak{A}_i$ is a sentential function; γ. $(\mathfrak{A}_i) \vee (\mathfrak{A}_j)$, where $\mathfrak{A}_i$ and $\mathfrak{A}_j$ are sentential functions containing the same variable.

An expression $\mathfrak{A}_k$ in S_6 is a *sentence* ($\mathfrak{S}$) in S_6 $=_{Df}$ $\mathfrak{A}_k$ has one of the following forms: a. $\mathfrak{pr}_i(\mathfrak{in}_j)$, where $\mathfrak{in}_j$ is a constant; b. $\sim(\mathfrak{S}_i)$; c. $(\mathfrak{S}_i) \vee (\mathfrak{S}_j)$; d. $(\mathfrak{i}_j)$ $(\mathfrak{A}_i)$, where $\mathfrak{A}_i$ is a sentential function containing $\mathfrak{i}_j$; e. $(\exists\mathfrak{i}_j)$ $(\mathfrak{A}_i)$, where $\mathfrak{A}_i$ is a sentential function containing $\mathfrak{i}_j$.

3A. Rules of designation. The same as in S_3. (We might, of course, add in S_6 more $\mathfrak{pr}$ and $\mathfrak{in}$ and then specify here the designatum of each of these signs.)

3B. Rules of determination. A sentential function $\mathfrak{A}_k$ *determines* in S_6 the property F $=_{Df}$ one of the following three conditions is fulfilled:

a. $\mathfrak{A}_k$ has the form $\mathfrak{pr}_i\,(\mathfrak{i}_j)$, and $\mathfrak{pr}_i$ designates F;
b. $\mathfrak{A}_k$ has the form $\sim(\mathfrak{A}_i)$, and F is the property of not having the property determined by $\mathfrak{A}_i$;
c. $\mathfrak{A}_k$ has the form $(\mathfrak{A}_i) \vee (\mathfrak{A}_j)$, and F is the property of having either the property determined by $\mathfrak{A}_i$ or that determined by $\mathfrak{A}_j$ or both.

3C. Rule of values. *Values* of the $\mathfrak{i}$ in S_6 are the towns in the United States.

4. Rules of truth. $\mathfrak{S}_k$ is *true* in $S_6 =_{Df}$ one of the following conditions is fulfilled:

(a), (b), and (c) as in S_3.
d. $\mathfrak{S}_k$ has the form $(\mathfrak{i}_j)\,(\mathfrak{A}_i)$ and every value of $\mathfrak{i}_j$ (i.e. every town in the United States) has the property determined by $\mathfrak{A}_i$.
e. $\mathfrak{S}_k$ has the form $(\exists\mathfrak{i}_j)\,(\mathfrak{A}_i)$ and at least one value of $\mathfrak{i}_j$ has the property determined by $\mathfrak{A}_i$.

The rules, especially those of determination, become more complicated in a system where operators within operands and therefore sentential functions of higher degree occur (e.g. '$(x)(\exists y)(\,.\,.\,x\,.\,.\,y\,.\,.)$'). Here, an order of the variables must be specified, an alphabetical order, so to speak. It is very convenient for many purposes, and especially for the formulation of rules for systems containing variables, to supplement the English word language (as metalanguage) by adding variables and the operators '(x)', '$(\exists x)$', and '(λx)'.

Examples of *rules of determination* ('M' is used as a '$^1p^n$').

1. If (the sentential function) $\mathfrak{A}_i$ determines (the attribute of degree n) M and if $\mathfrak{i}_k$ is the mth in alphabetical order among the n variables occurring freely in $\mathfrak{A}_i$, then the sentential function (of degree n–1) $(\mathfrak{i}_k)(\mathfrak{A}_i)$ determines $(\lambda x_1, x_2, \,.\,.\, x_{m-1}, x_{m+1}, \,.\,.\,.\, x_n)[(x_m)M(x_1, x_2, \,.\,.\, x_m, \,.\,.\, x_n)]$ (this is an attribute of degree n–1). (Formulated in words and variables but without symbolic operators, it would run like this: "$(\mathfrak{i}_k)(\mathfrak{A}_i)$ determines that relation which holds between $x_1, x_2, \,.\,.\, x_{m-1}, x_{m+1}, \,.\,.\, x_n$ if and only if for every individual x_m, M holds between $x_1, x_2, \,.\,.\,.\, x_m, \,.\,.\, x_n$".)
2. Under the same conditions $(\exists\mathfrak{i}_k)(\mathfrak{A}_i)$ determines $(\lambda x_1, x_2, \,.\,.\, x_{m-1}, x_{m+1}, \,.\,.\, x_n)\,[(\exists x_m)\,M(x_1, x_2, \,.\,.\, x_n)]$.

If S contains other kinds of variables, then the rules of values for these kinds are of course different from the examples given here (as shown by the examples given at the beginning of this section). But the form of rules of determination is in all essential respects similar to that of the examples just given.

The concept of fulfillment (or satisfaction) to be defined now is closely related to that of determination.

D11-1. *u* **fulfills** $\mathfrak{A}_i$ in $S =_{Df}$ there is an M such that $\mathfrak{A}_i$ determines M, and that $M(u)$ (i.e. there is an attribute M of degree n such that the sentential function $\mathfrak{A}_i$ of degree n determines M and that M holds for u, which is a sequence of degree n).

Examples. 1. The ordered pair (i.e. sequence of two members) Castor, Pollux (a pair of objects, not of names!) fulfills the sentential function 'x ist ein Bruder von y' in German. 2. Chicago fulfills '$P(x)$' in S_6. 3. Suppose that the system S_7 contains S_6 and, in addition, predicate variables ('F' etc.). The simple formulation "Chicago, large fulfills '$F(x)$' in S_7" is, unfortunately, not permitted by the traditional English grammar; therefore we have to replace it by the following clumsy formulation: "The pair consisting of Chicago and the property of being large fulfills '$F(x)$' in S_7".

D1 defines 'fulfillment' on the basis of 'determination'; the latter term is hereby supposed to be defined by rules of determination. The inverse procedure is also possible; 'determination' can be defined on the basis of 'fulfillment' (DA). $\mathfrak{A}_i$ is here a sentential function of degree n, M an attribute of degree n, u a sequence of degree n.

D11-A. $\mathfrak{A}_i$ *determines* M in $S =_{Df}$ for every u, $M(u)$ if and only if u fulfills $\mathfrak{A}_i$.

Thus fulfillment may serve as the basic concept in the construction of a semantical system, defined by rules of fulfillment instead of rules of determination. (For the formulation of rules of fulfillment, as for those of determination, it is convenient but not necessary to make use of the concept of designation to be defined by rules of designation.) Then determination would be defined on the basis of fulfillment as in DA, and truth on the basis of determination, as e.g. in the truth rules of S_6.

There is another way of defining truth directly on the basis of fulfillment without the use of the concept of determination. The definition can be given an especially simple form (DB below) if we make use of the concept of the null sequence (i.e. the sequence which has no members, analogous to the null class) and regard a sentence as a sentential function of degree zero. Analogously, we may regard a proposition as an attribute of degree zero. [This widening out of the concepts would of course involve certain modifications in previous explanations and definitions, especially with respect to the concept of fulfillment.]

D11-B. $\mathfrak{S}_i$ is *true* in S $=_{\text{Df}}$ the null sequence fulfills $\mathfrak{S}_i$.

Tarski [Wahrheitsbegriff] bases his definition of truth on the concept of fulfillment or satisfying (but in a way technically different from that indicated here). This procedure seems to have certain advantages in those cases where it can be applied, namely for languages containing variables.

In a later volume of these studies it is planned to make a systematic comparison of the different forms of bases for semantical systems.

Previously we defined 'universal', etc., for attributes (D10–1, etc.) and 'universal in S', etc., for predicates designating those attributes (D10–10, etc.). We now define the same terms for sentential functions determining those attributes.

D11-2. A *sentential function* $\mathfrak{A}_i$ is **a. universal** (**b. empty, c. non-empty**) in S $=_{\text{Df}}$ the attribute determined by $\mathfrak{A}_i$ in S is **a.** universal (**b.** empty, **c.** non-empty, respectively).

D11-3. A sentential function $\mathfrak{A}_i$ is **a.** an **implicate** of (**b. equivalent** to, **c. exclusive** of) a sentential function $\mathfrak{A}_j$ in S $=_{\text{Df}}$ the attribute determined by $\mathfrak{A}_i$ in S is **a.** an implicate of (**b.** equivalent to, **c.** exclusive of, respectively) the attribute determined by $\mathfrak{A}_j$ in S.

§ 12. The Relation of Designation

It is convenient to adopt for semantical discussions a use of the term 'designation' which is wider than the ordinary use, so that we may speak of the designata not only of individual constants and predicates but also of functors and sentences. A general convention for this wider use is laid down (D12–B).

To which signs and expressions of a semantical system S (i.e. of its object language) is it possible and advisable to apply the relation of designation? So far we have applied it to individual constants and predicates of different levels and degrees. In a similar way it may of course be applied to functors of any type occurring in S. But it is possible to enlarge the domain of application to a considerable extent, and it seems convenient to do so for the signs and expressions of S of all those types for which variables occur in the metalanguage, even if this includes the type of sentences and the types of sentential connectives. We use as metalanguage in this section the English language supplemented by variables, including propositional variables. Instead of 'u designates v in S' we write '$\text{Des}_S(u,v)$' or simply '$\text{Des}(u,v)$' where the context makes clear which system is meant.

Instead of, and in analogy to, the rules of truth based on the narrower concept of designation in the previous form of a semantical system (e.g. S_3 in § 8) we should have here rules of designation for sentences and, in addition, a general explicit definition for truth; the latter has the same form in all systems and may therefore be stated once for all in general semantics (D1).

D12-1. $\mathfrak{S}_i$ is *true* in S $=_{\text{Df}}$ there is a (proposition) p such that Des($\mathfrak{S}_i$,p) and p.

In order to satisfy the ordinary rule of types, we should have to use different terms for the relation of designation as applied to individuals, attributes (of different types), and propositions, e.g. 'DesInd', 'DesAttr', 'DesProp'. It is, however, much more convenient to use only one term 'Des'. This does not lead to ambiguities because the type of the second argument makes clear which kind of designation is meant. But this use presupposes a suitable structure of the metalanguage so as to avoid the restrictions by the ordinary rule of types in this point (see remark below).

Example. In order to reformulate the *system* S_3 (§ 8) in the way described, we replace (3) by (3A) and (3B), and (4) (§ 8) by (3C) and (4) (here). (3A) and (3B) are explicit definitions; (3C) is recursive, like (4) in § 8. 3A, B, and C could be combined into one recursive definition for 'Des_{S_3}'.

3. Rules of designation.

A. For individuals.

DesInd$_{S_3}$ ($\mathfrak{in}_i$,x) $=_{\text{Df}}$ one of the following three conditions is fulfilled:

a. $\mathfrak{in}_i$ = 'a', and x = Chicago,
b. $\mathfrak{in}_i$ = 'b', and x = New York,
c. $\mathfrak{in}_i$ = 'c', and x = Carmel.

B. For attributes.

DesAttr$_{S_3}$ ($\mathfrak{pr}_i$,F) $=_{\text{Df}}$ one of the following two conditions is fulfilled:

a. $\mathfrak{pr}_i$ = 'P', and F = (the property of being) large,
b. $\mathfrak{pr}_i$ = 'Q', and F = having a harbor.

C. For propositions.

DesProp$_{S_3}$ ($\mathfrak{S}_k$,p) $=_{\text{Df}}$ one of the following three conditions is fulfilled:

a. $\mathfrak{S}_k$ has the form $\mathfrak{pr}_i$ ($\mathfrak{in}_j$), and there is an F and an x such that DesAttr($\mathfrak{pr}_i$,F) and DesInd($\mathfrak{in}_j$,x), and p = (the proposition that) x is F.
b. $\mathfrak{S}_k$ has the form $\sim\mathfrak{S}_i$, and there is a q such that DesProp($\mathfrak{S}_i$,q), and p = not q.
c. $\mathfrak{S}_k$ has the form $\mathfrak{S}_i \vee \mathfrak{S}_j$, and there is a q and an r such that DesProp($\mathfrak{S}_i$,q) and DesProp($\mathfrak{S}_j$,r), and p = q or r.

4. Rule of truth.

$\mathfrak{S}_k$ is *true* in S_3 $=_{\text{Df}}$ there is a (proposition) p such that DesProp($\mathfrak{S}_k$,p) and p.

Application of the rules. It follows from (3Aa), (3Ba), (3Ca), that DesProp('P(a)', Chicago is large); and hence with (4), that 'P(a)' is true in S_3 if and only if Chicago is large. A similar result holds for each of the other sentences of S_3. Therefore, the definition of 'true in S_3' given by the rules stated above fulfills the requirement of adequacy (§ 7); it is merely another formulation for the same system S_3.

According to the ordinary *rule of types*, usually called the simple theory of types, a particular argument-place beside a particular predicate may be filled only by expressions which all have the same type and hence the same level and the same degree. Therefore, on the basis of this rule, we could not have 'x', 'F', and 'p' as second arguments to the same predicate 'Des', as we had above. [The same holds for 'Chicago' and 'the property - - -' as second arguments for 'designates' in the formulation of rule (3) for S_3 in § 8; that already was a violation of the rule of types.] We may, however, modify the rule of types by admitting transfinite levels; a predicate of level ω is allowed to take as arguments expressions of any finite level, including sentences, which we assign to the zero level. If we assign 'Des' to this level ω, then its use instead of 'DesInd', 'DesAttr', and 'DesProp' in the examples mentioned, and likewise its use with arguments of still other types, is correct. Another way of accommodating 'Des' as here used would be to use as metalanguage a language system without distinctions of types or levels; systems of this kind have recently been constructed especially by Quine [Math. Logic] and Bernays (*Journ. Symb. Logic*, vol. 2 (1937) and subsequent volumes).

Concerning the simple theory of types see [Syntax] § 27, [Logic] §§ 21b and 29b. Concerning transfinite levels see [Syntax] § 53 with references to Hilbert and Gödel, Tarski [Wahrheitsbegriff] § 7, Carnap [Logic] § 29b.

Sometimes objections are raised, especially by empiricists, against the wider use of the relation of designation and especially against its application as a relation between sentences and propositions. It is said that, while object names (individual constants) and predicates do designate something, namely objects and properties or relations, a sentence does not designate anything; it rather describes or states that something is the case. This may indeed be true with respect to the customary use of the words 'designation', 'to designate', etc., in English. It is obviously not in accordance with ordinary usage to say " 'P(a)' designates Chicago is large"; and the same holds for corresponding sentences in languages of similar structure. First, English grammar does not admit a sentence in the position of grammatical object. This difficulty, however, can easily be overcome by inserting 'that' after 'designates'. Second, 'to designate' would ordinarily not be used in this case. But this does not seem to me to be a sufficient reason against its wider use as a technical term. Very often, in transferring a word from the ordinary language into the language of science, we enlarge its domain of application. The only question in such a case seems to be a question of expediency; and the decision will depend chiefly upon whether the similarity between the cases of ordinary application and the new cases is strong enough for the enlargement to seem natural. In the case under discussion there seems to be a strong analogy between the different cases, in spite of the difference in types; this will soon become clear.

This analogy will also help us to remove from our path some other stumblingblocks. With respect to some of the types to which the relation of designation is here applied, the puzzling question is sometimes raised, what exactly is

the kind of designata of the expressions of the one type or the other? Thus it is e.g. discussed whether the designatum of a thing-name (e.g. 'Chicago') is the corresponding thing or its unit-class (e.g. whether it is Chicago or {Chicago}). Further, the question is discussed whether the designatum of a predicate of first degree is a property or a class. In both cases it is said as an argument in favor of the second answer that a designatum should always be a class. If designata of sentences are admitted at all, the question is raised whether they are states of affairs (or possible facts, conditions, etc., which seems chiefly a terminological difference) or rather thoughts.

Let us suppose for the moment that we understand a given object language *S*, say German or S_3 (§ 8), in such a way that we are able to translate its expressions and sentences into the metalanguage *M* used, say English (including some variables and symbols). It does not matter whether this understanding is based on the knowledge of semantical rules or is intuitive; it is merely supposed that, if an expression is given (say e.g. 'Pferd', 'drei' in German, 'P', 'P(a)' in S_3), for all practical purposes we know an English expression corresponding to it as its "literal translation" (e.g. 'horse', 'three'; 'large', 'Chicago is large'). Then we will lay down a definition of adequacy for the concept of designation, which is not itself a definition for a term 'Des_S' (or 'to designate in *S*') but a standard with which to compare proposed definitions. In a similar way, we had before a definition of adequacy for truth (D7–B), and later we shall have one for L-truth (D16–1). 'Adequacy' means here simply agreement with our intention for the use of the term.

D12-B. A predicate of second degree $\mathfrak{pr}_i$ in *M* is an *adequate* predicate for *designation* in *S* $=_{Df}$ every sentence in *M* of the form $\mathfrak{pr}_i$ ($\mathfrak{A}_j$, $\mathfrak{A}_k$) where $\mathfrak{A}_j$ is a name (or a syntactical description) in *M* of an expression $\mathfrak{A}_m$ of *S* (belonging to one of the kinds of expressions for which $\mathfrak{pr}_i$ is defined) and $\mathfrak{A}_k$ is a translation of $\mathfrak{A}_m$ into M, is true in M.

If $\mathfrak{pr}_i$ is adequate then we also call its definition and its designatum, i.e. the relation defined as designation, adequate. This definition of adequacy leaves open the question of which types are admitted as arguments for $\mathfrak{pr}_i$; it determines only *how* a predicate for designation is to be used for certain types *if* we choose to use it for these types. Hence we may, for instance, restrict its use, in the sense of the objection mentioned, to $\mathfrak{in}$ and $\mathfrak{pr}$. But it is proposed here to use it for all types for which there are variables in M, i.e. to admit as a second argument $\mathfrak{A}_k$ any value expression of any variable in M. The practical justification for the given definition of adequacy lies in these two facts: 1. It supplies a general rule for all the different types, in a simple way; 2. it seems to be in agreement with the ordinary use of 'designation' as far as this use goes.

On the basis of an adequate relation of designation, the question of the designatum of an object name is to be answered in favor of the object (see example 2a below) as against its unit class.

Examples. 1. If 'Des_G' is an adequate predicate (in M, i.e. English) for designation in German, then the following sentences are true: a. 'Des_G('Pferd', horse)'; b. 'Des_G('drei', three)'. 2. If 'Des_{S_3}' is defined as indicated above (taking the place of 'DesInd_{S_3}', 'DesAttr_{S_3}', and 'DesProp_{S_3}' simultaneously), then it is an adequate predicate for designation in S_3. Among other sentences, the following must become true: a. 'Des_{S_3}('a', Chicago)'; b. 'Des_{S_3}('P', large)'; c. 'Des_{S_3}('P(a)', Chicago is large)'; and they are indeed true, as we have seen before. We see that adequacy requires us to write in the argument-place of 'Des' 'large' instead of 'largeness' (as English grammar would demand after the word 'designates') or 'the property of being large' (as we formulated it previously) or 'the class of large things'; and likewise 'horse' instead of 'the property of being a horse' or 'the class of horses'. This shows that we can assign designata to predicates without using either the term 'property' or 'class'. [The question whether a designatum, e.g. large, is a property or a class will thus not disturb us in using the relation of designation, but it, too, must finally, of course, be answered. The answer will depend upon the structure of the languages used, especially with respect to extensionality. The same holds for the question whether sentential designata are truth-values or whatever else. It is planned to discuss these questions in a later volume of these studies in connection with the discussion of extensional and non-extensional language systems.]

We define 'synonymous' on the basis of 'designation' (D2). Thus the term 'synonymous' may be applied in a narrower or wider way according to the narrower or wider domain of application chosen for 'designation'.

D12-2. $\mathfrak{A}_i$ in S_m is **synonymous** with $\mathfrak{A}_j$ in S_n $=_{Df}$ $\mathfrak{A}_i$ designates in S_m the same entity as $\mathfrak{A}_j$ in S_n.

Thus the relation of synonymity is in general not restricted to the expressions of one system. Most of the semantical relations can be applied to expressions of *different systems*, even those which, for the sake of simplicity and in consideration of their most frequent use, we have defined with respect to one system.

Example. 'Gross' in German is synonymous with 'P' in S_3 because Des_G('gross', large) and Des_{S_3}('P', large).

Examples of other semantical relations for two systems. Instead of D9-4, we might take the following definition:

D12-C. $\mathfrak{T}_i$ in S_m is *equivalent* to $\mathfrak{T}_j$ in S_n $=_{Df}$ either $\mathfrak{T}_i$ is true in S_m and $\mathfrak{T}_j$ is true in S_n, or $\mathfrak{T}_i$ is false in S_m and $\mathfrak{T}_j$ is false in S_n.

The same could be done with 'implicate', 'exclusive', 'disjunct', and also with the corresponding L-terms (§ 14ff; see remark at the end of § 16), but not with the corresponding C-terms in syntax (§ 28).